14

Two Views of Renormalization

Here I discuss the relationship between two approaches to renormalization: the older one based on removing infinities in the quest for field theories in the continuum, and the more modern one due to Wilson based on obtaining effective theories. My focus will be on a few central questions. No elaborate calculations will be done.

14.1 Review of RG in Critical Phenomena

Let us recall the problem of critical phenomena and its resolution by the RG. Suppose we have some model on a lattice with some parameters, like K_1, K_2, ... of the Ising model. At very low and very high temperatures ($K \to \infty$ or $K \to 0$) we can employ perturbative methods like the low-temperature or high-temperature expansions to compute correlation functions. These series are predicated on the smooth change of physics as we move away from these extreme end points. By definition, these methods will fail at the critical point (and show signs of failing as we approach it) because there is a singular change of phase. One signature of trouble is the diverging correlation length ξ. The RG beats the problem by trading the original system near the critical point for one that is comfortably away from it (and where the series work) and things like ξ can be computed. The RG then provides a dictionary for translating quantities of original interest in terms of new ones. For example,

$$\xi(r_0) = 2^N \xi(r_N), \tag{14.1}$$

where $r_0 \simeq t$ is the deviation from criticality, N is the number of factor-of-2 RG transformations performed, and r_N the coupling that r_0 evolves into. At every step,

$$r_0 \to r_0 2^{ad} = r_0 2^{1/\nu}. \tag{14.2}$$

We keep renormalizing until r_N has grown to a safe value far from criticality, say

$$r_N = r_0 2^{N/\nu} = 1, \quad \text{that is} \tag{14.3}$$

$$2^N = r_0^{-\nu}. \tag{14.4}$$

Then, from Eq. (14.1),

$$\xi(r_0) = r_0^{-\nu}\xi(1). \tag{14.5}$$

The divergence in ξ is translated into the divergence in N, the number of steps needed to go from r_0 to $r_N = 1$ as r_0 approaches the critical value of 0.

In terms of the continuous scale s (which replaces 2^N), these relations take the form

$$\xi(r_0) = s\xi(r_s), \tag{14.6}$$

$$r_s = r_0 s^{1/\nu}. \tag{14.7}$$

Typically one finds some *approximate* flow equations, their fixed points $\boldsymbol{K}^*$, the linearized flow near $\boldsymbol{K}^*$, and, eventually, the exponents.

14.2 The Problem of Quantum Field Theory

Consider the field theory with action (with $c = 1 = \hbar$)

$$S = \int \left[\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}m_0^2\phi^2(x) + \frac{\lambda_0}{4!}\phi^4(x) \right] d^4x \tag{14.8}$$

$$= S_0 + S_{\mathrm{I}}, \tag{14.9}$$

where S_0 is the quadratic part. I have chosen $d = 4$, which is relevant to particle physics and serves to illustrate the main points, and a real scalar field to simplify the discussion. The parameters m_0 and λ_0 are to be determined by computing some measurable quantities and comparing to experiment.

To this end, we ask what is typically computed, how it is computed, and what information it contains.

Consider the two-point correlation function

$$G(x) = \langle\phi(x)\phi(0)\rangle \tag{14.10}$$

$$= \frac{\int[\mathcal{D}\phi]\phi(x)\phi(0)e^{-S}}{\int[\mathcal{D}\phi]e^{-S}}. \tag{14.11}$$

First, let us assume that $\lambda_0 = 0$. Doing the Gaussian functional integral we readily find

$$G(r) \simeq \frac{e^{-m_0 r}}{r^2}. \tag{14.12}$$

How do we determine m_0 from experiment? In the context of particle physics, m_0 would be the particle mass, measured the way masses are measured. If, instead, the ϕ^4 theory were being used to describe spins on a lattice of spacing a, we would first measure the dimensionless correlation length ξ (in lattice units) from the exponential decay of correlations and relate it to m_0 by the equation

$$m_0 = \frac{1}{a\cdot\xi}. \tag{14.13}$$

Sometimes I will discuss correlations of four ϕ's. They will also be called G, but will be shown with four arguments. If not, assume we are discussing the two-point function.

In momentum space we would consider the Fourier transform

$$\langle \phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\rangle = (2\pi)^4\delta^4(\boldsymbol{k}_1 - \boldsymbol{k}_2)G(k), \text{ where} \tag{14.14}$$

$$G(k) = G_0(k) = \frac{1}{k^2 + m_0^2}; \tag{14.15}$$

the subscript on G_0 reminds us that we are working with a free-field theory. Correlations with more fields can be computed as products of two-point functions $G_0(k)$ using Wick's theorem. If this explains the data, we are done.

