5
Perturbative renormalization group

5.1 Expectation values in the Gaussian model

Can we treat the Landau–Ginzburg Hamiltonian as a perturbation to the Gaussian model? In particular, for zero magnetic field, we shall examine

$$\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U} \equiv \int \mathrm{d}^d\mathbf{x}\left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots\right] + u\int \mathrm{d}^d\mathbf{x} m^4 + \cdots . \tag{5.1}$$

The *unperturbed* Gaussian Hamiltonian can be decomposed into independent Fourier modes, as

$$\begin{aligned}\beta\mathcal{H}_0 &= \frac{1}{V}\sum_{\mathbf{q}} \frac{t + Kq^2 + Lq^4 + \cdots}{2}|m(\mathbf{q})|^2 \\ &\equiv \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\frac{t + Kq^2 + Lq^4 + \cdots}{2}|m(\mathbf{q})|^2.\end{aligned} \tag{5.2}$$

The *perturbative interaction* which mixes up the normal modes has the form

$$\begin{aligned}\mathcal{U} &= u\int \mathrm{d}^d\mathbf{x} m(\mathbf{x})^4 + \cdots \\ &= u\int \mathrm{d}^d\mathbf{x}\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}\mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)} m_\alpha(\mathbf{q}_1)m_\alpha(\mathbf{q}_2)m_\beta(\mathbf{q}_3)m_\beta(\mathbf{q}_4) \\ &\quad + \cdots ,\end{aligned} \tag{5.3}$$

where summation over α and β is implicit. The integral over $\mathbf{x}$ sets $\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4 = \mathbf{0}$, and

$$\mathcal{U} = u\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}} m_\alpha(\mathbf{q}_1)m_\alpha(\mathbf{q}_2)m_\beta(\mathbf{q}_3)m_\beta(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) + \cdots . \tag{5.4}$$

From the variance of the Gaussian weights, the two-point expectation values in a finite sized system with discretized modes are easily obtained as

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 = \frac{\delta_{\mathbf{q},-\mathbf{q}'}\,\delta_{\alpha,\beta}\,V}{t + Kq^2 + Lq^4 + \cdots}. \tag{5.5}$$

In the limit of infinite size, the spectrum becomes continuous, and Eq. (5.5) goes over to

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 = \frac{\delta_{\alpha,\beta}(2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}')}{t + Kq^2 + Lq^4 + \cdots}. \tag{5.6}$$

The subscript 0 is used to indicate that the expectation values are taken with respect to the unperturbed (Gaussian) Hamiltonian. Expectation values involving any product of m's can be obtained starting from the identity

$$\left\langle \exp\left[\sum_i a_i m_i\right]\right\rangle_0 = \exp\left[\sum_{i,j} \frac{a_i a_j}{2} \left\langle m_i m_j\right\rangle_0\right], \tag{5.7}$$

which is valid for any set of Gaussian distributed variables $\{m_i\}$. (This is easily seen by "completing the square.") Expanding both sides of the equation in powers of $\{a_i\}$ leads to

$$\begin{aligned}
&1 + a_i\langle m_i\rangle_0 + \frac{a_i a_j}{2}\langle m_i m_j\rangle_0 + \frac{a_i a_j a_k}{6}\langle m_i m_j m_k\rangle_0 + \frac{a_i a_j a_k a_l}{24}\langle m_i m_j m_k m_k\rangle_0 + \cdots = \\
&1 + \frac{a_i a_j}{2}\langle m_i m_j\rangle_0 + \frac{a_i a_j a_k a_l}{24}\left(\langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0\right. \\
&\quad \left. + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0\right) + \cdots
\end{aligned} \tag{5.8}$$

Matching powers of $\{a_i\}$ on the two sides of the above equation gives

$$\left\langle \prod_{i=1}^{\ell} m_i \right\rangle_0 = \begin{cases} 0 & \text{for } \ell \text{ odd} \\ \text{sum over all pairwise contractions} & \text{for } \ell \text{ even.} \end{cases} \tag{5.9}$$

This result is known as *Wick's theorem*; and for example,

$$\langle m_i m_j m_k m_l\rangle_0 = \langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0.$$

5.2 Expectation values in perturbation theory

In the presence of an interaction $\mathcal{U}$, the expectation value of any operator $\mathcal{O}$ is computed perturbatively as

$$\begin{aligned}
\langle \mathcal{O}\rangle &= \frac{\int \mathcal{D}\vec{m}\, \mathcal{O} e^{-\beta\mathcal{H}_0 - \mathcal{U}}}{\int \mathcal{D}\vec{m}\, e^{-\beta\mathcal{H}_0 - \mathcal{U}}} = \frac{\int \mathcal{D}\vec{m}\, e^{-\beta\mathcal{H}_0}\mathcal{O}[1 - \mathcal{U} + \mathcal{U}^2/2 - \cdots]}{\int \mathcal{D}\vec{m}\, e^{-\beta\mathcal{H}_0}[1 - \mathcal{U} + \mathcal{U}^2/2 - \cdots]} \\
&= \frac{Z_0[\langle\mathcal{O}\rangle_0 - \langle\mathcal{O}\mathcal{U}\rangle_0 + \langle\mathcal{O}\mathcal{U}^2\rangle_0/2 - \cdots]}{Z_0[1 - \langle\mathcal{U}\rangle_0 + \langle\mathcal{U}^2\rangle_0/2 - \cdots]}.
\end{aligned} \tag{5.10}$$

Inverting the denominator by an expansion in powers of $\mathcal{U}$ gives

$$\begin{aligned}
\langle\mathcal{O}\rangle &= \left[\langle\mathcal{O}\rangle_0 - \langle\mathcal{O}\mathcal{U}\rangle_0 + \frac{1}{2}\langle\mathcal{O}\mathcal{U}^2\rangle_0 - \cdots\right]\left[1 + \langle\mathcal{U}\rangle_0 + \langle\mathcal{U}\rangle_0^2 - \frac{1}{2}\langle\mathcal{U}^2\rangle_0 - \cdots\right] \\
&= \langle\mathcal{O}\rangle_0 - (\langle\mathcal{O}\mathcal{U}\rangle_0 - \langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0) + \frac{1}{2}\left(\langle\mathcal{O}\mathcal{U}^2\rangle_0 - 2\langle\mathcal{O}\mathcal{U}\rangle_0\langle\mathcal{U}\rangle_0\right. \\
&\quad \left. + 2\langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0^2 - \langle\mathcal{O}\rangle_0\langle\mathcal{U}^2\rangle_0\right) + \cdots \\
&\equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\langle\mathcal{O}\mathcal{U}^n\rangle_0^c.
\end{aligned} \tag{5.11}$$

The *connected averages* (cumulants) are defined as the combination of unperturbed expectation values appearing at various orders in the expansion. Their significance will become apparent in diagrammatic representations, and from the following example.

Let us calculate the two-point correlation function of the Landau–Ginzburg model to first order in the parameter u. (In view of their expected irrelevance, we shall ignore higher order interactions, and also only keep the lowest order Gaussian terms.) Substituting Eq. (5.4) into Eq. (5.11) yields

$$\begin{aligned}\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 - u\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}}\\ \times[\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')m_i(\mathbf{q}_1)m_i(\mathbf{q}_2)m_j(\mathbf{q}_3)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0\\ -\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0\langle m_i(\mathbf{q}_1)m_i(\mathbf{q}_2)m_j(\mathbf{q}_3)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0]\\ +\mathcal{O}(u^2).\end{aligned} \tag{5.12}$$

To calculate $\langle\mathcal{O}\mathcal{U}\rangle_0$ we need the unperturbed expectation value of the product of six m's. This can be evaluated using Eq. (5.9) as the sum of all pair-wise contractions, 15 in all. Three contractions are obtained by first pairing m_α to m_β, and then the remaining four m's in $\mathcal{U}$. Clearly these contractions cancel exactly with corresponding ones in $\langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0$. The only surviving terms involve contractions that connect $\mathcal{O}$ to $\mathcal{U}$. This cancellation persists at all orders, and $\langle\mathcal{O}\mathcal{U}^n\rangle_0^c$ contains only terms in which all $n+1$ operators are connected by contractions. The remaining 12 pairings in $\langle\mathcal{O}\mathcal{U}\rangle_0$ fall into two classes:

(1) Four pairings involve contracting m_α and m_β to m's with the same index, e.g.

$$\begin{aligned}&\langle m_\alpha(\mathbf{q})\, m_i(\mathbf{q}_1)\rangle_0\langle m_\beta(\mathbf{q}')\, m_i(\mathbf{q}_2)\rangle_0\langle m_j(\mathbf{q}_3)\, m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0\\ &\quad=\frac{\delta_{\alpha i}\,\delta_{\beta i}\,\delta_{jj}\,(2\pi)^{3d}\,\delta^d(\mathbf{q}+\mathbf{q}_1)\,\delta^d(\mathbf{q}'+\mathbf{q}_2)\,\delta^d(\mathbf{q}_1+\mathbf{q}_2)}{(t+Kq^2)\,(t+Kq'^2)\,(t+Kq_3^2)},\end{aligned} \tag{5.13}$$

where we have used Eq. (5.6). After summing over i and j, and integrating over $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$, these terms make a contribution

$$-4u\frac{n\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{(t+Kq^2)^2}\int\frac{\mathrm{d}^d\mathbf{q}_3}{(2\pi)^d}\frac{1}{t+Kq_3^2}. \tag{5.14}$$

(2) Eight pairings involve contracting m_α and m_β to m's with different indices, e.g.

$$\begin{aligned}&\langle m_\alpha(\mathbf{q})m_i(\mathbf{q}_1)\rangle_0\langle m_\beta(\mathbf{q}')m_j(\mathbf{q}_3)\rangle_0\langle m_i(\mathbf{q}_2)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0\\ &\quad=\frac{\delta_{\alpha i}\delta_{\beta j}\delta_{ij}(2\pi)^{3d}\delta^d(\mathbf{q}+\mathbf{q}_1)\delta^d(\mathbf{q}'+\mathbf{q}_3)\delta^d(\mathbf{q}_1+\mathbf{q}_3)}{(t+Kq^2)(t+Kq'^2)(t+Kq_2^2)}.\end{aligned} \tag{5.15}$$

Summing over all indices, and integrating over the momenta leads to an overall contribution of

$$-8u\frac{\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{(t+Kq^2)^2}\int\frac{\mathrm{d}^d\mathbf{q}_2}{(2\pi)^d}\frac{1}{t+Kq_2^2}. \tag{5.16}$$

Adding up both contributions, we obtain

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \frac{\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{t+Kq^2}\left[1-\frac{4u(n+2)}{t+Kq^2}\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}+\mathcal{O}(u^2)\right]. \tag{5.17}$$

5.3 Diagrammatic representation of perturbation theory

The calculations become more involved at higher orders in perturbation theory. A diagrammatic representation can be introduced to help keep track of all possible contractions. To calculate the ℓ-point expectation value $\langle\prod_{i=1}^{\ell} m_{\alpha_i}(\mathbf{q}_i)\rangle$, at pth order in u, proceed according to the following rules:

(1) Draw ℓ *external points* labeled by $(\mathbf{q}_i, \alpha_i)$ corresponding to the coordinates of the required correlation function. Draw p *vertices* with four legs each, labeled by *internal* momenta and indices, e.g. $\{(\mathbf{k}_1, i), (\mathbf{k}_2, i), (\mathbf{k}_3, j), (\mathbf{k}_4, j)\}$. Since the four legs are not equivalent, the four point vertex is indicated by two solid branches joined by a dotted line. (The extension to higher order interactions is straightforward.)

