This is exactly the conclusion that we stated without proof in Section 4.1. In QED, the coupling constant e is dimensionless; thus QED is (at least superficially) renormalizable.

10.2 Renormalized Perturbation Theory

In the previous section we saw that a renormalizable quantum field theory contains only a small number of superficially divergent amplitudes. In QED, for example, there are three such amplitudes, containing four infinite constants. In Chapters 6 and 7 these infinities disappeared by the end of our computations: The infinity in the vertex correction diagram was canceled by the electron field-strength renormalization, while the infinity in the vacuum polarization diagram caused only an unobservable shift of the electron's charge. In fact, it is generally true that the divergences in a renormalizable quantum field theory never show up in observable quantities.

To obtain a finite result for an amplitude involving divergent diagrams, we have so far used the following procedure: Compute the diagrams using a regulator, to obtain an expression that depends on the bare mass (m_0), the bare coupling constant (e_0), and some ultraviolet cutoff (Λ). Then compute the physical mass (m) and the physical coupling constant (e), to whatever order is consistent with the rest of the calculation; these quantities will also depend on m_0, e_0, and Λ. To calculate an S-matrix element (rather than a correlation function), one must also compute the field-strength renormalization(s) Z (in accord with Eq. (7.45)). Combining all of these expressions, eliminate m_0 and e_0 in favor of m and e; this step is the "renormalization". The resulting expression for the amplitude should be finite in the limit $\Lambda \to \infty$.

The above procedure always works in a renormalizable quantum field theory. However, it can often be cumbersome, especially at higher orders in perturbation theory. In this section we will develop an alternative procedure which works more automatically. We will do this first for ϕ^4 theory, returning to QED in the next section.

The Lagrangian of ϕ^4 theory is

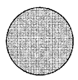
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4.$$

We now write m_0 and λ_0, to emphasize that these are the bare values of the mass and coupling constant, not the values measured in experiments.

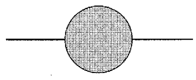
The superficial degree of divergence of a diagram with N external legs is, according to (10.13),

$$D = 4 - N.$$

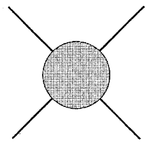
Since the theory is invariant under $\phi \to -\phi$, all amplitudes with an odd number of external legs vanish. The only divergent amplitudes are therefore



(unobservable vacuum energy shift);



$\sim\Lambda^2 + p^2 \log \Lambda +$ (finite terms);



$\sim \log \Lambda +$ (finite terms).

Ignoring the vacuum diagram, these amplitudes contain three infinite constants. Our goal is to absorb these constants into the three unobservable parameters of the theory: the bare mass, the bare coupling constant, and the field strength. To accomplish this goal, it is convenient to reformulate the perturbation expansion so that these unobservable quantities do not appear explicitly in the Feynman rules.

First we will eliminate the shift in the field strength. Recall from Section 7.1 that the exact two-point function has the form

$$\int d^4x \, \langle\Omega| T\phi(x)\phi(0) |\Omega\rangle \, e^{ip\cdot x} = \frac{iZ}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2), \tag{10.14}$$

where m is the physical mass. We can eliminate the awkward residue Z from this equation by rescaling the field:

$$\phi = Z^{1/2}\phi_r. \tag{10.15}$$

This transformation changes the values of correlation functions by a factor of $Z^{-1/2}$ for each field. Thus, in computing S-matrix elements, we no longer need the factors of Z in Eq. (7.45); a scattering amplitude is simply the sum of all connected, amputated diagrams, exactly as we originally guessed in Eq. (4.103).