14.3 Perturbation Series in λ_0: Mass Divergence

Let us say the $\lambda_0 = 0$ theory does not explain the data. For example, the particles could be found to scatter. The $\lambda_0 = 0$ theory cannot describe that. So we toss in a λ_0 and proceed to calculate correlation functions, and fit the results to the data to determine m_0 and λ_0.

When $\lambda_0 \neq 0$, we resort to perturbation theory. We bring the $\lambda_0\phi^4$ term in S downstairs as a power series in λ_0 and do the averages term-by-term using Wick's theorem. To order λ_0, we find

$$G(x) = \langle \phi(x)\phi(0)\rangle \tag{14.16}$$

$$= \frac{\int [\mathcal{D}\phi]\phi(x)\phi(0)e^{-S_0(\phi)}\left[1 - \frac{\lambda_0}{4!}\int \phi^4(y)d^4y\right]}{\int [\mathcal{D}\phi]e^{-S_0(\phi)}\left[1 - \frac{\lambda_0}{4!}\int \phi^4(y)d^4y\right]}. \tag{14.17}$$

In the denominator, we pair the four $\phi(y)$'s two-by-two to obtain

$$\text{denominator} = 1 - \frac{\lambda_0}{8}\int G_0^2(0)d^4y. \tag{14.18}$$

In the numerator, one option is to pair $\phi(x)$ and $\phi(0)$, which are being averaged, and pair the fields inside the y integral with each other. This will give $G_0(x)\cdot\left(1 - \frac{\lambda_0}{8}\int G_0^2(0)d^4y\right)$. The factor in parentheses will get canceled by the normalizing partition function in the denominator. This happens in general: any contribution in which the fields being averaged do not mingle with the ones in the interaction, the so-called *disconnected terms*, may be dropped.

This leaves us with contributions where $\phi(x)$ and $\phi(0)$ are paired with the $\phi(y)$'s. The result is, *to order* λ_0,

$$G(x) = G_0(x) - \frac{1}{2}\lambda_0\int G_0(x-y)G_0(y-y)G_0(y-0)d^4y. \tag{14.19}$$

Since the second term is of order λ_0, we may set the normalizing denominator $1 - \frac{\lambda_0}{8}\int G_0^2(0)d^4y$ to 1.

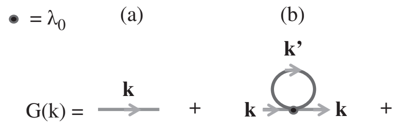

Figure 14.1 (a) $G_0(k) = G(k)$ in free-field theory. (b) One-loop correction to m_0^2. The lines with arrows denote the free propagator $G_0(k) = \frac{1}{k^2+m_0^2}$.

In momentum space,

$$G(k) = \frac{1}{k^2+m_0^2} - \frac{1}{k^2+m_0^2} \underbrace{\left[\frac{1}{2} \int_0^\infty \frac{\lambda_0}{k'^2+m_0^2} \frac{d^4k'}{(2\pi)^4} \right]}_{\delta m_0^2} \frac{1}{k^2+m_0^2} \cdots \tag{14.20}$$

This series is represented in Figure 14.1.

To this order in λ_0 we may rewrite this as

$$G(k) = \frac{1}{k^2+m_0^2+\delta m_0^2}. \tag{14.21}$$

We conclude that the mass squared in the interacting theory is

$$m^2 = m_0^2 + \delta m_0^2. \tag{14.22}$$

The next natural thing to do is compare the measured m^2 to this result and find a relation constraining m_0^2 and λ_0.

It is here we encounter the serious trouble with continuum field theory: δm_0^2 *is quadratically divergent in the ultraviolet*:

$$\delta m_0^2 = \frac{1}{2} \int_0^\infty \frac{\lambda_0}{k'^2+m_0^2} \frac{d^4k'}{(2\pi)^4}. \tag{14.23}$$

So no matter how small λ_0 is, the change in mass δm_0^2 is infinite. The infinity comes from working in the continuum with no limit on the momenta in Fourier expansions. The theory seems incapable of describing the experiment with a finite m, assuming m_0 and λ_0 are finite.

Let us set this aside and compute the scattering amplitude, to compare it with experiment to constrain m_0^2 and λ_0.

14.4 Scattering Amplitude and the Γ's

We must clearly begin with the correlation of four fields, two each for the incoming and outgoing particles. The momenta are positive flowing inwards and there is no difference between particles and antiparticles.The correlation function $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ is depicted in

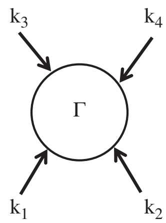

Figure 14.2 The scattering amplitude Γ is a function of the particle momenta, all chosen to point inwards. Their vector sum is zero.