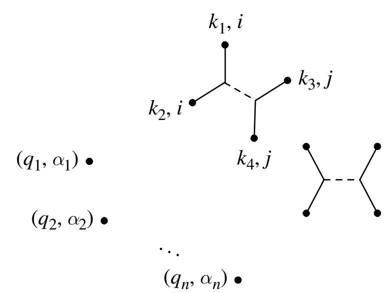
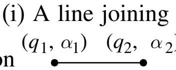
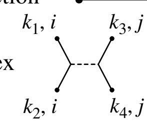


Fig. 5.1 Elements of the diagrammatic representation of perturbation theory.

(2) Each point of the graph now corresponds to one factor of $m_{\alpha_i}(\mathbf{q}_i)$, and the unperturbed average of the product is computed by Wick's theorem. This is implemented by joining all external and internal points *pair-wise*, by lines connecting one point to another, in all topologically distinct ways; see (5) below.

(3) The algebraic value of each such graph is obtained as follows: (i) A line joining a pair of points represents the two point average;[1] e.g. a connection 

, corresponds to $\delta_{\alpha_1\alpha_2}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2)/(t+Kq_1^2)$; (ii) A vertex 

stands for a term $u(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$ (the delta-function insures that momentum is conserved).

[1] Because of its original formulation in quantum field theory, the line joining two points is usually called a *propagator*. In this context, the line represents the world-line of a particle in time, while the perturbation $\mathcal{U}$ is an "interaction" between particles. For the same reason, the Fourier index is called a "momentum".

(4) Integrate over the $4p$ internal momenta $\{\mathbf{k}_i\}$, and sum over the $2p$ internal indices. Note that each closed loop produces a factor of $\delta_{ii} = n$ at this stage.

(5) There is a numerical factor of

$$\frac{(-1)^p}{p!} \times \text{number of different pairings leading to the same topology.}$$

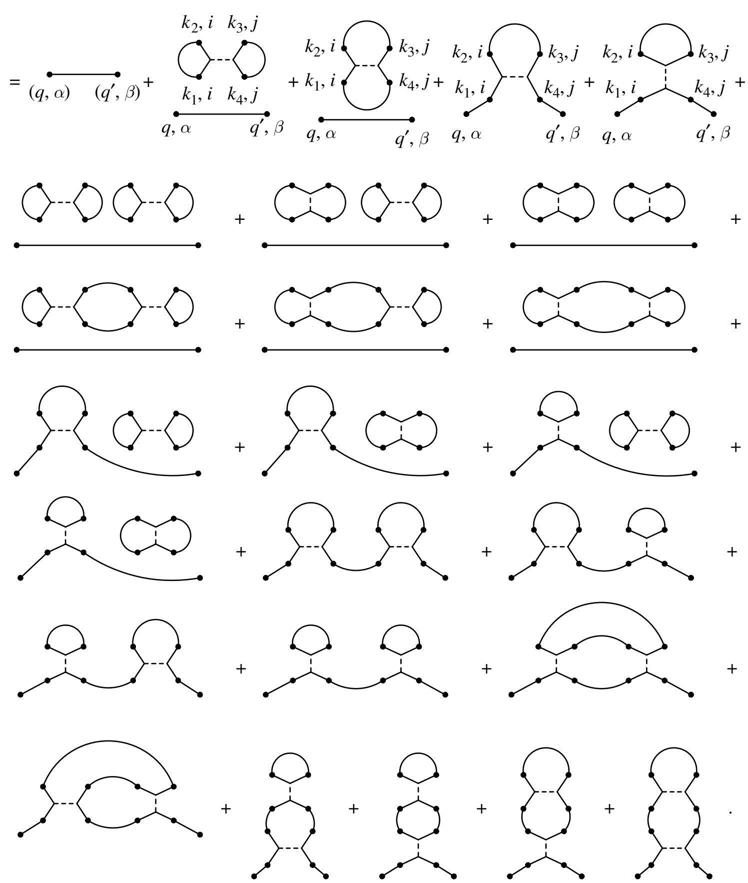
The first contribution comes from the expansion of the exponential; the second merely states that graphs related by symmetry give the same result, and can be calculated once.

(6) When calculating cumulants, only fully connected diagrams (without disjoint pieces) need to be included. This is a tremendous simplification.

For example, the diagrams appearing in the expansion for the propagator

$$\langle m_\alpha(q) m_\beta(q')\rangle \equiv \underset{(q,\alpha)\qquad(q^1,\beta)}{\bullet\!=\!=\!=\!=\!=\!\bullet},$$

to second order are



5.4 Susceptibility

It is no accident that the correction term in Eq. (5.17) is similar in form to the unperturbed value. This is because the form of the two point correlation function is constrained by symmetries, as can be seen from the identity

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \int \mathrm{d}^d\mathbf{x} \int \mathrm{d}^d\mathbf{x}' \mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{q}'\cdot\mathbf{x}'} \langle m_\alpha(\mathbf{x})\, m_\beta(\mathbf{x}')\rangle. \tag{5.18}$$

The two-point correlation function in real space must satisfy translation and rotation symmetry, and (in the high temperature phase)$\langle m_\alpha(\mathbf{x})\, m_\beta(\mathbf{x}')\rangle = \delta_{\alpha\beta}\langle m_1(\mathbf{x}-\mathbf{x}')\, m_1(\mathbf{0})\rangle$. Transforming to center of mass and relative coordinates, the above integral becomes,

$$\begin{aligned}
&\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle \\
&= \int \mathrm{d}^d\left(\frac{\mathbf{x}+\mathbf{x}'}{2}\right) \mathrm{d}^d\,(\mathbf{x}-\mathbf{x}')\, \mathrm{e}^{i(\mathbf{q}+\mathbf{q}')\cdot(\mathbf{x}+\mathbf{x}')/2} \mathrm{e}^{i(\mathbf{x}-\mathbf{x}')\cdot(\mathbf{q}-\mathbf{q}')/2} \delta_{\alpha\beta}\langle m_1(\mathbf{x}-\mathbf{x}')\, m_1(\mathbf{0})\rangle \\
&\equiv (2\pi)^d \delta^d(\mathbf{q}+\mathbf{q}')\delta_{\alpha\beta} S(q),
\end{aligned} \tag{5.19}$$

where

$$S(q) = \langle |m_1(\mathbf{q})|^2\rangle = \int \mathrm{d}^d\mathbf{x}\mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}} \langle m_1(\mathbf{x}-\mathbf{x}')\, m_1(\mathbf{0})\rangle \tag{5.20}$$

is the quantity observed in scattering experiments (Section 2.4).

From Eq. (5.17) we obtain

$$S(q) = \frac{1}{t+Kq^2}\left[1 - \frac{4u(n+2)}{t+Kq^2}\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2} + \mathcal{O}(u^2)\right]. \tag{5.21}$$

It is useful to examine the expansion of the inverse quantity

$$S(q)^{-1} = t + Kq^2 + 4u(n+2)\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2} + \mathcal{O}(u^2). \tag{5.22}$$

In the high-temperature phase, Eq. (5.20) indicates that the $q \to 0$ limit of $S(q)$ is just the magnetic susceptibility χ. For this reason, $S(q)$ is sometimes denoted by $\chi(q)$. From Eq. (5.22), the inverse susceptibility is given by

$$\chi^{-1}(t) = t + 4u(n+2)\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2} + \mathcal{O}(u^2). \tag{5.23}$$

The susceptibility no longer diverges at $t = 0$, since

$$\begin{aligned}
\chi^{-1}(0) &= 4u(n+2)\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{Kk^2} = \frac{4(n+2)u}{K}\frac{S_d}{(2\pi)^d}\int_0^\Lambda dk k^{d-3} \\
&= \frac{4(n+2)u}{K} K_d\left(\frac{\Lambda^{d-2}}{d-2}\right)
\end{aligned} \tag{5.24}$$

is a finite number ($K_d \equiv S_d/(2\pi)^d$). This is because in the presence of u the critical temperature is reduced to a negative value. The modified critical point is obtained by requiring $\chi^{-1}(t_c) = 0$, and hence from Eq. (5.23), to order of u,

$$t_c = -4u(n+2)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{t_c + Kk^2} \approx -\frac{4u(n+2)K_d\Lambda^{d-2}}{(d-2)K} < 0. \tag{5.25}$$

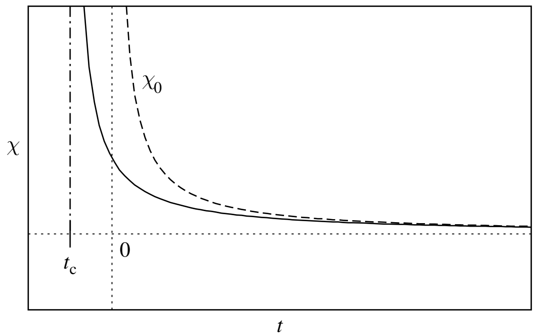


Fig. 5.2 The divergence of susceptibility occurs at a lower temperature due to the interaction u.

