We can turn the field $\phi_1(\mathbf{x}_1)$ into a Schrödinger operator using $\phi_S(\mathbf{x}_1)\,|\phi_1\rangle = \phi_1(\mathbf{x}_1)\,|\phi_1\rangle$. The completeness relation $\int \mathcal{D}\phi_1\,|\phi_1\rangle\langle\phi_1| = \mathbf{1}$ then allows us to eliminate the intermediate state $|\phi_1\rangle$. Similar manipulations work for ϕ_2, yielding the expression

$$\langle\phi_b|\,e^{-iH(T-x_2^0)}\,\phi_S(\mathbf{x}_2)\,e^{-iH(x_2^0-x_1^0)}\,\phi_S(\mathbf{x}_1)\,e^{-iH(x_1^0+T)}\,|\phi_a\rangle\,.$$

Most of the exponential factors combine with the Schrödinger operators to make Heisenberg operators. In the case $x_1^0 > x_2^0$, the order of x_1 and x_2 would simply be interchanged. Thus expression (9.15) is equal to

$$\langle\phi_b|\,e^{-iHT}\,T\big\{\phi_H(x_1)\phi_H(x_2)\big\}\,e^{-iHT}\,|\phi_a\rangle\,. \tag{9.17}$$

This expression is almost equal to the two-point correlation function. To make it more nearly equal, we take the limit $T \to \infty(1-i\epsilon)$. Just as in Section 4.2, this trick projects out the vacuum state $|\Omega\rangle$ from $|\phi_a\rangle$ and $|\phi_b\rangle$ (provided that these states have some overlap with $|\Omega\rangle$, which we assume). For example, decomposing $|\phi_a\rangle$ into eigenstates $|n\rangle$ of H, we have

$$e^{-iHT}\,|\phi_a\rangle = \sum_n e^{-iE_nT}\,|n\rangle\,\langle n|\phi_a\rangle \underset{T\to\infty(1-i\epsilon)}{\longrightarrow} \langle\Omega|\phi_a\rangle\,e^{-iE_0\cdot\infty(1-i\epsilon)}\,|\Omega\rangle\,.$$

As in Section 4.2, we obtain some awkward phase and overlap factors. But these factors cancel if we divide by the same quantity as (9.15) but without the two extra fields $\phi(x_1)$ and $\phi(x_2)$. Thus we obtain the simple formula

$$\langle\Omega|\,T\phi_H(x_1)\phi_H(x_2)\,|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int\mathcal{D}\phi\,\phi(x_1)\phi(x_2)\exp\!\left[i\int_{-T}^{T}d^4x\,\mathcal{L}\right]}{\int\mathcal{D}\phi\,\exp\!\left[i\int_{-T}^{T}d^4x\,\mathcal{L}\right]}. \tag{9.18}$$

This is our desired formula for the two-point correlation function in terms of functional integrals. For higher correlation functions, just insert additional factors of ϕ on both sides.

Feynman Rules

Our next task is to compute various correlation functions directly from the right-hand side of formula (9.18). In other words, we will now use (9.18) to derive the Feynman rules for a scalar field theory. We will begin by computing the two-point function in the free Klein-Gordon theory, then generalize to higher correlation functions in the free theory. Finally, we will consider ϕ^4 theory, in which we can perform a perturbation expansion to obtain the same Feynman rules as in Section 4.4.

Consider first a noninteracting real-valued scalar field:

$$S_0 = \int d^4x\,\mathcal{L}_0 = \int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2\right]. \tag{9.19}$$

Since $\mathcal{L}_0$ is quadratic in ϕ, the functional integrals in (9.18) take the form of generalized, infinite-dimensional Gaussian integrals. We will therefore be able to evaluate the functional integrals exactly.

Since this is our first functional integral computation, we will do it in a very explicit, but ugly, way. We must first define the integral $\mathcal{D}\phi$ over field configurations. To do this, we use the method of Eq. (9.4) in considering the continuous integral as a limit of a large but finite number of integrals. We thus replace the variables $\phi(x)$ defined on a continuum of points by variables $\phi(x_i)$ defined at the points x_i of a square lattice. Let the lattice spacing be ϵ, let the four-dimensional spacetime volume be L^4, and define

$$\mathcal{D}\phi = \prod_i d\phi(x_i), \tag{9.20}$$

up to an irrelevant overall constant.