Figure 14.2 and is defined as follows after pulling out the momentum-conserving δ function:

$$\langle \phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\phi(\boldsymbol{k}_3)\phi(\boldsymbol{k}_4)\rangle = (2\pi)^4\delta^4(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4). \tag{14.24}$$

To lowest order in λ_0, we get, upon pairing the four external ϕ's with the four ϕ's in the $\lambda_0\phi^4$ interaction,

$$G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = G(\boldsymbol{k}_1)G(\boldsymbol{k}_2)G(\boldsymbol{k}_3)G(\boldsymbol{k}_4)\lambda_0. \tag{14.25}$$

However, $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ is not the scattering amplitude which we should square to get the cross section. The four external propagators do not belong there. (In Minkowski space, the propagators will diverge because $k^2 = m^2$.) The scattering amplitude $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ is defined as follows:

$$G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = G(\boldsymbol{k}_1)G(\boldsymbol{k}_2)G(\boldsymbol{k}_3)G(\boldsymbol{k}_4)\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4). \tag{14.26}$$

That is,

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = G^{-1}(\boldsymbol{k}_1)G^{-1}(\boldsymbol{k}_2)G^{-1}(\boldsymbol{k}_3)G^{-1}(\boldsymbol{k}_4)G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4). \tag{14.27}$$

To lowest order,

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = \lambda_0. \tag{14.28}$$

Do we really need to bring in another function $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ if it is just $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ with the four external legs chopped off? Actually, we could get by with just the $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$'s, but in doing so would miss some important part of quantum field theory (QFT). First, $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ is not alone, it is part of a family of functions, as numerous as the G's. That is, there are entities $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ for all n. They provide an alternate, equally complete, description of the theory to the G's, just like the Hamiltonian formalism is an alternative to the Lagrangian formalism. They are better suited than the $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$'s for discussing renormalization. And they are not just $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$'s with the external legs amputated.

In view of time and space considerations, I will digress briefly to answer just two questions:

- Where do the Γ's come from?
- What are the Feynman diagrams that contribute to them?

Consider the partition function $Z(J)$ with a source:

$$Z(J) = \int [\mathcal{D}\phi]\, e^{-S} e^{\int J(x)\phi(x)dx} \equiv e^{-\boldsymbol{W}(J)}. \tag{14.29}$$

The *generating functional* $\boldsymbol{W}(J)$ yields $G_c(x_1,\ldots,x_n)$ upon repeated differentiation by $J(x)$, where the subscript c stands for *connected*:

$$\boldsymbol{W}(J) = -\int \frac{dx_1\cdots dx_n}{n!} G_c(x_1,\ldots,x_n)J(x_1)\cdots J(x_n). \tag{14.30}$$

In particular,

$$\bar{\phi}(x) \equiv \langle \phi(x)\rangle = -\frac{\partial \boldsymbol{W}}{\partial J(x)}. \tag{14.31}$$

(It is understood here and elsewhere that the derivatives are taken at $J=0$.) Taking one more derivative gives

$$\langle \phi(x)\phi(y)\rangle_c = -\frac{\partial^2 \boldsymbol{W}}{\partial J(x)\partial J(y)} = G_c(x,y). \tag{14.32}$$

Given this formalism, in which $\boldsymbol{W}(J)$ is a functional of J and $\bar{\phi}$ is its derivative, it is natural to consider a Legendre transform to a functional $\boldsymbol{\Gamma}(\bar{\phi})$ with J as its derivative. By the familiar route one follows to go from the Lagrangian to the Hamiltonian or from the energy to the free energy, we are led to

$$\boldsymbol{\Gamma}(\bar{\phi}) = \int J(y)\bar{\phi}(y)dy + \boldsymbol{W}(J). \tag{14.33}$$

By the usual arguments,

$$\frac{\partial \boldsymbol{\Gamma}(\bar{\phi})}{\partial \bar{\phi}(y)} = J(y). \tag{14.34}$$

The Taylor expansion

$$\boldsymbol{\Gamma}(\bar{\phi}) \stackrel{\text{def}}{=} \int \frac{dx_1\cdots dx_n}{n!}\Gamma(x_1,\ldots,x_n)\bar{\phi}(x_1)\cdots\bar{\phi}(x_n) \tag{14.35}$$

defines the Γ's with n arguments. A similar expansion in terms of $\bar{\phi}(\boldsymbol{k})$ defines $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$.

