Kardar 講義.

$$\langle \hat{O} \rangle = \frac{\int D\phi\, e^{-H_0-U}\, \hat{O}}{\int D\phi\, e^{-H_0-U}}$$ H_0 為可解型.

$$= \frac{\langle \hat{O}\, e^{-U} \rangle_0}{\langle e^{-U} \rangle_0} \quad 0 \leftarrow H_0.$$

$$\left(Z_0 = \int D\phi\, e^{-H_0} \right)$$

$$= \sum \frac{(-1)^\ell}{\ell!} \langle O U^\ell \rangle_0^{\text{connected}}.$$

重整化.

$g\,\delta^{(2)}(x)$ $\quad E \sim E(g, \frac{\hbar^2}{2m}).$

$$E = E(g, \frac{\hbar^2}{2m}, \Lambda)$$

有限 截斷

能標發散.

若g與能標有關,
則E亦會發散.

$$\therefore \frac{dg_n}{d\ln\Lambda} = \beta.$$

但只適合於少量模型.
若有相互作用問題,
更傾向於使用微擾方法.

參 Shankar Chapter 13, 14.
Peskin. ϕ^4理論.

Wick 收縮.

$$\frac{\int e^{-ax^2} x^{2n} dx}{\int e^{-ax^2} dx} = \frac{\frac{(2n)!}{n!2^n}}{a^n}$$

則兩兩配對.

$$\langle e^{ikx} \rangle = \sum \frac{(ik)^n}{n!} \langle x^n \rangle$$

$$\updownarrow$$

$$\langle e^{-V(\phi)} \rangle = \sum \frac{(-1)^n}{n!} \langle V^n \rangle$$

$$\langle x^2 \rangle = \langle x \rangle^2 + \sigma_2$$

$$\sigma_2 = \langle x^2 \rangle - \langle x \rangle^2$$

數學: 方差.
物理. 扣掉相互作用. 兩體之間的貢獻.

$\langle x^4 \rangle - \langle x \rangle^4$ - 兩體關聯 - 三體關聯 = 四體關聯

矩分解並不是好的分解.

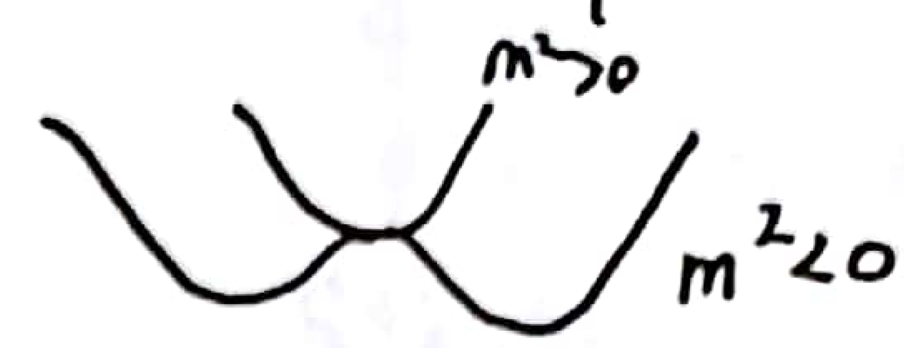
累積量分解:

$$\exp\left(\sum \frac{(ik)^\ell}{\ell!} \langle x^\ell \rangle_{\text{connected}}\right)$$

定義.

$$\begin{cases} k^{\mu}=(\omega,\vec{k}) \\ k_{\mu}=(\omega,-\vec{k}) \\ g^{\mu\nu}=(+,-,-,-). \end{cases} \sim \begin{cases} \partial^{\mu}=(\partial_t,\nabla) \\ \partial_{\mu}=(\partial_t,-\nabla). \end{cases}$$

$V(\phi)=\frac{m^2}{2}\phi^2+\frac{\lambda}{4}\phi^4$



與自發對稱破缺 Gold-stone、Soliton 等有關.

$$\mathcal{L}=\underbrace{\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi}_{(\partial_t\phi)^2-(\partial_x\phi)^2.}\underbrace{-\frac{m^2}{2}\phi^2-\frac{\lambda}{4!}\phi^4}_{V.}.$$

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right)=\frac{d}{dt}(\partial_t\phi)=\partial_t^2\phi. \\ &\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)=-\partial_x^2\phi. \end{aligned}\right\} \partial_t^2\phi-\partial_x^2\phi=0 \Rightarrow \phi\sim e^{i(\omega t-kx)}$$

配分函數.