The field values $\phi(x_i)$ can be represented by a discrete Fourier series:

$$\phi(x_i) = \frac{1}{V}\sum_n e^{-ik_n\cdot x_i}\phi(k_n), \tag{9.21}$$

where $k_n^\mu = 2\pi n^\mu/L$, with n^μ an integer, $|k^\mu| < \pi/\epsilon$, and $V = L^4$. The Fourier coefficients $\phi(k)$ are complex. However, $\phi(x)$ is real, and so these coefficients must obey the constraint $\phi^*(k) = \phi(-k)$. We will consider the real and imaginary parts of the $\phi(k_n)$ with $k_n^0 > 0$ as independent variables. The change of variables from the $\phi(x_i)$ to these new variables $\phi(k_n)$ is a unitary transformation, so we can rewrite the integral as

$$\mathcal{D}\phi(x) = \prod_{k_n^0>0} d\,\mathrm{Re}\,\phi(k_n)\,d\,\mathrm{Im}\,\phi(k_n).$$

Later, we will take the limit $L \to \infty$, $\epsilon \to 0$. The effect of this limit is to convert discrete, finite sums over k_n to continuous integrals over k:

$$\frac{1}{V}\sum_n \to \int\frac{d^4k}{(2\pi)^4}. \tag{9.22}$$

In the following discussion, this limit will produce Feynman perturbation theory in the form derived in Part I. We will not eliminate the infrared and ultraviolet divergences of Feynman diagrams that we encountered in Chapter 6, but at least the functional integral introduces no new types of singular behavior.

Having defined the measure of integration, we now compute the functional integral over ϕ. The action (9.19) can be rewritten in terms of the Fourier coefficients as

$$\begin{aligned}\int d^4x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2\right] &= -\frac{1}{V}\sum_n \tfrac{1}{2}(m^2-k_n^2)|\phi(k_n)|^2\\ &= -\frac{1}{V}\sum_{k_n^0>0}(m^2-k_n^2)\left[(\mathrm{Re}\,\phi_n)^2 + (\mathrm{Im}\,\phi_n)^2\right],\end{aligned}$$

where we have abbreviated $\phi(k_n)$ as ϕ_n in the second line. The quantity $(m^2 - k_n^2) = (m^2 + |\mathbf{k}_n|^2 - k_n^{02})$ is positive as long as k_n^0 is not too large. In the following discussion, we will treat this quantity as if it were positive. More precisely, we evaluate it by analytic continuation from the region where $|\mathbf{k}_n| > k_n^0$.

The denominator of formula (9.18) now takes the form of a product of Gaussian integrals:

$$\begin{aligned}
\int \mathcal{D}\phi\, e^{iS_0} &= \left(\prod_{k_n^0>0} \int d\,\mathrm{Re}\,\phi_n\, d\,\mathrm{Im}\,\phi_n\right) \exp\left[-\frac{i}{V}\sum_{k_n^0>0}(m^2-k_n^2)|\phi_n|^2\right] \\
&= \prod_{k_n^0>0}\left(\int d\,\mathrm{Re}\,\phi_n\, \exp\left[-\frac{i}{V}(m^2-k_n^2)(\mathrm{Re}\,\phi_n)^2\right]\right) \\
&\qquad\qquad \times \left(\int d\,\mathrm{Im}\,\phi_n\, \exp\left[-\frac{i}{V}(m^2-k_n^2)(\mathrm{Im}\,\phi_n)^2\right]\right) \\
&= \prod_{k_n^0>0} \sqrt{\frac{-i\pi V}{m^2-k_n^2}}\sqrt{\frac{-i\pi V}{m^2-k_n^2}} \\
&= \prod_{\text{all } k_n} \sqrt{\frac{-i\pi V}{m^2-k_n^2}}.
\end{aligned} \tag{9.23}$$

To justify using Gaussian integration formulae when the exponent appears to be purely imaginary, recall that the time integral in (9.18) is along a contour that is rotated clockwise in the complex plane: $t \to t(1-i\epsilon)$. This means that we should change $k^0 \to k^0(1+i\epsilon)$ in (9.21) and all subsequent equations; in particular, we should replace $(k^2 - m^2) \to (k^2 - m^2 + i\epsilon)$. The $i\epsilon$ term gives the necessary convergence factor for the Gaussian integrals. It also defines the direction of the analytic continuation that might be needed to define the square roots in (9.23).