How does the perturbed susceptibility diverge at the shifted critical point? From Eq. (5.23),

$$\begin{aligned}\chi^{-1}(t) - \chi^{-1}(t_c) &= t - t_c + 4u(n+2)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\left(\frac{1}{t+Kk^2} - \frac{1}{t_c + Kk^2}\right) \\ &= (t-t_c)\left[1 - \frac{4u(n+2)}{K^2}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{k^2(k^2+(t-t_c)/K)} + \mathcal{O}(u^2)\right].\end{aligned} \tag{5.26}$$

In going from the first equation to the second, we have changed the position of t_c from one denominator to another. Since $t_c = \mathcal{O}(u)$, the corrections due to this change only appear at $\mathcal{O}(u^2)$. The final integral has dimensions of $\left[k^{d-4}\right]$. For $d > 4$ it is dominated by the largest momenta and scales as Λ^{d-4}. For $2 < d < 4$, the integral is convergent at both limits. Its magnitude is therefore set by the momentum scale $\xi^{-1} = \sqrt{(t-t_c)/K}$, which can be used to make the integrand dimensionless. Hence, in these dimensions,

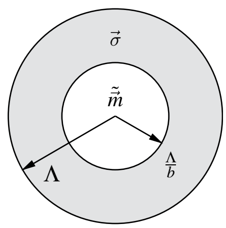
$$\chi^{-1}(t) = (t-t_c)\left[1 - \frac{4u(n+2)}{K^2}c\left(\frac{K}{t-t_c}\right)^{2-d/2} + \mathcal{O}(u^2)\right], \tag{5.27}$$

where c is a constant. For $d < 4$, the correction term at the order of u diverges at the phase transition, masking the unperturbed singularity of χ with $\gamma = 1$. Thus the perturbation series is inherently inapplicable for describing the divergence of susceptibility in $d < 4$. The same conclusion arises in calculating any other quantity perturbatively. Although we start by treating u as the perturbation parameter, it is important to realize that it is not dimensionless; u/K^2 has dimensions of (length)$^{d-4}$. The perturbation series for any quantity then takes the form $X(t, u) = X_0(t)[1 + f(ua^{4-d}/K^2, u\xi^{4-d}/K^2)]$, where f is a power series. The two length scales a and ξ are available to construct dimensionless variables. Since ξ diverges close to the critical point, there is an inherent failure of the perturbation series. The effective (dimensionless) perturbation parameter diverges at t_c and is not small, making it an inherently ineffective expansion parameter.

5.5 Perturbative RG (first order)

The last section demonstrates how various expectation values associated with the Landau–Ginzburg Hamiltonian can be calculated perturbatively in powers of u. However, the perturbative series is inherently divergent close to the critical point and cannot be used to characterize critical behavior in dimensions $d \leq 4$. K.G. Wilson showed that it is possible to combine perturbative and renormalization group approaches into a systematic method for calculating critical exponents. Accordingly, we shall extend the RG calculation of Gaussian model in Section 3.7 to the Landau–Ginzburg Hamiltonian, by treating $\mathcal{U} = u\int \mathrm{d}^d\mathbf{x} m^4$ as a perturbation.

(1) ***Coarse grain:*** This is the most difficult step of the RG procedure. As before, subdivide the fluctuations into two components as,



$$\vec{m}(\mathbf{q}) = \begin{cases} \tilde{\vec{m}}(\mathbf{q}) & \text{for } 0 < q < \Lambda/b \\ \vec{\sigma}(\mathbf{q}) & \text{for } \Lambda/b < q < \Lambda. \end{cases} \tag{5.28}$$

In the partition function,

$$Z = \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q})\mathcal{D}\vec{\sigma}(\mathbf{q}) \exp\left\{-\int_0^\Lambda \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)\right. \\ \left.\left(|\tilde{m}(\mathbf{q})|^2 + |\sigma(\mathbf{q})|^2\right) - \mathcal{U}[\tilde{\vec{m}}(\mathbf{q}), \vec{\sigma}(\mathbf{q})]\right\}, \tag{5.29}$$

the two sets of modes are mixed by the operator $\mathcal{U}$. Formally, the result of integrating out $\{\vec{\sigma}(\mathbf{q})\}$ can be written as

$$\begin{aligned} Z &= \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q}) \exp\left\{-\int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)|\tilde{m}(\mathbf{q})|^2\right\} \\ &\quad\times \exp\left\{-\frac{nV}{2}\int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \ln\left(t+Kq^2\right)\right\}\left\langle \mathrm{e}^{-\mathcal{U}[\tilde{\vec{m}},\vec{\sigma}]}\right\rangle_\sigma \\ &\equiv \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q})\mathrm{e}^{-\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}]}. \end{aligned} \tag{5.30}$$

Here we have defined the partial averages

$$\langle \mathcal{O} \rangle_\sigma \equiv \int \frac{\mathcal{D}\vec{\sigma}(\mathbf{q})}{Z_\sigma}\mathcal{O}\exp\left[-\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)|\sigma(\mathbf{q})|^2\right], \tag{5.31}$$

with $Z_\sigma = \int \mathcal{D}\vec{\sigma}(\mathbf{q})\exp\{-\beta\mathcal{H}_0[\vec{\sigma}]\}$, being the *Gaussian* partition function associated with the short wavelength fluctuations. From Eq. (5.30), we obtain

$$\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}] = V\delta f_b^0 + \int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)|\tilde{m}(\mathbf{q})|^2 - \ln\left\langle \mathrm{e}^{-\mathcal{U}[\tilde{\vec{m}},\vec{\sigma}]}\right\rangle_\sigma. \tag{5.32}$$

The final expression can be calculated perturbatively as,

$$\ln\left\langle e^{-\mathcal{U}}\right\rangle_\sigma = -\langle\mathcal{U}\rangle_\sigma + \frac{1}{2}\left(\left\langle\mathcal{U}^2\right\rangle_\sigma - \langle\mathcal{U}\rangle_\sigma^2\right) + \cdots \\ + \frac{(-1)^\ell}{\ell!} \times \ell\text{th cumulant of } \mathcal{U} + \cdots \tag{5.33}$$

The cumulants can be computed using the rules set in the previous sections. For example, at the first order we need to compute

$$\left\langle\mathcal{U}\left[\tilde{\vec{m}}, \vec{\sigma}\right]\right\rangle_\sigma = u\int \frac{d^d\mathbf{q}_1 d^d\mathbf{q}_2 d^d\mathbf{q}_3 d^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4) \\ \left\langle\left[\tilde{\vec{m}}(\mathbf{q}_1)+\vec{\sigma}(\mathbf{q}_1)\right]\cdot\left[\tilde{\vec{m}}(\mathbf{q}_2)+\vec{\sigma}(\mathbf{q}_2)\right]\right. \\ \times \left.\left[\tilde{\vec{m}}(\mathbf{q}_3)+\vec{\sigma}(\mathbf{q}_3)\right]\cdot\left[\tilde{\vec{m}}(\mathbf{q}_4)+\vec{\sigma}(\mathbf{q}_4)\right]\right\rangle_\sigma . \tag{5.34}$$

The following types of terms result from expanding the product:

[1] 1 $\left\langle \tilde{\vec{m}}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\ \tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma$

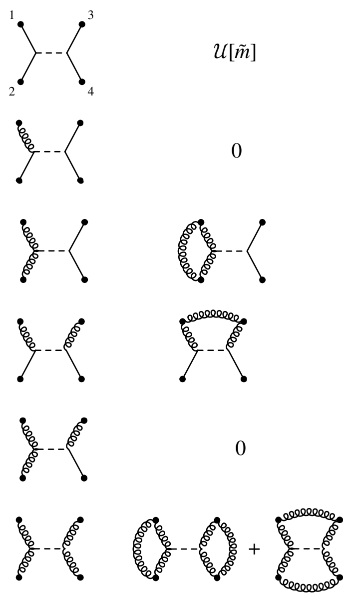
[2] 4 $\left\langle \vec{\sigma}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\ \tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma$

[3] 2 $\left\langle \vec{\sigma}(\mathbf{q}_1)\cdot\vec{\sigma}(\mathbf{q}_2)\ \tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma$

[4] 4 $\left\langle \vec{\sigma}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\ \vec{\sigma}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma$

[5] 4 $\left\langle \vec{\sigma}(\mathbf{q}_1)\cdot\vec{\sigma}(\mathbf{q}_2)\ \vec{\sigma}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\right\rangle_\sigma$

[6] 1 $\left\langle \vec{\sigma}(\mathbf{q}_1)\cdot\vec{\sigma}(\mathbf{q}_2)\ \vec{\sigma}(\mathbf{q}_3)\cdot\vec{\sigma}(\mathbf{q}_4)\right\rangle_\sigma$



(5.35)

The second element in each line is the number of terms with a given "symmetry". The total of these coefficients is $2^4 = 16$. Since the averages $\langle\mathcal{O}\rangle_\sigma$ involve only the short wavelength fluctuations, only contractions with $\vec{\sigma}$ appear. The resulting internal momenta are integrated from Λ/b to Λ.