Given this definition, and a lot of work, one can show that $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ will have the following diagrammatic expansion:

- Draw the connected diagrams that contribute to $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n)$ with the same incoming lines, except for those diagrams that can be split into two disjoint parts by cutting just one internal line. For this reason the Γ's are called 1*PI* or *one-particle irreducible* correlation functions.
- Append a factor $G^{-1}(\boldsymbol{k})$ for every incoming particle of momentum $\boldsymbol{k}$.

To get acquainted with this formalism, let us derive the relation between $\Gamma(k)$ and $G(k)$ that it implies. Given that $J(x)$ and $J(y)$ are independent, it follows that

$$\delta(x-y) = \frac{\partial J(x)}{\partial J(y)} \tag{14.36}$$

$$= \frac{\partial^2 \boldsymbol{\Gamma}}{\partial J(y)\partial\bar{\phi}(x)} \tag{14.37}$$

$$= \frac{\partial^2 \boldsymbol{\Gamma}}{\partial\bar{\phi}(x)\partial J(y)} \tag{14.38}$$

$$= \int dz \frac{\partial^2 \boldsymbol{\Gamma}}{\partial\bar{\phi}(x)\partial\bar{\phi}(z)} \frac{\partial\bar{\phi}(z)}{\partial J(y)} \tag{14.39}$$

$$= -\int dz \frac{\partial^2 \boldsymbol{\Gamma}}{\partial\bar{\phi}(x)\partial\bar{\phi}(z)} \frac{\partial^2 \boldsymbol{W}}{\partial J(z)\partial J(y)} \tag{14.40}$$

$$= \int dy \Gamma(x,z) G(z,y), \tag{14.41}$$

which leads to the very interesting result that the matrices Γ and G with elements $\Gamma(x,z)$ and $G(z,y)$ are inverses:

$$\Gamma = G^{-1}. \tag{14.42}$$

This agrees with the rules given above for computing the two-point function $\Gamma(k)$ from $G(k)$: If we take the two-point function $G(k)$ and multiply by two inverse powers of $G(k)$ (one for each incoming line) we get $\Gamma(k) = G^{-1}(k)$.

Upon further differentiation with respect to $\bar{\phi}(\boldsymbol{k})$, one can deduce the relation between the G's and Γ's and the Feynman rules stated above.

14.4.1 Back to Coupling Constant Renormalization

Let us now return to the scattering amplitude $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$. To lowest order in λ_0,

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = \lambda_0. \tag{14.43}$$

It is $|\lambda_0|^2$ you must use to compute cross sections.

In general, $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ will depend on the external momenta. However, to this order in λ_0, we find Γ does not have any momentum dependence and coincides with the coupling λ_0 in the action.

As we go to higher orders, $\Gamma(0,0,0,0)$ will be represented by a power series in λ_0. *We will then define* $\Gamma(0,0,0,0)$ *as the coupling* λ*, not the* λ_0 *in the action.* This fixes the interaction strength completely. I am not saying that the external momenta vanish in every scattering event, but that in any one theory, given $\Gamma(0,0,0,0)$, a unique $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ is given by Feynman diagrams.

The trick of comparing the observed scattering rate to the one calculated from Eq. (14.43) to extract λ_0 will work only if λ_0 is small and higher-order corrections are negligible. Let us assume that λ_0 is very small, just like in electrodynamics where the analog of $\lambda_0 \simeq \frac{1}{137}$.

We will now consider scattering to order λ_0^2, even though it is one order higher than the correction to m_0^2. The reason is that it is also given by a one-loop graph, as shown in Figure 14.3, and *the systematic way to organize perturbation theory is in the number of loops*. If we restore the $\frac{1}{\hbar}$ in front of the action, we will find (Exercise 14.4.1) that the tree-level diagram, which is zeroth order in the loop expansion, is of order $\frac{1}{\hbar}$ and that each additional loop brings in one more positive power of $\hbar$. The loop expansion is therefore an $\hbar$ expansion. (During Christmas, we have a tree in our house but no wreath on the door, making us Christians at tree level but not one-loop level.)

Exercise 14.4.1 *Introduce* $\hbar^{-1}$ *in front of the action and see how this modifies* G_0 *and* λ_0*. Look at the diagrams for* G *and* Γ *to one loop and see how the loop brings in an extra* $\hbar$*.*

The one-loop corrections to scattering are depicted in Figure 14.3. They correspond to the following expression:

$$\Gamma(0,0,0,0) = \lambda_0 - 3\lambda_0^2 \int_0^\infty \frac{1}{(k^2+m_0^2)^2} \frac{d^4k}{(2\pi)^4} \equiv \lambda_0 + \delta\lambda_0 \equiv \lambda. \tag{14.44}$$

This defines the coupling λ to next order.