The Lagrangian is much uglier after the rescaling:

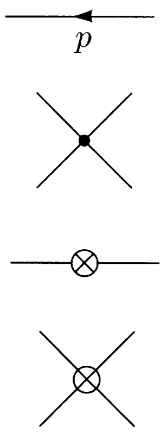
$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\phi_r)^2 - \frac{1}{2}m_0^2 Z\phi_r^2 - \frac{\lambda_0}{4!}Z^2\phi_r^4. \tag{10.16}$$

The bare mass and coupling constant still appear in $\mathcal{L}$, but they can be eliminated as follows. Define

$$\delta_Z = Z - 1, \qquad \delta_m = m_0^2 Z - m^2, \qquad \delta_\lambda = \lambda_0 Z^2 - \lambda, \tag{10.17}$$

where m and λ are the physically measured mass and coupling constant. Then the Lagrangian becomes

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4 \\ &+ \frac{1}{2}\delta_Z(\partial_\mu\phi_r)^2 - \frac{1}{2}\delta_m\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4. \end{aligned} \tag{10.18}$$



$$= \frac{i}{p^2 - m^2 + i\epsilon}$$

$$= -i\lambda$$

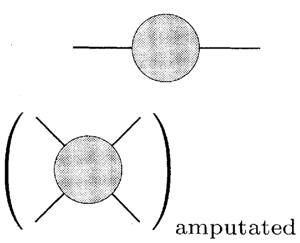
$$= i(p^2\delta_Z - \delta_m)$$

$$= -i\delta_\lambda$$

Figure 10.3. Feynman rules for ϕ^4 theory in renormalized perturbation theory.

The first line now looks like the familiar ϕ^4-theory Lagrangian, but is written in terms of the physical mass and coupling. The terms in the second line, known as *counterterms*, have absorbed the infinite but unobservable shifts between the bare parameters and the physical parameters. It is tempting to say that we have "added" these counterterms to the Lagrangian, but in fact we have merely split each term in (10.16) into two pieces.

The definitions in (10.17) are not useful unless we give precise definitions of the physical mass and coupling constant. Equation (10.14) defines m^2 as the location of the pole in the propagator. There is no obviously best definition of λ, but a perfectly good definition would be obtained by setting λ equal to the magnitude of the scattering amplitude at zero momentum. Thus we have the two defining relations,



$$= \frac{i}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2);$$

$$= -i\lambda \qquad \text{at } s = 4m^2,\ t = u = 0. \tag{10.19}$$

These equations are called *renormalization conditions.* (The first equation actually contains two conditions, specifying both the location of the pole and its residue.)

Our new Lagrangian, Eq. (10.18), gives a new set of Feynman rules, shown in Fig. 10.3. The propagator and the first vertex come from the first line of (10.18), and are identical to the old rules except for the appearance of the physical mass and coupling in place of the bare values. The counterterms in the second line of (10.18) give two new vertices (also called counterterms).

We can use these new Feynman rules to compute any amplitude in ϕ^4 theory. The procedure is as follows. Compute the desired amplitude as the sum of all possible diagrams created from the propagator and vertices shown in Fig. 10.3. The loop integrals in the diagrams will often diverge, so one must introduce a regulator. The result of this computation will be a function of the three unknown parameters δ_Z, δ_m, and δ_λ. Adjust (or "renormalize") these three parameters as necessary to maintain the renormalization conditions (10.19). After this adjustment, the expression for the amplitude should be finite and independent of the regulator.

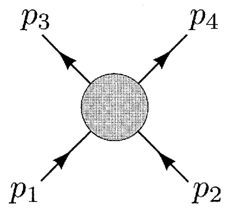
This procedure, using Feynman rules with counterterms, is known as *renormalized perturbation theory.* It should be contrasted with the procedure we used in Part 1, outlined at the beginning of this section, which is called *bare perturbation theory* (since the Feynman rules involve the bare mass and coupling constant). The two methods are completely equivalent. The differences between them are purely a matter of bookkeeping. You will get the same answers using either procedure, so you may choose whichever you find more convenient. In general, renormalized perturbation theory is technically easier to use, especially for multiloop diagrams; however, bare perturbation theory is sometimes easier for complicated one-loop calculations. We will use renormalized perturbation theory in most of the rest of this book.