Term [1] has no $\vec{\sigma}$ factors and evaluates to $\mathcal{U}[\tilde{\vec{m}}]$. The second and fifth terms involve an odd number of $\vec{\sigma}$s and their average is zero. Term [3] has one contraction and evaluates to

$$-\,u\times 2\int\frac{\mathrm{d}^d\mathbf{q}_1\cdots\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1+\cdots+\mathbf{q}_4)\frac{\delta_{jj}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2)}{t+Kq_1^2}\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)$$
$$=-2nu\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}|\tilde{m}(\mathbf{q})|^2\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}. \tag{5.36}$$

Term [4] also has one contraction but there is no closed loop (the factor δ_{jj}) and hence no factor of n. The various contractions of 4 $\vec{\sigma}$ in term [6] lead to a number of terms with no dependence on $\tilde{\vec{m}}$. We shall denote the sum of these terms by $uV\delta f_b^1$. Collecting all terms, the coarse-grained Hamiltonian at order of u is given by

$$\begin{aligned}\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}]=&\,V\left(\delta f_b^0+u\delta f_b^1\right)+\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{\tilde{t}+Kq^2}{2}\right)|\tilde{m}(\mathbf{q})|^2\\&+u\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}}\tilde{\vec{m}}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3),\end{aligned} \tag{5.37}$$

where

$$\tilde{t}=t+4u(n+2)\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}. \tag{5.38}$$

The coarse-grained Hamiltonian is thus again described by three parameters $\tilde{t}$, $\tilde{K}$, and $\tilde{u}$. The last two parameters are unchanged, and

$$\tilde{K}=K,\quad\text{and}\quad\tilde{u}=u. \tag{5.39}$$

(2) ***Rescale*** by setting $\mathbf{q}=b^{-1}\mathbf{q}'$, and
(3) ***Renormalize***, $\tilde{\vec{m}}=z\vec{m}'$, to get

$$\begin{aligned}(\beta\mathcal{H})'[m']=&\,V\left(\delta f_b^0+u\delta f_b^1\right)+\int_0^{\Lambda}\frac{\mathrm{d}^d\mathbf{q}'}{(2\pi)^d}b^{-d}z^2\left(\frac{\tilde{t}+Kb^{-2}q'^2}{2}\right)|m'(\mathbf{q}')|^2\\&+uz^4b^{-3d}\int_0^{\Lambda}\frac{\mathrm{d}^d\mathbf{q}'_1\mathrm{d}^d\mathbf{q}'_2\mathrm{d}^d\mathbf{q}'_3}{(2\pi)^{3d}}\vec{m}'(\mathbf{q}'_1)\cdot\vec{m}'(\mathbf{q}'_2)\vec{m}'(\mathbf{q}'_3)\cdot\vec{m}'(-\mathbf{q}'_1-\mathbf{q}'_2-\mathbf{q}'_3).\end{aligned} \tag{5.40}$$

The renormalized Hamiltonian is characterized by the triplet of interactions (t',K',u'), such that

$$t'=b^{-d}z^2\tilde{t},\quad K'=b^{-d-2}z^2K,\quad u'=b^{-3d}z^4u. \tag{5.41}$$

As in the Gaussian model there is a fixed point at $t^*=u^*=0$, provided that we set $z=b^{1+\frac{d}{2}}$, such that $K'=K$. The recursion relations for t and u in the vicinity of this point are given by

$$\begin{cases}t'_b=b^2\left[t+4u(n+2)\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}\right]\\u'_b=b^{4-d}u.\end{cases} \tag{5.42}$$

While the recursion relation for u at this order is identical to that obtained by dimensional analysis, the one for t is different. It is common to convert the discrete recursion relations to continuous differential flow equations by setting $b = e^{\ell}$, such that for an infinitesimal $\delta\ell$,

$$t'_b \equiv t(b) = t(1+\delta\ell) = t + \delta\ell\frac{dt}{d\ell} + \mathcal{O}(\delta\ell^2), \quad u'_b \equiv u(b) = u + \delta\ell\frac{du}{d\ell} + \mathcal{O}(\delta\ell^2).$$

Expanding Eqs. (5.42) to order of $\delta\ell$, gives

$$\begin{cases} t + \delta\ell\dfrac{dt}{d\ell} = (1+2\delta\ell)\left(t + 4u(n+2)\dfrac{S_d}{(2\pi)^d}\dfrac{1}{t+K\Lambda^2}\Lambda^d\delta\ell\right) \\ u + \delta\ell\dfrac{du}{d\ell} = (1+(4-d)\delta\ell)\,u. \end{cases} \tag{5.43}$$

The differential equations governing the evolution of t and u under rescaling are then

$$\begin{cases} \dfrac{dt}{d\ell} = 2t + \dfrac{4u(n+2)K_d\Lambda^d}{t+K\Lambda^2} \\ \dfrac{du}{d\ell} = (4-d)u. \end{cases} \tag{5.44}$$

The recursion relation for u is easily integrated to give $u(\ell) = u_0 e^{(4-d)\ell} = u_0 b^{(4-d)}$.

The recursion relations can be linearized in the vicinity of the fixed point $t^* = u^* = 0$, by setting $t = t^* + \delta t$ and $u = u^* + \delta u$, as

$$\frac{d}{d\ell}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix} = \begin{pmatrix} 2 & \dfrac{4(n+2)K_d\Lambda^{d-2}}{K} \\ 0 & 4-d \end{pmatrix}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix}. \tag{5.45}$$

In the differential form of the recursion relations, the eigenvalues of the matrix determine the relevance of operators. Since the above matrix has zero elements on one side, its eigenvalues are the diagonal elements, and as in the Gaussian model we can identify $y_t = 2$, and $y_u = 4-d$. The results at this order are identical to those obtained from dimensional analysis on the Gaussian model. The only difference is in the eigendirections. The exponent $y_t = 2$ is still associated with $u = 0$, while $y_u = 4-d$ is actually associated with the direction $t = -4u(n+2)K_d\Lambda^{d-2}/K$. This agrees with the shift in the transition temperature calculated to order of u from the susceptibility.

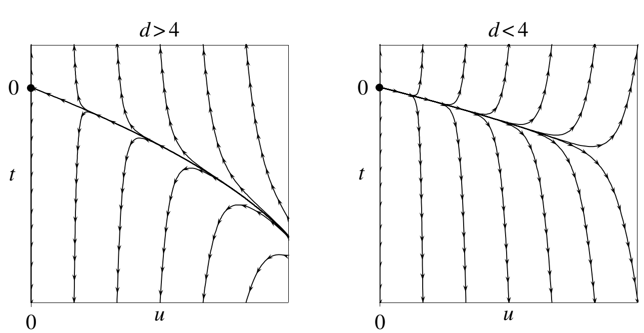


Fig. 5.3 RG flows obtained perturbatively to first order.

For $d > 4$ the Gaussian fixed point has only one unstable direction associated with y_t. It thus correctly describes the phase transition. For $d < 4$ it has two relevant directions and is unstable. Unfortunately, the recursion relations have no other fixed point at this order and it appears that we have learned little from the perturbative RG. However, since we are dealing with an alternating series we can *anticipate* that the recursion relations at the next order are modified to

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}\ell} = 2t + \dfrac{4u(n+2)K_d\Lambda^d}{t+K\Lambda^2} - Au^2 \\ \dfrac{\mathrm{d}u}{\mathrm{d}\ell} = (4-d)u - Bu^2, \end{cases} \tag{5.46}$$

with A and B positive. There is now an additional fixed point at $u^* = (4-d)/B$ for $d < 4$. For a systematic perturbation theory we need to keep the parameter u small. Thus the new fixed point can be explored systematically only for small $\epsilon = 4 - d$; we are led to consider an expansion in the dimension of space in the vicinity of $d = 4$! For a calculation valid at $\mathcal{O}(\epsilon)$ we have to keep track of terms of second order in the recursion relation for u, but only to first order in that of t. It is thus unnecessary to calculate the term A in the above recursion relation.

5.6 Perturbative RG (second order)

The coarse-grained Hamiltonian at second order in $\mathcal{U}$ is

$$\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}] = V\delta f_b^0 + \int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \left(\frac{t+Kq^2}{2}\right) |\tilde{m}(\mathbf{q})|^2 + \langle\mathcal{U}\rangle_\sigma - \frac{1}{2}\left(\left\langle\mathcal{U}^2\right\rangle_\sigma - \langle\mathcal{U}\rangle_\sigma^2\right) + O(\mathcal{U}^3). \tag{5.47}$$

To calculate $\left(\left\langle\mathcal{U}^2\right\rangle_\sigma - \langle\mathcal{U}\rangle_\sigma^2\right)$ we need to consider all possible decompositions of two $\mathcal{U}$s into $\tilde{\vec{m}}$ and $\vec{\sigma}$ as in Eq. (5.34). Since each $\mathcal{U}$ can be broken up into six types of terms as in Eq. (5.35), there are 36 such possibilities for two $\mathcal{U}$s which can be arranged in a 6×6 matrix, as below. Many of the elements of this matrix are either zero, or can be neglected at this stage, due to a number of considerations:

(1) All the 11 terms involving at least one factor of type [1] are zero because they cannot be contracted into a *connected* piece, and the disconnected elements cancel in calculating the cumulant.
(2) An additional 12 terms (such as $[2] \times [3]$) involve an *odd* number of $\vec{\sigma}$s and are zero due to their *parity*.
(3) Two terms, $[2] \times [5]$ and $[5] \times [2]$, involve a vertex where two $\vec{\sigma}$s are contracted together, leaving a $\tilde{\vec{m}}(\mathbf{q}^<)$ and a $\vec{\sigma}(\mathbf{q}^>)$. This configuration is not allowed by the δ-function which ensures momentum conservation for the vertex, as by construction $\mathbf{q}^> + \mathbf{q}^< \neq \mathbf{0}$.

(4) Terms $[3]\times[6]$, $[4]\times[6]$, and their partners by exchange have two factors of $\tilde{\tilde{m}}$. They involve *two-loop* integrations, and appear as corrections to the coefficient $\tilde{t}$. We shall denote their net effect by A, which as noted earlier does not need to be known precisely at this order.

(5) The term $[5]\times[5]$ also involves two factors of $\tilde{\tilde{m}}$, while $[2]\times[2]$ includes six such factors. The latter is important as it indicates that the space of parameters *is not closed* at this order. Even if initially zero, a term proportional to m^6 is generated under RG. In fact, considerations of momentum conservation indicate that both these terms are zero for $\mathbf{q}=0$, and are thus contributions to q^2m^2 and q^2m^6, respectively. We shall comment on their effect later on.

(6) The contributions resulting from $[6]\times[6]$ are constants, and will be collectively denoted by $u^2V\delta f_b^2$.