(統計物理、$t\in[0,\beta]$、需有空間週期條件)

$Z=\int D\phi\, e^{i\int dx\,dt\,\mathcal{L}}$. $i\to it$ Wick Rotation. 傳播子 ~ 配分函數

Path Integral ~ Statistic Mechanics

$$\langle\hat{O}\rangle=\frac{1}{Z}\mathrm{Tr}(Oe^{-\beta H})$$
$$=\frac{\cancel{1}}{\cancel{Z}}\frac{\int D\phi\,\hat{O}\,e^{i\int dt\,dx\,\mathcal{L}}}{\int D\phi\, e^{i\int dt\,dx\,\mathcal{L}}}$$

$\mathcal{L}=\mathcal{L}_0-V.$

$V=\frac{\lambda}{4!}\phi^4=u.$

$i\int dt\,dx\,\mathcal{L}\sim\int d\tau\,dx\,\mathcal{L}'$.

$\mathcal{L}=\frac{1}{2}(\partial_t\phi)^2-\frac{1}{2}(\nabla\phi)^2-\frac{m^2}{2}\phi^2-\frac{\lambda}{4}\phi^4.$

$\Rightarrow -\left(\frac{1}{2}(\partial_\tau\phi)^2+\frac{1}{2}(\nabla\phi)^2+\frac{m^2}{2}\phi+\frac{\lambda}{4}\phi^4\right)$

$=-H.$

令 $\tau=\beta$.

$e^{iHt}\sim e^{-\beta H}$

(與 g-2 重整化、具體處理方法有關)

有強作用、高階累積量將非常重要、

僅適以微擾處理、微擾結果會發散.

不得不引入 m 與 λ 有關.

相互作用 coupling constant 影響.

(HW.

$H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2+\lambda x^4$.)

求本徵值

令 $H=\underbrace{\frac{p^2}{2m_R}+\frac{1}{2}m_R\omega_R^2x^2}_{H_0}+\underbrace{\delta_m p^2+\delta_\omega x^2+\lambda x^4}_{微擾}$、

$$\begin{cases} H_0|0\rangle=\frac{\hbar\omega_R}{2} \\ H_0|1\rangle=\frac{3}{2}\hbar\omega_R \end{cases}$$

假設 $\langle 0|V|0\rangle+\cdots=0$

$\langle 1|V|1\rangle+\cdots=0.$

微擾.

$$\begin{cases} E_0=\frac{\hbar\omega_R}{2}+\langle 0|V|0\rangle+\cdots \\ E=\frac{3}{2}\hbar\omega_R+\langle 1|V|1\rangle+\cdots \end{cases}$$

定 m_R、ω_R 參數、

定下 H, H_0 關係

~~假設二、多個、對令 m_R ω_R 求相應解~~

$$\frac{\int D\phi\, e^{-H_0-u}\,\hat{o}}{\int D\phi\, e^{-H_0-u}} = \langle \hat{o} \rangle = \langle \hat{o} e^{-u} \rangle_0$$

ϕ^4理論中. $-H_0 = \int dt\,dx\,\mathcal{L}$ 二次型
$u = \int dt\,dx\,V$. 相互作用

$\int dt\,dx\,\mathcal{L} = ?$
需轉入動量空間下計算.

參考. Peskin Chapter 9.
Kleinert. ~~critical~~ critical propertie of ϕ^4 theory. chapter 6.
Shanker Chapter 13-14
Kardar.

ϕ is Real.

$$\frac{\int dx\, e^{-\frac{1}{2}Ax^2}\cdot x^2 dx}{\int dx\, e^{-\frac{1}{2}Ax^2}} = \frac{1}{A}.$$

ϕ is Complex.

$$\frac{\int D\bar{z}Dz\, e^{-A\bar{z}z}\cdot \bar{z}z}{\int D\bar{z}Dz\, e^{-A\bar{z}z}} = \frac{1}{A}$$

$$\begin{cases} \phi(x) = \frac{1}{V}\sum_k \phi(k) e^{ikx} \\ \phi(x)\in\mathbb{R},\quad \phi(k)\in\mathbb{C}. \\ \quad\text{并有 } \phi(-k) = \phi^*(k). \end{cases}$$

$$\langle \phi(x)\phi(y)\rangle_0 = \frac{1}{V^2}\sum_{k,q}\langle \phi(k)\phi(q)\rangle e^{ikx+iqy}.$$

$$\int dt\,dx\,\mathcal{L} = \frac{1}{2}\int dt\,dx\,\left((\partial_t\phi)^2 - (\nabla\phi)^2 - m^2\phi^2\right)$$

$$\int (\partial_t\phi)^2 dt\,dx = \int \frac{1}{V^2}\sum_{kq} e^{ikx+iqx}\phi(k)\phi(q)\cdot(ik^0)(iq_0)\,dx\,dt.$$