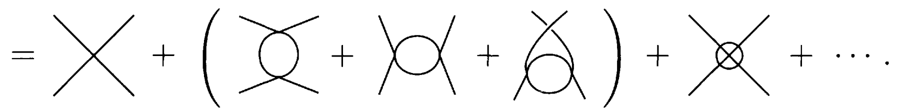
One-Loop Structure of ϕ^4 Theory

To make more sense of the renormalization procedure, let us carry it out explicitly at the one-loop level.

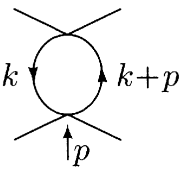
First consider the basic two-particle scattering amplitude,

$i\mathcal{M}(p_1 p_2 \to p_3 p_4) =$





If we define $p = p_1 + p_2$, then the second diagram is



$$= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2}$$

$$\equiv (-i\lambda)^2 \cdot iV(p^2). \tag{10.20}$$

Note that p^2 is equal to the Mandelstam variable s. The next two diagrams are identical, except that s will be replaced by t and u. The entire amplitude is therefore

$$i\mathcal{M} = -i\lambda + (-i\lambda)^2\big[iV(s) + iV(t) + iV(u)\big] - i\delta_\lambda. \tag{10.21}$$

According to our renormalization condition (10.19), this amplitude should equal $-i\lambda$ at $s = 4m^2$ and $t = u = 0$. We must therefore set

$$\delta_\lambda = -\lambda^2\big[V(4m^2) + 2V(0)\big]. \tag{10.22}$$

(At higher orders, δ_λ will receive additional contributions.)

We can compute $V(p^2)$ explicitly using dimensional regularization. The procedure is exactly the same as in Section 7.5: Introduce a Feynman parameter, shift the integration variable, rotate to Euclidean space, and perform the momentum integral. We obtain

$$\begin{aligned}
V(p^2) &= \frac{i}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{\big[k^2 + 2xk\cdot p + xp^2 - m^2\big]^2} \\
&= \frac{i}{2}\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\big[\ell^2 + x(1-x)p^2 - m^2\big]^2} \qquad (\ell = k + xp) \\
&= -\frac{1}{2}\int_0^1 dx \int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{\big[\ell_E^2 - x(1-x)p^2 + m^2\big]^2} \qquad (\ell_E^0 = -i\ell^0) \\
&= -\frac{1}{2}\int_0^1 dx\, \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\,\frac{1}{\big[m^2 - x(1-x)p^2\big]^{2-d/2}} \\
&\underset{d\to 4}{\longrightarrow} -\frac{1}{32\pi^2}\int_0^1 dx\Big(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log\big[m^2 - x(1-x)p^2\big]\Big),
\end{aligned} \tag{10.23}$$

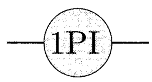
where $\epsilon = 4 - d$. The shift in the coupling constant (10.22) is therefore

$$\begin{aligned}
\delta_\lambda &= \frac{\lambda^2}{2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\int_0^1 dx\bigg(\frac{1}{[m^2 - x(1-x)4m^2]^{2-d/2}} + \frac{2}{[m^2]^{2-d/2}}\bigg) \\
&\underset{d\to 4}{\longrightarrow} \frac{\lambda^2}{32\pi^2}\int_0^1 dx\Big(\frac{6}{\epsilon} - 3\gamma + 3\log(4\pi) - \log\big[m^2 - x(1-x)4m^2\big] - 2\log\big[m^2\big]\Big).
\end{aligned} \tag{10.24}$$

These expressions are divergent as $d \to 4$. But if we combine them according to (10.21), we obtain the finite (if rather complicated) result,

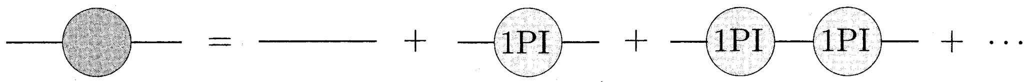
$$\begin{aligned}
i\mathcal{M} = -i\lambda - \frac{i\lambda^2}{32\pi^2}\int_0^1 dx\bigg[&\log\Big(\frac{m^2 - x(1-x)s}{m^2 - x(1-x)4m^2}\Big) + \log\Big(\frac{m^2 - x(1-x)t}{m^2}\Big) \\
&+ \log\Big(\frac{m^2 - x(1-x)u}{m^2}\Big)\bigg].
\end{aligned} \tag{10.25}$$