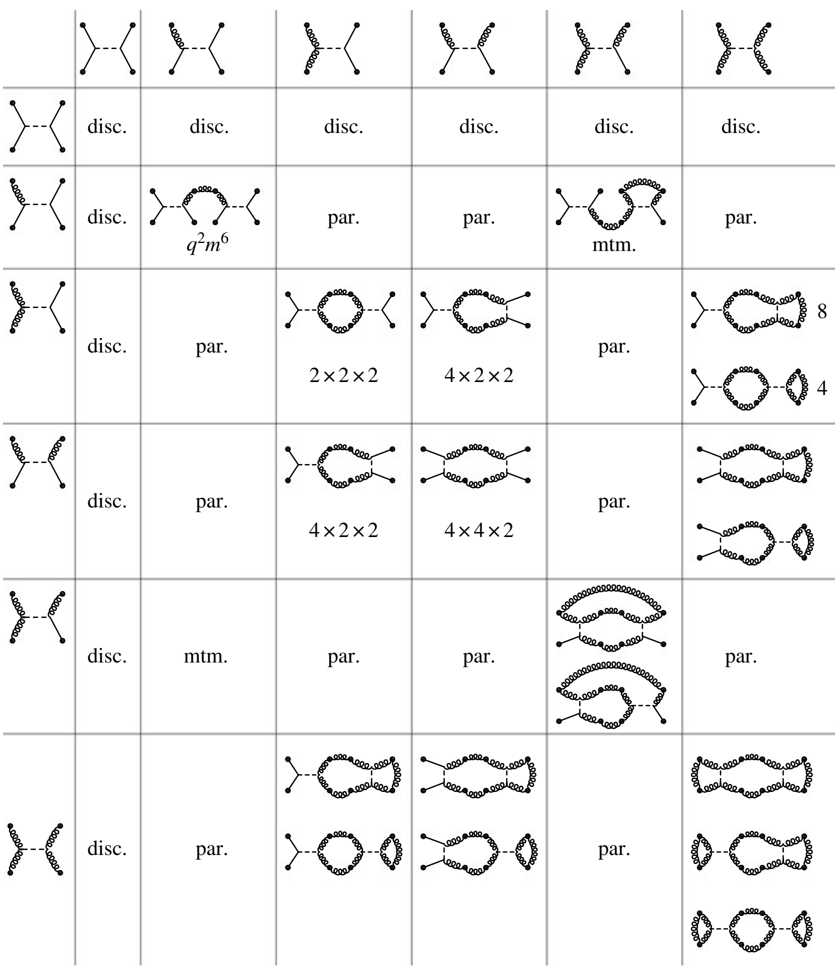
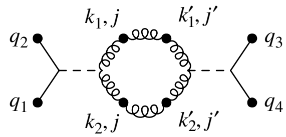


Fig. 5.4 Diagrams appearing in the second-order RG calculation (par. and disc. indicate contributions that are zero due to parity considerations, or being disconnected and mtm. is used to label diagrams that appear at higher order in q^2 due to momentum conservation).

(7) The terms [3] × [3], [3] × [4], [4] × [3], and [4] × [4] contribute to $\tilde{\vec{m}}^4$. For example, [3] × [3] results in



$$\frac{u^2}{2}\times 2\times 2\times 2\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}_1\cdots\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}_1\mathrm{d}^d\mathbf{k}_2\mathrm{d}^d\mathbf{k}_1'\mathrm{d}^d\mathbf{k}_2'}{(2\pi)^{4d}}$$
$$\times(2\pi)^{2d}\delta^d(\mathbf{q}_1+\mathbf{q}_2+\mathbf{k}_1+\mathbf{k}_2)\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{q}_3+\mathbf{q}_4)$$
$$\times\frac{\delta_{\alpha\alpha'}(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_1')}{t+Kk_1^{2'}}\frac{\delta_{\alpha\alpha'}(2\pi)^d\delta^d(\mathbf{k}_2+\mathbf{k}_2')}{t+Kk_2^{2'}}\tilde{\vec{m}}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)$$
$$=4nu^2\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}_1\cdots\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\tilde{\vec{m}}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)$$
$$\times\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t+Kk^2)\,(t+K(\mathbf{q}_1+\mathbf{q}_2-\mathbf{k})^2)}. \tag{5.48}$$

The contractions from terms [3] × [4], [4] × [3], and [4] × [4] lead to similar expressions with prefactors of 8, 8, and 16 respectively. Apart from the dependence on $\mathbf{q}_1$ and $\mathbf{q}_2$, the final result has the form of $\mathcal{U}[\tilde{\vec{m}}]$. In fact the last integral can be expanded as

$$f(\mathbf{q}_1+\mathbf{q}_2)=\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t+Kk^2)^2}\left[1-\frac{2K\mathbf{k}\cdot(\mathbf{q}_1+\mathbf{q}_2)-K(\mathbf{q}_1+\mathbf{q}_2)^2}{(t+Kk^2)}+\cdots\right]. \tag{5.49}$$

After fourier transforming back to real space we find in addition to m^4, such terms as $m^2(\nabla m)^2$, $m^2\nabla^2 m^2, \cdots$.

Putting all contributions together, the coarse grained Hamiltonian at order of u^2 takes the form

$$\beta\tilde{\mathcal{H}}=V\left(\delta f_b^0+u\delta f_b^1+u^2\delta f_b^2\right)+\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}|\tilde{m}(\mathbf{q})|^2$$
$$\left[\frac{t+Kq^2}{2}+2u(n+2)\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}-\frac{u^2}{2}A(t,K,q^2)\right]$$
$$+\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}}\tilde{\vec{m}}(\mathbf{q}_1)\cdot\tilde{\vec{m}}(\mathbf{q}_2)\tilde{\vec{m}}(\mathbf{q}_3)\cdot\tilde{\vec{m}}(\mathbf{q}_4)\times\left[u-\frac{u^2}{2}(8n+64)\right.$$
$$\left.\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t+Kk^2)^2}+\mathcal{O}(u^2q^2)\right]+\mathcal{O}(u^2\tilde{m}^6q^2,\cdots)+\mathcal{O}(u^3). \tag{5.50}$$

5.7 The ϵ-expansion

The parameter space (K, t, u) is no longer closed at this order; several new interactions proportional to m^2, m^4, and m^6, all consistent with symmetries of the problem, appear in the coarse-grained Hamiltonian at second order in u. Ignoring these interactions for the time being, the coarse grained parameters are given by

$$\begin{cases} \tilde{K} = K - u^2 A''(0) \\ \tilde{t} = t + 4(n+2)\, u \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{t+Kk^2} - u^2 A(0) \\ \tilde{u} = u - 4(n+8)u^2 \int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \frac{1}{(t+Kk^2)^2}, \end{cases} \tag{5.51}$$

where $A(0)$ and $A''(0)$ correspond to the first two terms in the expansion of $A(t, K, q^2)$ in Eq. (5.50) in powers of q.

After the *rescaling* $\mathbf{q} = b^{-1}\mathbf{q}'$, and *renormalization* $\tilde{\vec{m}} = z\vec{m}'$, steps of the RG procedure, we obtain

$$K' = b^{-d-2} z^2 \tilde{K}, \quad t' = b^{-d} z^2 \tilde{t}, \quad u' = b^{-3d} z^4 \tilde{u}. \tag{5.52}$$

As before, the renormalization parameter z is chosen such that $K' = K$, leading to

$$z^2 = \frac{b^{d+2}}{(1 - u^2 A''(0)/K)} = b^{d+2}\left(1 + O(u^2)\right). \tag{5.53}$$

The value of z does depend on the fixed point position u^*. But as u^* is of the order of ϵ, $z = b^{1+\frac{d}{2}+\mathcal{O}(\epsilon^2)}$, it is not changed at the lowest order. Using this value of z, and following the previous steps for constructing differential recursion relations, we obtain

$$\begin{cases} \frac{\mathrm{d}t}{\mathrm{d}\ell} = 2t + \frac{4u(n+2)K_d\Lambda^d}{t+K\Lambda^2} - A(t, K, \Lambda)u^2 \\ \frac{\mathrm{d}u}{\mathrm{d}\ell} = (4-d)u - \frac{4(n+8)K_d\Lambda^d}{(t+K\Lambda^2)^2} u^2. \end{cases} \tag{5.54}$$

The fixed points are obtained from $\mathrm{d}t/\mathrm{d}\ell = \mathrm{d}u/\mathrm{d}\ell = 0$. In addition to the Gaussian fixed point at $u^* = t^* = 0$, discussed in the previous section, there is now a non-trivial fixed point located at

$$\begin{cases} u^* = \frac{(t^* + K\Lambda^2)^2}{4(n+8)K_d\Lambda^d}\epsilon = \frac{K^2}{4(n+8)K_4}\epsilon + \mathcal{O}(\epsilon^2) \\ t^* = -\frac{2u^*(n+2)K_d\Lambda^d}{t^*+K\Lambda^2} = -\frac{(n+2)}{2(n+8)} K\Lambda^2\,\epsilon + \mathcal{O}(\epsilon^2). \end{cases} \tag{5.55}$$

The above expressions have been further simplified by systematically keeping terms to first order in $\epsilon = 4 - d$.

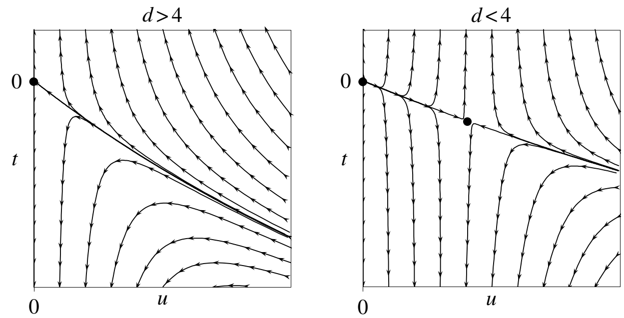


Fig. 5.5 RG flows obtained perturbatively to second order.