~~$(\partial_t\phi)^2$~~ ~~$t\partial_t$~~

$$\left(\begin{aligned} &\int dx\, e^{ikx} = \int_0^\infty e^{ikx-\varepsilon x}dx + \int_{-\infty}^0 e^{ikx+\varepsilon x}dx. = \frac{1}{\varepsilon - ik} + \frac{1}{\varepsilon+ik} = \frac{2\varepsilon}{\varepsilon^2+k^2} \\ &\int \frac{2\varepsilon}{\varepsilon^2+k^2}dk \xrightarrow{\text{令 } k=\varepsilon q} \int \frac{2\varepsilon^2 dq}{\varepsilon^2(1+q^2)} \\ &\qquad = 2\pi \end{aligned} \right)$$

$$= \cancel{\frac{1}{V^2}\int dt\,dx \sum_{kq}\int}\ \frac{1}{V^2}\sum_{kq}\underbrace{\int dt\,dx\, e^{i(k+q)x}}_{(2\pi)^d\delta(k+q)}\phi(k)\phi(q)\underbrace{iq^0 ik^0}_{k_0^2}.$$

$$= \frac{1}{V^2}\sum_{kq}(2\pi)^d\,\delta(k+q)\,\phi(k)\phi(q)\,k_0^2$$

$$= \frac{1}{V^2}\sum_{kq}(2\pi)^d\delta(k+q)\,\phi(k)\phi(q)\,k^{02}.$$

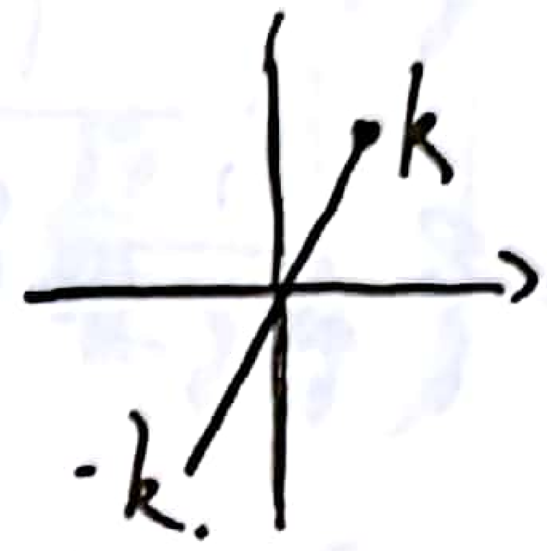
$$= \frac{1}{V^2}\cdot\frac{V}{(2\pi)^d}\,\frac{V}{(2\pi)^d}\int dk\,dq\,\delta(k+q)\,\phi(k)\phi(q)\,k^{0\,2}.$$

$$= \frac{1}{(2\pi)^d}\int dk\,\underbrace{\phi(k)\phi(-k)}_{\phi^*(k)\phi(k)}\,k^{0\,2} \qquad \int \delta(k+q)\,dq = 1.$$

$\phi(-k)\phi(k) = \phi^*(k)\phi(k)$

$\phi(k)\phi(-k) = \phi^*(k)\phi(k).$

$$= \frac{2}{(2\pi)^d}\int_{k^0>0} dk\,\phi^*(k)\phi(k)\,k^{0\,2}$$



$$\therefore \frac{1}{2}\int dt\,dx\,(\partial_t\phi)^2 = \frac{1}{(2\pi)^d}\int_{k^0>0} dk\,\phi^*(k)\phi(k)\,k^{2}.$$

(Peskin P287)

$\left(\int f^2(x)\,dx = \int f^*(k)\,f(k)\,dk.\right.$
实空间粒子数．动量空间粒子数
$\left.\text{帕塞瓦定理}.\right)$