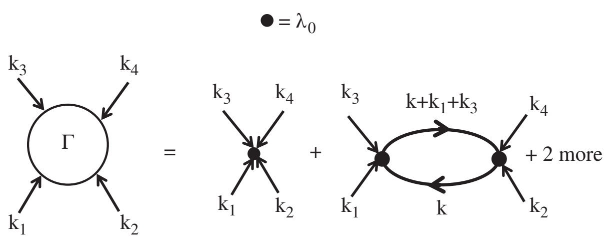

Figure 14.3 One-loop correction to Γ and $\lambda = \Gamma(0,0,0,0)$. Two more diagrams with external momenta connected to the vertices differently are not shown. They make the same contributions when external momenta vanish. The incoming arrows denote momenta and not propagators of that momentum (which have been amputated).

The factor of 3 comes from three loops with different routing of external momenta to the interaction vertices. Since all external momenta vanish, the graphs make identical contributions. *Unfortunately,* $\delta\lambda_0$ *is logarithmically divergent.*

14.5 Perturbative Renormalization

How do we reconcile these infinities in mass and coupling with the fact that actual masses and cross sections are finite? We employ the notion of *renormalization*.

First, we introduce a large momentum cut-off Λ in the loop integrals so that everything is finite but Λ-*dependent*:

$$\delta m_0^2(\Lambda) = \frac{\lambda_0}{2}\int_0^\Lambda \frac{1}{k^2+m_0^2}\frac{d^4k}{(2\pi)^4}, \tag{14.45}$$

$$\delta\lambda_0(\Lambda) = -3\lambda_0^2\int_0^\Lambda \frac{1}{(k^2+m_0^2)^2}\frac{d^4k}{(2\pi)^4}. \tag{14.46}$$

Then we identify the perturbatively corrected quantities with the measured ones. That is, we say

$$m^2 = m_0^2(\Lambda) + \delta m_0^2(\Lambda) \tag{14.47}$$

is the finite measured or *renormalized mass*, and that $m_0^2(\Lambda)$ is the *bare mass*, with an Λ-dependence chosen to ensure that m^2 equals the measured value. This means that we must choose

$$m_0^2(\Lambda) = m^2 - \frac{\lambda_0}{2}\int_0^\Lambda \frac{1}{k^2+m_0^2}\frac{d^4k}{(2\pi)^4} \tag{14.48}$$

$$= m^2 - \frac{\lambda}{2}\int_0^\Lambda \frac{1}{k^2+m^2}\frac{d^4k}{(2\pi)^4}, \tag{14.49}$$

where I have replaced the bare mass and coupling by the physical mass and coupling with errors of higher order.

Likewise, we must go back to Eq. (14.44) and choose

$$\lambda_0(\Lambda) = \lambda + 3\lambda_0^2\int_0^\Lambda \frac{1}{(k^2+m_0^2)^2}\frac{d^4k}{(2\pi)^4} \tag{14.50}$$

$$= \lambda + 3\lambda^2\int_0^\Lambda \frac{1}{(k^2+m^2)^2}\frac{d^4k}{(2\pi)^4}, \tag{14.51}$$

where I have replaced the bare mass squared by the physical mass squared and ł$_0^2$ by λ^2 with errors of higher order.

Equations (14.49) and 14.51 specify the requisite bare mass $m_0^2(\lambda, m, \Lambda)$ *and bare coupling* $\lambda_0(\lambda, m, \Lambda)$ *corresponding to the experimentally determined values of* λ *and* m

for any given Λ. If we choose the bare parameters as above, we will end up with physical mass and coupling that are finite and independent of Λ, *to this order*.

What about the scattering amplitude for non-zero external momenta? What about its divergences? We find that

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = \lambda_0 - \lambda_0^2\left[\int_0^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m_0^2)(|\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_3|^2+m_0^2)} + \text{ two more contributions}\right] \tag{14.52}$$

is logarithmically divergent as $\Lambda \to \infty$. Don't panic yet! We first replace m_0^2 by m^2 everywhere, due to the λ_0^2 in front of the integral. Next, we use Eq. (14.51) to replace the first λ_0 by

$$\lambda_0 = \lambda + 3\lambda^2 \int_0^\Lambda \frac{1}{(k^2+m^2)^2}\frac{d^4k}{(2\pi)^4}, \tag{14.53}$$

and the λ_0^2 in front of the integral by λ^2 (with errors of higher order), to arrive at

$$\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = \lambda + \lambda^2\left[\int_0^\Lambda \left[\frac{1}{(k^2+m^2)(|\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_3|^2+m^2)} - \frac{1}{(k^2+m^2)^2}\right]\frac{d^4k}{(2\pi)^4} + \text{ two more contributions}\right]. \tag{14.54}$$

I have divided the $3\lambda^2$ term in Eq. (14.53) into three equal parts and lumped them with the three integrals in large square brackets.