To understand the result of (9.23), consider as an analogy the general Gaussian integral

$$\left(\prod_k \int d\xi_k\right) \exp\left[-\xi_i B_{ij} \xi_j\right],$$

where B is a symmetric matrix with eigenvalues b_i. To evaluate this integral we write $\xi_i = O_{ij} x_j$, where O is the orthogonal matrix of eigenvectors that diagonalizes B. Changing variables from ξ_i to the coefficients x_i, we have

$$\begin{aligned}
\left(\prod_k \int d\xi_k\right) \exp\left[-\xi_i B_{ij}\xi_j\right] &= \left(\prod_k \int dx_k\right) \exp\left[-\sum_i b_i x_i^2\right] \\
&= \prod_i \left(\int dx_i\, \exp\left[-b_i x_i^2\right]\right)
\end{aligned}$$

$$= \prod_i \sqrt{\frac{\pi}{b_i}}$$

$$= \text{const} \times \left[\det B\right]^{-1/2}. \qquad (9.24)$$

The analogy is clearer if we perform an integration by parts to write the Klein-Gordon action as

$$S_0 = \tfrac{1}{2}\int d^4x\, \phi\,(-\partial^2 - m^2)\,\phi \;+\; (\text{surface term}).$$

Thus the matrix B corresponds to the operator $(m^2+\partial^2)$, and we can formally write our result as

$$\int \mathcal{D}\phi\, e^{iS_0} = \text{const} \times \left[\det(m^2+\partial^2)\right]^{-1/2}. \qquad (9.25)$$

This object is called a *functional determinant*. The actual result (9.23) looks quite ill-defined, and in fact all of these factors will cancel in Eq. (9.18). However, in many circumstances, the functional determinant itself has physical meaning. We will see examples of this in Sections 9.5 and 11.4.

Now consider the numerator of formula (9.18). We need to Fourier-expand the two extra factors of ϕ:

$$\phi(x_1)\phi(x_2) = \frac{1}{V}\sum_m e^{-ik_m\cdot x_1}\phi_m\, \frac{1}{V}\sum_l e^{-ik_l\cdot x_2}\phi_l.$$

Thus the numerator is

$$\begin{aligned}\frac{1}{V^2}\sum_{m,l} e^{-i(k_m\cdot x_1 + k_l\cdot x_2)}&\biggl(\prod_{k_n^0>0}\int d\,\mathrm{Re}\,\phi_n\, d\,\mathrm{Im}\,\phi_n\biggr) \qquad (9.26)\\ &\times (\mathrm{Re}\,\phi_m + i\,\mathrm{Im}\,\phi_m)(\mathrm{Re}\,\phi_l + i\,\mathrm{Im}\,\phi_l)\\ &\times \exp\Bigl[-\frac{i}{V}\sum_{k_n^0>0}(m^2-k_n^2)\bigl[(\mathrm{Re}\,\phi_n)^2 + (\mathrm{Im}\,\phi_n)^2\bigr]\Bigr].\end{aligned}$$

For most values of k_m and k_l this expression is zero, since the extra factors of ϕ make the integrand odd. The situation is more complicated when $k_m = \pm k_l$. Suppose, for example, that $k_m^0 > 0$. Then if $k_l = +k_m$, the term involving $(\mathrm{Re}\,\phi_m)^2$ is nonzero, but is exactly canceled by the term involving $(\mathrm{Im}\,\phi_m)^2$. If $k_l = -k_m$, however, the relation $\phi(-k) = \phi^*(k)$ gives an extra minus sign on the $(\mathrm{Im}\,\phi_m)^2$ term, so the two terms add. When $k_m^0 < 0$ we obtain the same expression, so the numerator is

$$\text{Numerator} = \frac{1}{V^2}\sum_m e^{-ik_m\cdot(x_1-x_2)}\biggl(\prod_{k_n^0>0}\frac{-i\pi V}{m^2-k_n^2}\biggr)\frac{-iV}{m^2-k_m^2-i\epsilon}.$$

The factor in parentheses is identical to the denominator (9.23), while the rest of this expression is the discretized form of the Feynman propagator. Taking the continuum limit (9.22), we find

$$\langle 0| T\phi(x_1)\phi(x_2) |0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x_1-x_2)}}{k^2 - m^2 + i\epsilon} = D_F(x_1-x_2). \tag{9.27}$$

This is exactly right, including the $+i\epsilon$.

Next we would like to compute higher correlation functions in the free Klein-Gordon theory.

Inserting an extra factor of ϕ in (9.18), we see that the three-point function vanishes, since the integrand of the numerator is odd. All other odd correlation functions vanish for the same reason.