To determine δ_Z and δ_m we must compute the two-point function. As in Section 7.2, let us define $-iM^2(p^2)$ as the sum of all one-particle-irreducible insertions into the propagator:



$$= -iM^2(p^2). \tag{10.26}$$

Then the full two-point function is given by the geometric series,



$$= \frac{i}{p^2 - m^2 - M^2(p^2)}. \tag{10.27}$$

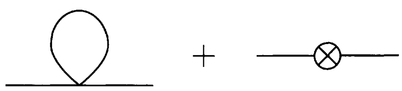
The renormalization conditions (10.19) require that the pole in this full propagator occur at $p^2 = m^2$ and have residue 1. These two conditions are equivalent, respectively, to

$$M^2(p^2)\big|_{p^2=m^2} = 0 \qquad \text{and} \qquad \frac{d}{dp^2}M^2(p^2)\big|_{p^2=m^2} = 0. \tag{10.28}$$

(To check the latter condition, expand M^2 about $p^2 = m^2$ in Eq. (10.27).)

Explicitly, to one-loop order,

$$-iM^2(p^2) =$$



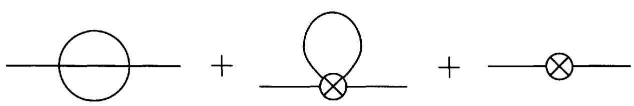
$$\begin{aligned} &= -i\lambda \cdot \frac{1}{2} \cdot \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m^2} + i(p^2\delta_Z - \delta_m) \\ &= -\frac{i\lambda}{2} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}} + i(p^2\delta_Z - \delta_m). \end{aligned} \tag{10.29}$$

Since the first term is independent of p^2, the result is rather trivial: Setting

$$\delta_Z = 0 \qquad \text{and} \qquad \delta_m = -\frac{\lambda}{2(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}} \tag{10.30}$$

yields $M^2(p^2) = 0$ for all p^2, satisfying both of the conditions in (10.28).

The first nonzero contributions to $M^2(p^2)$ and δ_Z are proportional to λ^2, coming from the diagrams

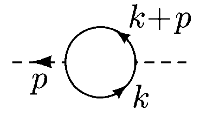
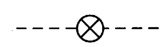


(10.31)

The second diagram contains the δ_λ counterterm, which we have already computed. It cancels ultraviolet divergences in the first diagram that occur when one of the loop momenta is large and the other is small. The third diagram is again the $(p^2\delta_Z - \delta_m)$ counterterm, and is fixed to order λ^2 by requiring that the remaining divergences (when both loop momenta become large) cancel. In Section 10.4 we will see an explicit example of the interplay of various counterterms in a two-loop calculation.

The vanishing of δ_Z at one-loop order is a special feature of ϕ^4 theory, which does not occur in more general theories of scalar fields. The Yukawa theory described in Section 4.7 gives an explicit example of a one-loop correction for which this counterterm is required.

In the Yukawa theory, the scalar field propagator receives corrections at order g^2 from a fermion loop diagram and the two propagator counterterms. Using the Feynman rules on p. 118 to compute the loop diagram, we find

$-iM^2(p^2) =$  $+$ 

$$
\begin{aligned}
-iM^2(p^2) &= -(-ig)^2 \int \frac{d^dk}{(2\pi)^d} \operatorname{tr}\left[\frac{i(\not k+\not p+m_f)}{(k+p)^2-m_f^2}\frac{i(\not k+m_f)}{k^2-m_f^2}\right] + i(p^2\delta_Z - \delta_m) \\
&= -4g^2 \int \frac{d^dk}{(2\pi)^d} \frac{k\cdot(p+k)+m_f^2}{((p+k)^2-m_f^2)(k^2-m_f^2)} + i(p^2\delta_Z-\delta_m), \qquad (10.32)
\end{aligned}
$$