The integrals are now convergent because as $k \to \infty$, the integrand in the diagram shown goes as

$$\frac{(q^2+2\boldsymbol{k}\cdot\boldsymbol{q})k^3}{k^6}, \tag{14.55}$$

where $\boldsymbol{q} = \boldsymbol{k}_1 + \boldsymbol{k}_3$ is the external momentum flowing in. Because the $\boldsymbol{k}\cdot\boldsymbol{q}$ term does not contribute due to rotational invariance, the integrand has lost two powers of k due to renormalization. The other two diagrams are also finite for the same reason. In short, *once* $\Gamma(0,0,0,0)$ *is rendered finite, so is* $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$.

The moral of the story is that, *to one-loop order*, the quantities considered so far are free of divergences when written in terms of the renormalized mass and coupling.

14.6 Wavefunction Renormalization

However, at next order a new kind of trouble pops up that calls for more renormalization. I will describe this in terms of

$$\Gamma(k) = G^{-1}(k). \tag{14.56}$$

In free-field theory,

$$\Gamma(k) = k^2 + m_0^2 \tag{14.57}$$

and, to one-loop order (consult Figure 14.4(a) and (b)),

$$\Gamma(k) = k^2 + m_0^2 + \delta m_0^2, \tag{14.58}$$

where δm_0^2 is the one-loop contribution that we encountered in Eq. (14.20).

To next order in the loop expansion, we see two more diagrams shown in Figure 14.4(c) and (d). (Check that the two-loop diagram has one more power of $\hbar$ than the one-loop diagram.) Let us represent their (divergent) contributions as follows:

$$\Gamma(k) = k^2 + m_0^2 + \delta m_0^2 + \lambda_0^2 A(m_0,\Lambda) + \lambda_0^2 B(m_0,\Lambda,k). \tag{14.59}$$

The term $\lambda_0^2 A(m_0,\Lambda)$, being k-independent, makes a contribution to mass renormalization and we can deal with it as before.

By contrast, B depends on the external momentum k and is given, up to constants, by

$$B(m_0^2,\Lambda,k) = \int_0^\Lambda \frac{d^4k_1 d^4k_2}{(k_1^2+m_0^2)(k_2^2+m_0^2)(|k_1+k_2+k|^2+m_0^2)}. \tag{14.60}$$

Consider the expansion of B in a series in k^2. The zeroth-order term $\lambda_0^2 B(m_0^2,\Lambda,0)$ also contributes to mass renormalization.

The next term, proportional to k^2, modifies the k^2 term from free-field theory:

$$k^2 \to k^2 + \lambda_0^2 \frac{dB(m_0,\Lambda,k^2)}{dk^2}\bigg|_0 k^2 \equiv k^2\left(1 + c\lambda_0^2 \ln\frac{\Lambda^2}{m_0^2}\right) \equiv k^2 Z^{-1}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right), \tag{14.61}$$

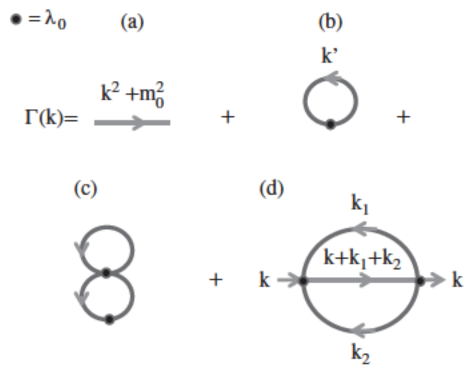

Figure 14.4 (a) $\Gamma(k)$ in free-field theory. (b) One-loop correction to m_0^2. (c) A k-independent two-loop correction $\lambda_0^2 A$ that renormalizes the mass. (d) A k-dependent two-loop correction $\lambda_0^2 B$ that renormalizes the k^2 part. The external legs have been amputated.

where c is some constant and I have introduced the *field renormalization factor*:

$$Z^{-\frac{1}{2}}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right) = \left(1 + c\lambda_0^2 \ln \frac{\Lambda^2}{m_0^2}\right)^{\frac{1}{2}}. \tag{14.62}$$

Because Z^{-1} diverges, the k^2 term in $\Gamma(k)$ now has a divergent coefficient.