Linearizing the recursion relations in the vicinity of the fixed point results in

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}=\begin{pmatrix}2-\dfrac{4(n+2)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^*-A'u^{*2} & \dfrac{4(n+2)K_d\Lambda^d}{t^*+K\Lambda^2}-2Au^*\\ \dfrac{8(n+8)K_d\Lambda^d}{(t^*+K\Lambda^2)^3}u^{*2} & \epsilon-\dfrac{8(n+8)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^*\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}. \quad (5.56)$$

At the Gaussian fixed point, $t^*=u^*=0$, and Eq. (5.45) is reproduced. At the new fixed point of Eqs. (5.55),

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}=\begin{pmatrix}2-\dfrac{4(n+2)K_4\Lambda^4}{K^2\Lambda^4}\dfrac{K^2\epsilon}{4(n+8)K_4} & \cdots\cdots\\ \mathcal{O}(\epsilon^2) & \epsilon-\dfrac{8(n+8)K_4\Lambda^4}{K^2\Lambda^4}\dfrac{K^2\epsilon}{4(n+8)K_4}\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}. \quad (5.57)$$

We have not explicitly calculated the top element of the second column as it is not necessary for calculating the eigenvalues. This is because the lower element of the first column is zero to order of ϵ. Hence the eigenvalues are determined by the diagonal elements alone. The first eigenvalue is positive, controlling the instability of the fixed point,

$$y_t=2-\frac{(n+2)}{(n+8)}\epsilon+\mathcal{O}(\epsilon^2). \quad (5.58)$$

The second eigenvalue,

$$y_u=-\epsilon+\mathcal{O}(\epsilon^2), \quad (5.59)$$

is negative for $d<4$. The new fixed point thus has co-dimension of one and can describe the phase transition in these dimensions. It is quite satisfying that while various intermediate results, such as the position of the fixed point, depend on such microscopic parameters as K and Λ, the final eigenvalues are pure numbers, only depending on n and $d=4-\epsilon$. These eigenvalues characterize the *universality classes* of rotational symmetry breaking in $d<4$, with short-range interactions. (As discussed in the problem section, long-range interaction may lead to new universality classes.)

The divergence of the correlation length, $\xi \sim (\delta t)^{-\nu}$, is controlled by the exponent

$$\nu = \frac{1}{y_t} = \left\{2\left[1 - \frac{(n+2)}{2(n+8)}\epsilon\right]\right\}^{-1} = \frac{1}{2} + \frac{1}{4}\frac{n+2}{n+8}\epsilon + \mathcal{O}(\epsilon^2). \tag{5.60}$$

The singular part of the free energy scales as $f \sim (\delta t)^{2-\alpha}$, and the heat capacity diverges with the exponent

$$\alpha = 2 - d\nu = 2 - \frac{(4-\epsilon)}{2}\left[1 + \frac{1}{2}\frac{n+2}{n+8}\epsilon\right] = \frac{4-n}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2). \tag{5.61}$$

To complete the calculation of critical exponents, we need the eigenvalue associated with the (relevant) symmetry breaking field h. This is easily found by adding a term $-\vec{h}\cdot\int \mathrm{d}^d\mathbf{x}\vec{m}(\mathbf{x}) = -\vec{h}\cdot\vec{m}(\mathbf{q}=\mathbf{0})$ to the Hamiltonian. This term is not affected by coarse graining or rescaling, and after the renormalization step changes to $-z\vec{h}\cdot\vec{m}'(\mathbf{q}'=\mathbf{0})$, implying

$$h' = zh = b^{1+\frac{d}{2}}h, \quad \Longrightarrow \quad y_h = 1 + \frac{d}{2} + \mathcal{O}(\epsilon^2) = 3 - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2). \tag{5.62}$$

The vanishing of magnetization as $T \to T_c^-$ is controlled by the exponent

$$\begin{aligned}\beta &= \frac{d - y_h}{y_t} = \left(\frac{4-\epsilon}{2} - 1\right) \times \frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2)\right) \\ &= \frac{1}{2} - \frac{3}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2),\end{aligned} \tag{5.63}$$

while the susceptibility diverges as $\chi \sim (\delta t)^{-\gamma}$, with

$$\gamma = \frac{2y_h - d}{y_t} = 2 \times \frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\epsilon\right) = 1 + \frac{n+2}{2(n+8)}\epsilon + O(\epsilon^2). \tag{5.64}$$

Using the above results, we can estimate various exponents as a function of d and n. For example, for $n = 1$, by setting $\epsilon = 1$ or 2 in Eqs. (5.60) and Eqs. (5.63) we obtain the values $\nu(1) \approx 0.58$, $\nu(2) \approx 0.67$, and $\beta(1) \approx 0.33$, $\beta(2) \approx 0.17$. The best estimates of these exponents in $d = 3$ are $\nu \approx 0.63$, and $\beta \approx 0.32$. In $d = 2$ the exact values are known to be $\nu = 1$ and $\beta = 0.125$. The estimates for β are quite good, while those for ν are less reliable. It is important to note that in all cases these estimates are an improvement over the mean field (saddle point) values. Since the expansion is around four dimensions, the results are more reliable in $d = 3$ than in $d = 2$. In any case, they correctly describe the decrease of β with lowering dimension, and the increase of ν. They also correctly describe the trends with varying n at a fixed d as indicated by the following table of exponents $\alpha(n)$.

Although the sign of α is incorrectly predicted at this order for $n = 2$ and 3, the decrease of α with increasing n is correctly described.

	$n=1$	$n=2$	$n=3$	$n=4$
$\mathcal{O}(\epsilon)$ at $\epsilon=1$	0.17	0.11	0.06	0
Experiments in $d=3$	0.11	−0.01	−0.12	–

5.8 Irrelevance of other interactions

The fixed point Hamiltonian at $\mathcal{O}(\epsilon)$ (from Eqs. 5.55) has only three terms

$$\beta\mathcal{H}^* = \frac{K}{2}\int_\Lambda \mathrm{d}^d\mathbf{x}\left[(\nabla m)^2 - \frac{(n+2)}{(n+8)}\epsilon\Lambda^2 m^2 + \frac{\epsilon\Lambda^{-\epsilon}}{2(n+8)}\frac{K}{K_4}m^4\right], \tag{5.65}$$

and explicitly depends on the imposed cutoff $\Lambda \sim 1/a$ (unlike the exponents). However, as described in Section 3.4, the starting point for RG must be the most general Hamiltonian consistent with symmetries. We also discovered that even if some of these terms are left out of the original Hamiltonian, they are generated under coarse graining. At second order in u, terms proportional to m^6 were generated; higher powers of m will appear at higher orders in u.

Let us focus on a rotationally symmetric Hamiltonian for $\vec{h}=0$. We can incorporate all terms consistent with this symmetry in a perturbative RG by setting $\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U}$, where

$$\beta\mathcal{H}_0 = \int \mathrm{d}^d\mathbf{x}\left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots\right] \tag{5.66}$$

includes all quadratic (Gaussian terms), while the remaining higher order terms are placed in the perturbation

$$\mathcal{U} = \int \mathrm{d}^d\mathbf{x}\left[um^4 + vm^2(\nabla m)^2 + \cdots + u_6 m^6 + \cdots + u_8 m^8 + \cdots\right]. \tag{5.67}$$

After coarse graining, and steps (ii) and (iii) of RG in real space, $\mathbf{x} = b\mathbf{x}'$ and $\tilde{\vec{m}} = \zeta\vec{m}'$, the renormalized weight depends on the parameters

$$\begin{cases} t \mapsto & b^d\zeta^2\tilde{t} = b^2\tilde{t} \\ K \mapsto & b^{d-2}\zeta^2\tilde{K} = K \\ L \mapsto & b^{d-4}\zeta^2\tilde{L} = b^{-2}\tilde{L} \\ & \vdots \\ u \mapsto & b^d\zeta^4\tilde{u} = b^{4-d}\tilde{u} \\ v \mapsto & b^{d-2}\zeta^4\tilde{v} = b^{2-d}\tilde{v} \\ & \vdots \\ u_6 \mapsto & b^d\zeta^6\tilde{u}_6 = b^{6-2d}\tilde{u}_6 \\ u_8 \mapsto & b^d\zeta^8\tilde{u}_8 = b^{8-3d}\tilde{u}_8 \\ & \vdots \end{cases} \tag{5.68}$$

The second set of equalities are obtained by choosing $\zeta^2 = b^{2-d} K/\tilde{K} = b^{2-d}[1 + \mathcal{O}(u^2, uv, v^2, \cdots)]$, such that $K' = K$. By choosing an infinitesimal rescaling, the recursion relations take the differential forms

$$\begin{cases} \dfrac{dt}{d\ell} = & 2t + \mathcal{O}(u, v, u_6, u_8, \cdots) \\ \dfrac{dK}{d\ell} = & 0 \\ \dfrac{dL}{d\ell} = & -2L + \mathcal{O}(u^2, uv, v^2, \cdots) \\ & \vdots \\ \dfrac{du}{d\ell} = & \epsilon u - Bu^2 + \mathcal{O}(uv, v^2, \cdots) \\ \dfrac{dv}{d\ell} = & (-2+\epsilon)v + \mathcal{O}(u^2, uv, v^2, \cdots) \\ & \vdots \\ \dfrac{du_6}{d\ell} = & (-2+2\epsilon)u_6 + \mathcal{O}(u^3, u_6^2, \cdots) \\ \dfrac{du_8}{d\ell} = & (-4+3\epsilon)u_8 + \mathcal{O}(u^3, u^2 u_6, \cdots) \\ & \vdots \end{cases} \tag{5.69}$$

These recursion relations describe two fixed points:

(1) The Gaussian fixed point, $t^* = L^* = u^* = v^* = \cdots = 0$, and $K \neq 0$, has eigenvalues

$$\begin{aligned} &y_t^0 = 2,\ y_L^0 = -2,\ \cdots,\ y_u^0 = +\epsilon,\ y_v^0 = -2+\epsilon,\ \cdots, \\ &y_6^0 = -2+2\epsilon,\ y_8^0 = -4+3\epsilon,\ \cdots. \end{aligned} \tag{5.70}$$

(2) Setting Eqs. (5.69) to zero, a non-trivial fixed point is located at

$$t^* \sim u^* \sim \mathcal{O}(\epsilon), \quad L^* \sim v^* \sim \cdots \sim \mathcal{O}(\epsilon^2), \quad u_6^* \sim \cdots \sim \mathcal{O}(\epsilon^3), \quad \cdots. \tag{5.71}$$

The stability of this fixed point is determined by the matrix,

$$\frac{d}{d\ell}\begin{pmatrix} \delta t \\ \delta L \\ \vdots \\ \delta u \\ \delta v \\ \vdots \end{pmatrix} = \begin{pmatrix} 2-\mathcal{O}(u^*) & \mathcal{O}(\epsilon) & \cdots & \mathcal{O}(1) & \mathcal{O}(1) & \cdots \\ \mathcal{O}(\epsilon^2) & -2+\mathcal{O}(\epsilon) & & & & \\ \vdots & & \ddots & & & \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & & & & \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & & & & \\ \vdots & & & & & \end{pmatrix} \begin{pmatrix} \delta t \\ \delta L \\ \vdots \\ \delta u \\ \delta v \\ \vdots \end{pmatrix}. \tag{5.72}$$

Note that as $\epsilon \to 0$, the non-trivial fixed part, its eigenvalues and eigendirections continuously go over to the Gaussian fixed points. Hence the eigenvalues can only be corrected by order of ϵ, and Eq. (5.70) is modified to

$$y_t = 2 - \frac{n+2}{n+8}\epsilon + \mathcal{O}(\epsilon^2), \qquad y_L = -2 + \mathcal{O}(\epsilon), \cdots,$$
$$y_u = -\epsilon + \mathcal{O}(\epsilon^2), y_v = -2 + \mathcal{O}(\epsilon), \cdots, y_6 = -2 + \mathcal{O}(\epsilon), y_8 = -4 + \mathcal{O}(\epsilon), \cdots \tag{5.73}$$

While the eigenvalues are still labeled with the coefficients of the various terms in the Landau–Ginzburg expansion, we must remember that the actual eigendirections are now rotated away from the axes of this parameter space, although their largest projection is still parallel to the corresponding axis.