where m_f is the mass of the fermion that couples to the Yukawa field. To evaluate the integral, combine denominators and shift as in Eq. (10.23). Then the first term in the last line becomes

$$
\begin{aligned}
-4g^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} & \frac{\ell^2 - x(1-x)p^2 + m_f^2}{(\ell^2 + x(1-x)p^2 - m_f^2)^2} \\
&= -4g^2 \int_0^1 dx \frac{-i}{(4\pi)^{d/2}} \left(\frac{\frac{d}{2}\Gamma(1-\frac{d}{2})}{\Delta^{1-d/2}} - \frac{\Delta\,\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \right) \\
&= \frac{4ig^2(d-1)}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-d/2}}, \qquad (10.33)
\end{aligned}
$$

where $\Delta = m_f^2 - x(1-x)p^2$.

Now we can see that both of the counterterms δ_m and δ_Z must take nonzero values in order to satisfy the renormalization conditions (10.28). To determine δ_m, we subtract the value of the loop diagram at $p^2 = m^2$ as before, so that

$$
\delta_m = \frac{4g^2(d-1)}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(1-\frac{d}{2})}{[m_f^2 - x(1-x)m^2]^{1-d/2}} + m^2\delta_Z. \qquad (10.34)
$$

To determine δ_Z, we cancel also the first derivative with respect to p^2 of the loop integral (10.33). This gives

$$\delta_Z = -\frac{4g^2(d-1)}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{x(1-x)\Gamma(2-\frac{d}{2})}{[m_f^2 - x(1-x)m^2]^{2-d/2}}$$
$$\underset{d\to 4}{\longrightarrow} -\frac{3g^2}{4\pi^2} \int_0^1 dx\, x(1-x)\Big(\frac{2}{\epsilon} - \gamma - \frac{2}{3} + \log(4\pi) - \log\big[m_f^2 - x(1-x)m^2\big]\Big). \tag{10.35}$$

Thus, in Yukawa theory, the propagator corrections at one-loop order require a quadratically divergent mass renormalization and a logarithmically divergent field strength renormalization. This is the usual situation in scalar field theories.

10.3 Renormalization of Quantum Electrodynamics

The procedure we followed in the previous section, yielding a "renormalized" perturbation theory formulated in terms of physically measurable parameters, can be summarized as follows:

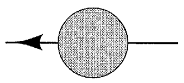
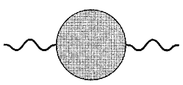
1. Absorb the field-strength renormalizations into the Lagrangian by rescaling the fields.
2. Split each term of the Lagrangian into two pieces, absorbing the infinite and unobservable shifts into counterterms.
3. Specify the renormalization conditions, which define the physical masses and coupling constants and keep the field-strength renormalizations equal to 1.
4. Compute amplitudes with the new Feynman rules, adjusting the counterterms as necessary to maintain the renormalization conditions.

Let us now use this procedure to construct a renormalized perturbation theory for Quantum Electrodynamics.

The original QED Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\partial\!\!\!/ - m_0)\psi - e_0\bar{\psi}\gamma^\mu\psi A_\mu .$$

Computing the electron and photon propagators with this Lagrangian, we would find expressions of the general form

 $= \dfrac{iZ_2}{p\!\!\!/ - m} + \cdots ;$  $= \dfrac{-iZ_3 g_{\mu\nu}}{q^2} + \cdots .$

(We found just such expressions in the explicit one-loop calculations of Chapter 7.) To absorb Z_2 and Z_3 into $\mathcal{L}$, and hence eliminate them from formula (7.45) for the S-matrix, we substitute $\psi = Z_2^{1/2}\psi_r$ and $A^\mu = Z_3^{1/2}A_r^\mu$. Then the Lagrangian becomes

$$\mathcal{L} = -\tfrac{1}{4}Z_3(F_r^{\mu\nu})^2 + Z_2\bar{\psi}_r(i\partial\!\!\!/ - m_0)\psi_r - e_0 Z_2 Z_3^{1/2}\bar{\psi}_r\gamma^\mu\psi_r A_{r\mu}. \tag{10.36}$$