Let us first handle this divergence and then interpret our actions. We begin with

$$\Gamma(k) = Z^{-1}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right) k^2 + k\text{-independent term } m_1^2 + \mathcal{O}(k^4). \tag{14.63}$$

Multiplying both sides by Z, we arrive at

$$Z\Gamma(k) = k^2 + Zm_1^2 + \mathcal{O}(k^4) \equiv k^2 + m^2 + \mathcal{O}(k^4), \tag{14.64}$$

where we have finally defined the quantity m^2 that is identified with the experimentally measured renormalized mass to this order.

The renormalized function

$$\Gamma_{\mathrm{R}} = Z\Gamma \tag{14.65}$$

now has a finite value and finite derivative at $k^2 = 0$:

$$\Gamma_{\mathrm{R}}(0) = m^2, \tag{14.66}$$

$$\left.\frac{d\Gamma_{\mathrm{R}}(k^2)}{dk^2}\right|_{k^2=0} = 1. \tag{14.67}$$

What does $\Gamma \to \Gamma_{\mathrm{R}}$ imply for G? Since $\Gamma = G^{-1}$, it follows that the *renormalized propagator*

$$G_{\mathrm{R}}(k) = Z^{-1}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right) G(k) \tag{14.68}$$

is divergence free. As Z is independent of momentum we may also assert that the Fourier transform to real space given by

$$G_{\mathrm{R}}(\boldsymbol{r}) = Z^{-1}\left(\lambda_0, \frac{\Lambda^2}{m_0^2}\right) G(\boldsymbol{r}) \tag{14.69}$$

is also divergence free. But

$$G(\boldsymbol{r}) = \langle \phi(\boldsymbol{r})\phi(0)\rangle, \tag{14.70}$$

which means that

$$G_{\mathrm{R}}(\boldsymbol{r}) = \langle Z^{-\frac{1}{2}}\phi(\boldsymbol{r})Z^{-\frac{1}{2}}\phi(0)\rangle \equiv \langle \phi_{\mathrm{R}}(\boldsymbol{r})\phi_{\mathrm{R}}(0)\rangle \tag{14.71}$$

is divergence free. Above, we have defined a *renormalized field*

$$\phi_{\mathrm{R}} = Z^{-\frac{1}{2}}\phi \tag{14.72}$$

in coordinate or momentum space, which has divergence-free correlations when everything is expressed in terms of renormalized mass and coupling (except for the unavoidable momentum-conservation δ-function in front of $G(k)$). One refers to Eq. (14.72) as *field renormalization*.

Several questions arise at this point:

- Since our original task was to compute correlations of ϕ, what good is it to have correlations of ϕ_{R}, even if the latter are finite?
- Renormalization looks like a Ponzi scheme, wherein we keep shoving problems to higher and higher orders. How many more new infinities will arise as we go to higher orders in λ_0 and k^2 and consider correlation functions of more than two fields? Will all the infinities be removed by simply renormalizing the mass, coupling, and field?

As to the first point, it turns out that the overall scale of ϕ does not affect any physical quantity: one will infer the same particle masses and physical scattering matrix elements before and after rescaling. This is not obvious, and I will not try to show that here.

As for the second set of points, it is the central claim of renormalization theory that no more quantities need to be renormalized (though the amount of renormalization will depend on the order of perturbation theory), and that the renormalized correlation function of rescaled fields

$$\phi_{\mathrm{R}} = Z^{-\frac{1}{2}}\phi, \tag{14.73}$$

expressed in terms of the renormalized mass and coupling,

$$G_{\mathrm{R}}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,m,\lambda) = Z^{-M/2}G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,m_0,\lambda_0,\Lambda), \tag{14.74}$$

are finite and independent of Λ as $\Lambda \to \infty$.

(New divergences arise if the spatial arguments of any two or more ϕ's in G_{R} coincide to form the operators like ϕ^2. We will not discuss that here.)

The proof of renormalizability is very complicated. To anyone who has done the calculations, it is awesome to behold the cancellation of infinities in higher-loop diagrams as we rewrite everything in terms of quantities renormalized at lower orders. It seems miraculous and mysterious.

While all this is true for the theory we just discussed, ϕ^4 interaction in $d = 4$, referred to as ϕ^4_4, there are also *non-renormalizable theories*. For example, if we add a ϕ^6 interaction in $d = 4$, the infinities that arise cannot be fixed by renormalizing any finite number of parameters. Here it should be borne in mind that in quantum field theory one adds this term with a coefficient, $\lambda_6 = w_6/\mu^2$, where μ is some fixed mass (say 1 GeV) introduced to define a dimensionless w_6. In the post-Wilson era one adds the ϕ^6 term with coupling $\lambda_6 = w_6/\Lambda^2$, which is more natural. Its impact is benign and will be explained later.