Whereas the Gaussian fixed point has two relevant directions in $d < 4$, the generalized $O(n)$ fixed point has only one relevant direction corresponding to y_t. At least perturbatively, this fixed point has a basin of attraction of co-dimension one, and thus describes the phase transition. The original concept of Kadanoff scaling is thus explicitly realized and the universality of exponents is traced to the irrelevance (at least perturbatively) of the multitude of other possible interactions. The perturbative approach does not exclude the existence of other fixed points at finite values of these parameters. The uniqueness of the critical exponents observed so far for each universality class, and their proximity to the values calculated from the ϵ-expansion, suggests that postulating such *non-perturbative* fixed points is unnecessary.

5.9 Comments on the ϵ-expansion

The perturbative implementation of RG for the Landau–Ginzburg Hamiltonian was achieved by K.G. Wilson in the early 1970s; the ϵ-expansion was developed jointly with M.E. Fisher. This led to a flurry of activity in the topic which still continues. Wilson was awarded the Nobel Prize in 1982. Historical details can be found in his Nobel lecture reprinted in Rev. Mod. Phys. **55**, 583 (1983). A few comments on the ϵ-expansion are in order at this stage.

(1) ***Higher orders, and convergence of the series:*** Calculating the exponents to $\mathcal{O}(\epsilon)^2$ and beyond, by going to order of $\mathcal{U}^3$ and higher, is quite complicated as we have to keep track of many more interactions. It is in fact quite unappealing that the intermediate steps of the RG explicitly keep track of the cutoff scale Λ, while the final exponents must be independent of it. In fact there are a number of field theoretical RG schemes (dimensional regularization, summing leading divergences, etc.) that avoid many of these difficulties. These methods are harder to visualize and will not be described here. All higher order calculations are currently performed using one of these schemes. It is sometimes (but not always) possible to prove that these approaches are consistent with each other, and can be carried out to all orders. In principle, the problem of evaluating critical exponents in $d = 3$ is now solved: simple computations lead to approximate results, while more refined calculations should provide better answers. The situation is somewhat like finding the energy levels of a He atom, which cannot be done exactly, but which may be obtained with sufficient accuracy using various approximation methods.

To estimate how reliable the exponents are, we need some information on the convergence of the series. The ϵ expansion has been carried out to the fifth order, and the results for the exponent γ, for $n=1$ at $d=3$, are

$$\begin{aligned}\gamma &= 1+0.167\epsilon+0.077\epsilon^2-0.049\epsilon^3+0.180\epsilon^4-0.415\epsilon^5\\ 1.2385\pm 0.0025 &= 1.000,\ 1.167,\ 1.244,\ 1.195,\ 1.375,\ 0.96.\end{aligned} \tag{5.74}$$

The second line compares the values obtained at different orders by substituting $\epsilon=1$, with the best estimate of $\gamma\approx 1.2385$ in $d=3$. Note that the elements of the series have alternating signs. The truncated series evaluated at $\epsilon=1$ improves up to third order, beyond which it starts to oscillate, and deviates from the left hand side. These are characteristics of an *asymptotic series*. It can be proved that for large p, the coefficients in the expansion of most quantities scale as $|f_p|\sim cp!a^{-p}$. As a result, the ϵ-expansion series is *non-convergent*, but can be evaluated by the *Borel summation* method, using the identity $\int_0^\infty \mathrm{d}x\, x^p \mathrm{e}^{-x}=p!$, as

$$f(\epsilon)=\sum_p f_p\epsilon^p=\sum_p f_p\epsilon^p\frac{1}{p!}\int_0^\infty \mathrm{d}x\, x^p\mathrm{e}^{-x}=\int_0^\infty \mathrm{d}x\mathrm{e}^{-x}\sum_p\frac{f_p(\epsilon x)^p}{p!}. \tag{5.75}$$

The final summation (which is convergent) results in a function of x which can be integrated to give $f(\epsilon)$. Very good estimates of exponents in $d=3$, such as the one for γ quoted above, are obtained by this summation method. There is no indication of any singularity in the exponents up to $\epsilon=2$, corresponding to the lower critical dimension $d=2$.

(2) ***The*** $\mathbf{1/n}$ ***expansion:*** The fixed point position,

$$u^*=\frac{(t^*+K\Lambda^2)^2(4-d)}{4(n+8)K_d\Lambda^d},$$

vanishes as $n\to\infty$. This suggests that a controlled $1/n$ expansion of the critical exponents is also possible. Indeed such an expansion can be developed by a number of methods, such as a saddle point expansion that takes advantage of the exponential dependence of the Hamiltonian on n, or by an exact resummation of the perturbation series. Equation (5.58) in this limit gives,

$$y_t=\lim_{n\to\infty}\left[2-\frac{n+2}{n-8}(4-d)\right]=d-2 \quad\Longrightarrow\quad \nu=\frac{1}{d-2}. \tag{5.76}$$

This result is exact in dimensions $4<d<2$. Above four dimensions the mean field value of 1/2 is recovered, while for $d<2$ there is no order.

Problems for chapter 5

1. *Longitudinal susceptibility:* While there is no reason for the longitudinal susceptibility to diverge at the mean-field level, it in fact does so due to fluctuations in dimensions $d<4$. This problem is intended to show you the origin of this divergence in perturbation theory. There are actually a number of subtleties in this calculation which you are instructed to ignore at various steps. You may want to think about why they are justified.

Consider the Landau–Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int \mathrm{d}^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2\right],$$

describing an n-component magnetization vector $\vec{m}(\mathbf{x})$, in the ordered phase for $t<0$.

(a) Let $\vec{m}(\mathbf{x}) = \big(\overline{m} + \phi_\ell(\mathbf{x})\big)\hat{e}_\ell + \vec{\phi}_t(\mathbf{x})\hat{e}_t$, and expand $\beta\mathcal{H}$ *keeping all terms in the expansion.*

(b) Regard the quadratic terms in ϕ_ℓ and $\vec{\phi}_t$ as an unperturbed Hamiltonian $\beta\mathcal{H}_0$, and the lowest order term coupling ϕ_ℓ and $\vec{\phi}_t$ as a perturbation U; i.e.

$$U = 4u\overline{m}\int \mathrm{d}^d\mathbf{x}\phi_\ell(\mathbf{x})\vec{\phi}_t(\mathbf{x})^2.$$

Write U in Fourier space in terms of $\phi_\ell(\mathbf{q})$ and $\vec{\phi}_t(\mathbf{q})$.

(c) Calculate the Gaussian (bare) expectation values $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle_0$ and $\langle\phi_{t,\alpha}(\mathbf{q})\phi_{t,\beta}(\mathbf{q}')\rangle_0$, and the corresponding momentum dependent susceptibilities $\chi_\ell(\mathbf{q})_0$ and $\chi_t(\mathbf{q})_0$.

(d) Calculate $\langle\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\ \ \vec{\phi}_t(\mathbf{q}_1')\cdot\vec{\phi}_t(\mathbf{q}_2')\rangle_0$ using Wick's theorem. (Don't forget that $\vec{\phi}_t$ is an $(n-1)$ component vector.)

(e) Write down the expression for $\langle\phi_\ell(\mathbf{q})\phi_\ell(\mathbf{q}')\rangle$ to second order in the perturbation U. Note that since U is odd in ϕ_ℓ, only two terms at the second order are non-zero.

(f) Using the form of U in Fourier space, write the correction term as a product of two four-point expectation values similar to those of part (d). Note that only connected terms for the longitudinal four-point function should be included.

(g) Ignore the disconnected term obtained in (d) (i.e. the part proportional to $(n-1)^2$), and write down the expression for $\chi_\ell(\mathbf{q})$ in second order perturbation theory.

(h) Show that for $d<4$, the correction term diverges as q^{d-4} for $q\to 0$, implying an infinite longitudinal susceptibility.

2. *Crystal anisotropy:* Consider a ferromagnet with a tetragonal crystal structure. Coupling of the spins to the underlying lattice may destroy their full rotational symmetry. The resulting anisotropies can be described by modifying the Landau–Ginzburg Hamiltonian to

$$\beta\mathcal{H} = \int \mathrm{d}^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u\left(\vec{m}^2\right)^2 + \frac{r}{2}m_1^2 + v\,m_1^2\,\vec{m}^2\right],$$

where $\vec{m}\equiv(m_1,\cdots,m_n)$, and $\vec{m}^2=\sum_{i=1}^n m_i^2$ ($d=n=3$ for magnets in three dimensions). Here $u>0$, and *to simplify calculations we shall set* $v=0$ throughout.

(a) For a fixed magnitude $|\vec{m}|$, what directions in the n component magnetization space are selected for $r>0$, and for $r<0$?

(b) Using the saddle point approximation, calculate the free energies ($\ln Z$) for phases uniformly magnetized *parallel* and *perpendicular* to direction 1.