$(\partial_t\phi)^2 \Rightarrow k^{0\,2}\,\phi^*(k)\phi(k).$

$(\partial_x\phi)^2 \Rightarrow k^{x\,2}\,\phi^*(k)\phi(k).$

$\phi^2 \longrightarrow \phi^*(k)\phi(k)$

$$\int dt\,dx\,\mathcal{L}_0 = \frac{1}{(2\pi)^d}\int_{k^0>0} dk\,(k^{0\,2}-\vec{k}^2-m^2)\,\phi^*(k)\phi(k)$$

$$= \frac{1}{V}\sum_{k^0>0}(k^{0\,2}-k^2-m^2)\,\phi^*(k)\phi(k)$$

平均值

$$\frac{\int D\phi\, e^{iS_0}\phi(x)\phi(y)}{\int D\phi\, e^{iS_0}} \qquad \int D\phi \equiv \prod_i \int D\phi(x_i) \equiv J\prod_k \int D\phi(k) = J\prod_{k^0>0}\int D\phi^*(k)\,D\phi(k)$$

分别均会产生Jacobi，会互相抵消．

$\int D\phi(x_i) \longrightarrow$ 动量空间、需离散化．

$$S_0 = \int dt\,dx\,\mathcal{L}_0$$
$$= \frac{1}{(2\pi)^d}\int_{k^0>0} dk\,(k^{0\,2}-k^2-m^2)\,\phi^*(k)\phi(k)$$
$$= \frac{1}{V}\sum_{k^0>0}(k^{0\,2}-k^2-m^2)\,\phi^*(k)\phi(k)$$

$$= \frac{1}{V^2}\,e^{ikx+iqy}\,\frac{\int D\phi^*(k)\,D\phi(k)\cdot\phi(k)\,\phi(q)\;e^{iS_0}}{\int D\phi^*(k)\,D\phi(k)\;e^{iS_0}}$$

$= \sum_q \left(\frac{1}{V^2} e^{iq(x-y)}\right)$

$= \sum_q \frac{1}{V^2} e^{-iq(x-y)} \frac{iV^2}{q^{0 2}-q^2-m^2(+i\epsilon)}$

$= \sum_q \frac{i}{q^{0 2}-q^2-m^2} e^{-iq(x-y)}$

$\equiv D_{Free}(x-y).$

例.

$\frac{\int dx dy\, e^{-ax^2-by^2} x^2}{\int dx dy\, e^{-ax^2-by^2}} = \frac{\int dy\, e^{-by^2} \int dx\, e^{-ax^2} x^2}{\int dy\, e^{-by^2} \int e^{-ax^2} dx}$

$= \frac{\int dx\, e^{-ax^2} x^2}{\int dx\, e^{-ax^2}}$

求 $\frac{\int d\bar{z} dz\, e^{i \frac{1}{V} A \bar{z} z} \bar{z} z}{\int d\bar{z} dz\, e^{i\frac{1}{V} A \bar{z} z}}$ 令 $\begin{cases} z = \phi(q) \\ \bar{z} = \phi^*(q) \\ A = q^{0 2} - \vec{q}^2 - m^2 \end{cases}$

$= V^2 \frac{i}{A}.$

$D_F(x-y) = \sum_q \frac{i}{q^{0 2}-q^2-m^2} e^{-iq(x-y)}$

相互作用.

$D(x-y) = \langle \phi(x)\phi(y)\rangle = \frac{\langle \phi(x)\phi(y) e^{-u}\rangle_0}{\langle e^{-u}\rangle_0} = \cancel{\sum_l \langle O u^l\rangle_0^c \cdot \frac{(-1)^l}{l!}}$ $\sum_l \frac{(-1)^l}{l!}\langle O u^l\rangle_0^c$

其中 $e^{-u} = e^{-i\int dt dx \frac{\lambda}{4!}\phi^4} = e^{i\int dz \frac{\lambda}{4!}\phi^4(z)}.$

$l=0$ $\quad D_F(x-y)$ (x —— y).

$l=1.$ $\quad (-1)\langle O u\rangle_0^c$

$\langle O u\rangle_0^c = ?$ $\quad u = i\int dz \frac{\lambda}{4!}\phi^4$

$\langle O u\rangle_0^c = \cancel{\int \langle \phi(x)\phi(y)\rangle}$

$\int \langle \phi(x)\phi(y)\, i\frac{\lambda}{4!}\phi^4(z)\rangle_0^c dz.$

$= \frac{i\lambda}{4!}\int dz \langle \phi(x)\phi(y)\phi^4(z)\rangle_0^c.$

Gauss function 只有二阶积累量.