The four-point function has four factors of ϕ in the numerator. Fourier-expanding the fields, we obtain an expression similar to Eq. (9.26), but with a quadruple sum over indices that we will call m, l, p, and q. The integrand contains the product

$$(\text{Re}\,\phi_m + i\,\text{Im}\,\phi_m)(\text{Re}\,\phi_l + i\,\text{Im}\,\phi_l)(\text{Re}\,\phi_p + i\,\text{Im}\,\phi_p)(\text{Re}\,\phi_q + i\,\text{Im}\,\phi_q).$$

Again, most of the terms vanish because the integrand is odd. One of the nonvanishing terms occurs when $k_l = -k_m$ and $k_q = -k_p$. After the Gaussian integrations, this term of the numerator is

$$\frac{1}{V^4}\sum_{m,p} e^{-ik_m\cdot(x_1-x_2)}e^{-ik_p\cdot(x_3-x_4)}\left(\prod_{k_n^0>0}\frac{-i\pi V}{m^2-k_n^2}\right)\frac{-iV}{m^2-k_m^2-i\epsilon}\,\frac{-iV}{m^2-k_p^2-i\epsilon}$$

$$\underset{V\to\infty}{\longrightarrow}\left(\prod_{k_n^0>0}\frac{-i\pi V}{m^2-k_n^2}\right)D_F(x_1-x_2)D_F(x_3-x_4).$$

The factor in parentheses is again canceled by the denominator. We obtain similar terms for each of the other two ways of grouping the four momenta in pairs. To keep track of the groupings, let us define the *contraction* of two fields as

$$\overbrace{\phi(x_1)\phi(x_2)} = \frac{\int \mathcal{D}\phi\, e^{iS_0}\phi(x_1)\phi(x_2)}{\int \mathcal{D}\phi\, e^{iS_0}} = D_F(x_1 - x_2). \tag{9.28}$$

Then the four-point function is simply

$$\begin{aligned}\langle 0| T\phi_1\phi_2\phi_3\phi_4 |0\rangle &= \text{sum of all full contractions}\\ &= D_F(x_1-x_2)D_F(x_3-x_4)\\ &\quad + D_F(x_1-x_3)D_F(x_2-x_4)\\ &\quad\quad + D_F(x_1-x_4)D_F(x_2-x_3),\end{aligned} \tag{9.29}$$

the same expression that we obtained using Wick's theorem in Eq. (4.40).

The same method allows us to compute still higher correlation functions. In each case the answer is just the sum of all possible full contractions of the fields. This result, identical to that obtained from Wick's theorem in Section 4.3, arises here from the simple rules of Gaussian integration.

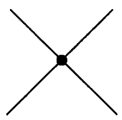
We are now ready to move from the free Klein-Gordon theory to ϕ^4 theory. Add to $\mathcal{L}_0$ a ϕ^4 interaction:

$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4.$$

Assuming that λ is small, we can expand

$$\exp\left[i\int d^4x\,\mathcal{L}\right] = \exp\left[i\int d^4x\,\mathcal{L}_0\right]\left(1 - i\int d^4x\,\frac{\lambda}{4!}\phi^4 + \cdots\right).$$

Making this expansion in both the numerator and the denominator of (9.18), we see that each is (aside from the constant factor (9.23), which again cancels) expressed entirely in terms of free-field correlation functions. Moreover, since $i\int d^3x\,\mathcal{L}_{\text{int}} = -iH_{\text{int}}$, we obtain exactly the same expansion as in Eq. (4.31). We can express both the numerator and the denominator in terms of Feynman diagrams, with the fundamental interaction again given by the vertex



$$= -i\lambda\,(2\pi)^4\delta^{(4)}(\textstyle\sum p). \tag{9.30}$$

All of the combinatorics work the same as in Section 4.4. In particular, the disconnected vacuum bubble diagrams exponentiate and factor from the numerator of (9.18), and are canceled by the denominator, just as in Eq. (4.31).

The vertex rule for ϕ^4 theory follows from the Lagrangian in an exceedingly simple way, and this simple procedure will turn out to be valid for other quantum field theories as well. Once the quadratic terms in the Lagrangian are properly understood and the propagators of the theory are computed, the vertices can be read directly from the Lagrangian as the coefficients of the cubic and higher-order terms.

Functional Derivatives and the Generating Functional

To conclude this section, we will now introduce a slicker, more formal, method for computing correlation functions. This method, based on an object called the *generating functional*, avoids the awkward Fourier expansions of the preceding derivation.

First we define the *functional derivative*, $\delta/\delta J(x)$, as follows. The functional derivative obeys the basic axiom (in four dimensions)

$$\frac{\delta}{\delta J(x)}\,J(y) = \delta^{(4)}(x-y) \qquad \text{or} \qquad \frac{\delta}{\delta J(x)}\int d^4y\,J(y)\phi(y) = \phi(x). \tag{9.31}$$

This definition is the natural generalization, to continuous functions, of the rule for discrete vectors,

$$\frac{\partial}{\partial x_i}\,x_j = \delta_{ij} \qquad \text{or} \qquad \frac{\partial}{\partial x_i}\sum_j x_j k_j = k_i.$$