(c) Sketch the phase diagram in the (t, r) plane, and indicate the phases (type of order), and the nature of the phase transitions (continuous or discontinuous).

(d) Are there Goldstone modes in the ordered phases?

(e) For $u = 0$, and positive t and r, calculate the unperturbed averages $\langle m_1(\mathbf{q})m_1(\mathbf{q}')\rangle_0$ and $\langle m_2(\mathbf{q})m_2(\mathbf{q}')\rangle_0$, where $m_i(\mathbf{q})$ indicates the Fourier transform of $m_i(\mathbf{x})$.

(f) Write the fourth order term $\mathcal{U} \equiv u \int \mathrm{d}^d\mathbf{x}(\vec{m}^2)^2$, in terms of the Fourier modes $m_i(\mathbf{q})$.

(g) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_1(\mathbf{q})m_1(\mathbf{q}')\rangle$. (You can leave your answers in the form of some integrals.)

(h) Treating $\mathcal{U}$ as a perturbation, calculate the *first order* correction to $\langle m_2(\mathbf{q})m_2(\mathbf{q}')\rangle$.

(i) Using the above answer, identify the inverse susceptibility χ_{22}^{-1}, and then find the transition point, t_c, from its vanishing to first order in u.

(j) Is the critical behavior different from the isotropic $O(n)$ model in $d < 4$? In RG language, is the parameter r *relevant* at the $\mathcal{O}(n)$ fixed point? In either case indicate the universality classes expected for the transitions.

3. *Cubic anisotropy – mean-field treatment:* Consider the modified Landau–Ginzburg Hamiltonian

$$\beta\mathcal{H} = \int \mathrm{d}^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{m})^2 + \frac{t}{2}\vec{m}^2 + u(\vec{m}^2)^2 + v\sum_{i=1}^{n} m_i^4\right],$$

for an n-component vector $\vec{m}(\mathbf{x}) = (m_1, m_2, \cdots, m_n)$. The "cubic anisotropy" term $\sum_{i=1}^{n} m_i^4$ breaks the full rotational symmetry and selects specific directions.

(a) For a fixed magnitude $|\vec{m}|$, what directions in the n component magnetization space are selected for $v > 0$ and for $v < 0$? What is the degeneracy of easy magnetization axes in each case?

(b) What are the restrictions on u and v for $\beta\mathcal{H}$ to have finite minima? Sketch these regions of stability in the (u, v) plane.

(c) In general, higher order terms (e.g. $u_6(\vec{m}^2)^3$ with $u_6 > 0$) are present and insure stability in the regions not allowed in part (b) (as in the case of the tricritical point discussed in earlier problems). With such terms in mind, sketch the saddle point phase diagram in the (t, v) plane for $u > 0$; clearly identifying the phases, and order of the transition lines.

(d) Are there any Goldstone modes in the ordered phases?

4. *Cubic anisotropy ε-expansion:*

(a) By looking at diagrams in a second order perturbation expansion in both u and v show that the recursion relations for these couplings are

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}\ell} = \varepsilon u - 4C\left[\,(n+8)u^2 + 6uv\,\right] \\[2mm] \dfrac{\mathrm{d}v}{\mathrm{d}\ell} = \varepsilon v - 4C\left[\,12uv + 9v^2\,\right], \end{cases}$$

where $C = K_d\Lambda^d/(t+K\Lambda^2)^2 \approx K_4/K^2$ is approximately a constant.

(b) Find all fixed points in the (u, v) plane, and draw the flow patterns for $n < 4$ and $n > 4$. Discuss the relevance of the cubic anisotropy term near the stable fixed point in each case.

(c) Find the recursion relation for the reduced temperature, t, and calculate the exponent ν at the stable fixed points for $n < 4$ and $n > 4$.

(d) Is the region of stability in the (u, v) plane calculated in part (b) of the previous problem enhanced or diminished by inclusion of fluctuations? Since in reality higher order terms will be present, what does this imply about the nature of the phase transition for a small negative v and $n > 4$?

(e) Draw schematic phase diagrams in the (t, v) plane $(u > 0)$ for $n > 4$ and $n < 4$, identifying the ordered phases. Are there Goldstone modes in any of these phases close to the phase transition?

5. *Exponents:* Two critical exponents at second order are,

$$\begin{cases} \nu = \dfrac{1}{2} + \dfrac{(n+2)}{4(n+8)}\epsilon + \dfrac{(n+2)(n^2+23n+60)}{8(n+8)^3}\epsilon^2 \quad , \\ \eta = \dfrac{(n+2)}{2(n+8)^2}\epsilon^2 . \end{cases}$$

Use scaling relations to obtain ϵ-expansions for two or more of the remaining exponents $\alpha,\ \beta,\ \gamma,\ \delta$ and Δ. Make a table of the results obtained by setting $\epsilon = 1,\ 2$ for $n = 1,\ 2$ and 3; and compare to the best estimates of these exponents that you can find by other sources (series, experiments, etc.).

---

6. *Anisotropic criticality:* A number of materials, such as liquid crystals, are anisotropic and behave differently along distinct directions, which shall be denoted parallel and perpendicular, respectively. Let us assume that the d spatial dimensions are grouped into n parallel directions $\mathbf{x}_\parallel$, and $d-n$ perpendicular directions $\mathbf{x}_\perp$. Consider a one-component field $m(\mathbf{x}_\parallel, \mathbf{x}_\perp)$ subject to a Landau–Ginzburg Hamiltonian, $\beta\mathcal{H} = \beta\mathcal{H}_0 + U$, with

$$\beta\mathcal{H}_0 = \int \mathrm{d}^n\mathbf{x}_\parallel \mathrm{d}^{d-n}\mathbf{x}_\perp \left[\frac{K}{2}(\nabla_\parallel m)^2 + \frac{L}{2}(\nabla_\perp^2 m)^2 + \frac{t}{2}m^2 - hm\right],$$

$$\text{and} \quad U = u\int \mathrm{d}^n\mathbf{x}_\parallel \mathrm{d}^{d-n}\mathbf{x}_\perp \; m^4 \; .$$

(Note that $\beta\mathcal{H}$ depends on the **first** *gradient in the $\mathbf{x}_\parallel$ directions, and on the* **second** *gradient in the $\mathbf{x}_\perp$ directions.)*

(a) Write $\beta\mathcal{H}_0$ in terms of the Fourier transforms $m(\mathbf{q}_\parallel, \mathbf{q}_\perp)$.

(b) Construct a renormalization group transformation for $\beta\mathcal{H}_0$, by rescaling coordinates such that $\mathbf{q}_\parallel' = b\,\mathbf{q}_\parallel$ and $\mathbf{q}_\perp' = c\,\mathbf{q}_\perp$ and the field as $m'(\mathbf{q}') = m(\mathbf{q})/z$. *Note that parallel and perpendicular directions are scaled differently.* Write down the recursion relations for $K,\ L,\ t$, and h in terms of $b,\ c$, and z. (The exact shape of the Brillouin zone is immaterial at this stage, and you do not need to evaluate the integral that contributes an additive constant.)

(c) Choose $c(b)$ and $z(b)$ such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues y_t and y_h for the rescalings of t and h.
(d) Write the relationship between the (singular parts of) free energies $f(t, h)$ and $f'(t', h')$ in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form $f(t, h) = t^{2-\alpha} g_f(h/t^{\Delta})$, and identify the exponents α and Δ.
(e) How does the unperturbed zero-field susceptibility $\chi(t, h=0)$ diverge as $t \to 0$?

In the remainder of this problem set $h = 0$, and treat U as a perturbation.

(f) In the unperturbed Hamiltonian calculate the expectation value $\langle m(q)m(q')\rangle_0$, and the corresponding susceptibility $\chi_0(q) = \langle |m_q|^2\rangle_0$, where q stands for $(\mathbf{q}_\parallel, \mathbf{q}_\perp)$.
(g) Write the perturbation U, in terms of the normal modes $m(q)$.
(h) Using RG, or any other method, find the upper critical dimension d_u, for validity of the Gaussian exponents.
(i) Write down the expansion for $\langle m(q)m(q')\rangle$, to first order in U, and reduce the correction term to a product of two point expectation values.
(j) Write down the expression for $\chi(q)$, in first order perturbation theory, and identify the transition point t_c at order of u. (Do not evaluate any integrals explicitly.)

7. *Long-range interactions* between spins can be described by adding a term

$$\int \mathrm{d}^d\mathbf{x} \int \mathrm{d}^d\mathbf{y} J(|\mathbf{x}-\mathbf{y}|)\vec{m}(\mathbf{x}) \cdot \vec{m}(\mathbf{y}),$$

to the usual Landau–Ginzburg Hamiltonian.

(a) Show that for $J(r) \propto 1/r^{d+\sigma}$, the Hamiltonian can be written as

$$\beta\mathcal{H} = \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d} \frac{t + K_2 q^2 + K_\sigma q^\sigma + \cdots}{2} \vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q})$$
$$+ u \int \frac{\mathrm{d}^d\mathbf{q}_1 \mathrm{d}^d\mathbf{q}_2 \mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}} \vec{m}(\mathbf{q}_1) \cdot \vec{m}(\mathbf{q}_2)\, \vec{m}(\mathbf{q}_3) \cdot \vec{m}(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \quad .$$

(b) For $u = 0$, construct the recursion relations for (t, K_2, K_σ) and show that K_σ is irrelevant for $\sigma > 2$. What is the fixed Hamiltonian in this case?
(c) For $\sigma < 2$ and $u = 0$, show that the spin rescaling factor must be chosen such that $K'_\sigma = K_\sigma$, in which case K_2 is irrelevant. What is the fixed Hamiltonian now?
(d) For $\sigma < 2$, calculate the generalized Gaussian exponents ν, η, and γ from the recursion relations. Show that u is irrelevant, and hence the Gaussian results are valid, for $d > 2\sigma$.
(e) For $\sigma < 2$, use a perturbation expansion in u to construct the recursion relations for (t, K_σ, u) as in the text.
(f) For $d < 2\sigma$, calculate the critical exponents ν and η to first order in $\epsilon = d - 2\sigma$.

[See M.E. Fisher, S.-K. Ma and B.G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).]

(g) What is the critical behavior if $J(r) \propto \exp(-r/a)$? Explain!