$\frac{\int D\phi\, \phi(q_1)\phi(q_2)\phi(q_3)\phi(q_4)\phi(q_5)\phi(q_6) e^{iS_0}}{\int D\phi\, e^{iS_0}} = \frac{i\lambda}{4!}\cdot 12$

(考虑 $\int dx\, x^{2n} e^{-ax^2} = \frac{N}{a^n}$)

Number.

$= i\frac{\lambda}{2}\int dz\, D_F(x-z)\underbrace{D_F(z-z)}\, D_F(z-y)$

$D(z-z) = \sum_q e^{iq(z-z)}\left(\frac{i}{q^{0 2}-q^2-m^2}\right)$

$= \underbrace{\sum_q \frac{i}{q^{0 2}-q^2-m^2}}$ 发散

$\Sigma \frac{i}{q_0^2-q^2-m^2}$

回顾. $\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+g\delta(x)\right)\psi = E\psi$.

$\psi = \sum_k C_k e^{ikx}$

$C_k \cdot \frac{\hbar^2}{2m}k^2 + \psi_0 = \cdot E C_k$

$C_k = -\frac{1}{E+\frac{\hbar^2 k^2}{2m}}\psi_0 \quad \Leftrightarrow \quad g\sum_k \frac{1}{E+\frac{\hbar^2 k^2}{2m}} = 1$

~~$D(z)$~~ $D(z-z) = \sum_q e^{iq(z-z)} \frac{i}{q_0^2-q^2-m^2}$

$=$

$\therefore = D_F(x-y) - \frac{i\lambda}{2}\int dz\, \underbrace{D_F(x-z)\, D_F(z-z)\, D_F(z-y)}_{\text{卷積}}$.

$\downarrow$ FT

$D(k) = D_F(k) - \frac{i\lambda}{2} D_F(k) D_F(z-z) D_F(k)$

$\therefore D(k) = D_F(k) - \frac{i\lambda}{2} D_F(z-z) D_F(k) D_F(k)$.

$= D_F(k) - \frac{i\lambda}{2} D_F(k) D_F(0) D_F(k)$

$-\frac{i\lambda}{2}D_F(z-z) = \Sigma$. self energy

$D_F(k) = \frac{i}{k_0^2-k^2-m_0^2}$.

$D(k) = \frac{i}{k_0^2-k^2-m^2} + \frac{i}{k_0^2-k^2-m_0^2} + \frac{i}{k_0^2+k^2-m_0^2}\Sigma\frac{i}{k_0^2-k^2-m_0^2} + \cdots$

$\Leftrightarrow m^2 = m_0^2 + \Sigma$.

(類比. $\frac{1}{1-x} = 1+x+x^2+\cdots$),

$\therefore D_F + D_F \cdot \Sigma D_F + D_F \Sigma D_F \Sigma D_F + \cdots$

$= D_F(1+\Sigma D_F + \Sigma D_F \Sigma D_F + \cdots)$

$= \frac{D_F}{1-\Sigma D_F} = \frac{1}{D_F^{-1}-\Sigma}$. (~~Dyson equa~~ Dyson equation).

$\therefore D = \frac{1}{D_F^{-1} - \Sigma}$.

$\frac{1}{k^{0^2}-k^2-m^2} = \frac{1}{k^{0^2}-k^2-m_0^2-\Sigma}$

$\underbrace{m}^2 = m_0^2 + \Sigma$.
可觀測.

$m^2 = m_0^2 + \cancel{\frac{\lambda}{2}} - \frac{i\lambda}{2}\int dq \frac{1}{q^{0^2}-q^2-m_0^2}$

$q_0 \to i q_0$.

$\Rightarrow \underbrace{m_0^2 - \frac{\lambda}{2}\int d^d q \frac{1}{q^2+m_0^2}}$ 發散
需正規化.
①. G.H. 的
②. 維度正規化

類似於 $E = E(g, \frac{\lambda^2}{2m}, \Lambda)$

參. Shanker chapter 14. (14.3).

頻率值與能標有關.
既是可觀測結果, 若 m_0 是有限值, 則觀測的 m 必會因之發散.

$\therefore \begin{cases} m_0 = m_0(\lambda) \\ \lambda = \lambda_\Lambda \end{cases}$

求可能修正.
$m^2 = m_0^2 + \delta m$. $\therefore \delta m = \frac{\lambda}{2}\int d^D q \frac{1}{q^2+m^2}$.
~~低維發散不嚴重~~.

$q^{d-3} dq \sim \Lambda^{d-2}$.
除非一維, 否則發散