What is the diagnostic for renormalizability? The answer is that *any interaction that requires a coupling constant with inverse dimensions of mass is non-renormalizable*. The couplings of ϕ^2 and ϕ^4 have dimensions m^2 and m^0, while a ϕ^6 coupling would have dimension m^{-2} in $d = 4$. These dimensions are established (in units of $\hbar = 1 = c$) by demanding that the kinetic term $\int (\nabla\phi)^2 d^d x$ be dimensionless and using that to fix the dimension of ϕ as

$$[\phi(x)] = \left(\frac{d}{2} - 1\right). \tag{14.75}$$

I invite you to show that

$$[\lambda] = 4 - d, \tag{14.76}$$

which means that λ is marginal in $d = 4$ and renormalizable in $d < 4$. Likewise, try showing that λ_6, the coupling for the ϕ^6 interaction, has dimension

$$[\lambda_6] = 6 - 2d, \tag{14.77}$$

which makes it non-renormalizable in $d = 4$ but renormalizable for $d \leq 3$.

You must have noticed the trend: *The renormalizable couplings are the ones which are relevant or marginal at the Gaussian fixed point.*

That the Gaussian fixed point plays a central role is to be expected in all old treatments of QFT because they were based on perturbation theory about the free-field theory. These topics are treated nicely in many places; a sample [1–6] is given at the end of this chapter. The relation between relevance and renormalizability can be readily understood in Wilson's approach to renormalization, which I will now describe. His approach gives a very transparent non-perturbative explanation of the "miracle" of canceling infinities in renormalizable theories.

14.7 Wilson's Approach to Renormalizing QFT

Compared to the diagrammatic and perturbative proof of renormalization in QFT, Wilson's approach [7, 8] is simplicity itself.

Recall our goal: to define a QFT in the continuum with the following properties:

- All quantities of physical significance – correlation functions, masses, scattering amplitudes, and so on – must be finite.
- There should be no reference in the final theory to a lattice spacing a or an ultraviolet momentum cut-off Λ.

Of course, at intermediate stages a cut-off will be needed and the continuum theory will be defined as the $\Lambda \to \infty$ limit of such cut-off theories.

Wilson's approach is structured around a fixed point of the RG. Every relevant direction will yield an independent parameter.

It is assumed that we know the eigenvectors and eigenvalues of the flow near this fixed point.

Even if we cannot find such fixed points explicitly, the RG provides a *framework* for understanding renormalizability, just as it provides a framework for understanding critical phenomena and demystifying universality in terms of flows, fixed points, scaling operators, and so on, even without explicit knowledge of these quantities.

Consider a scalar field theory. By assumption, we are given complete knowledge of a fixed point action S^* that lives in some infinite-dimensional space of *dimensionless couplings* such as r_0, u_0, and so forth. The values of these couplings are what we previously referred to as $\boldsymbol{K}^*$. Let the fixed point have one relevant direction, labeled by a coordinate t. As t increases from 0, the representative point moves from S^* to $S^* + tS_{\mathrm{rel}}$, where S_{rel} is the relevant perturbation, a particular combination of ϕ^2,ϕ^4, and so on. Once we go a finite distance from S^* the flow may not be along the direction of the relevant eigenvector at S^*, but along its continuation, a curve called the *renormalized trajectory* (RT).

Let us say that our goal is to describe physics in the 1 GeV scale using a continuum theory. (In terms of length, 1 GeV corresponds to roughly 1 fermi, a natural unit for nuclear physics. More precisely, $1\,\mathrm{GeV}\cdot 1\mathrm{fermi} \simeq 5 \simeq 1$ in units $\hbar = c = 1$.) Although we limit our interest to momenta within the cut-off of 1 GeV, we want the correlations to be exactly those of an underlying theory with a cut-off that approaches infinity, a theory that knows all about the extreme-short-distance physics. The information from very short distances is not discarded, but encoded in the renormalized couplings that flow under the RG.

Notice the change in language: we are speaking of a very large cut-off Λ. We are therefore using laboratory units in contrast to the Wilsonian language in which the cut-off is always unity. (For example, when we performed decimation, the new lattice size a served as the unit of length in terms of which the dimensionless correlation length ξ was measured.)

To make contact with QFT, we too will carry out the following discussion *in fixed laboratory units*. In these units the allowed momenta will be reduced from a huge sphere of radius Λ GeV to smaller and smaller spheres of radius Λ/s GeV. The surviving momenta will range over smaller and smaller values, and they will be a small subset of the original set $k < \Lambda$.

We have had this discussion about laboratory versus running units before in discussing the continuum limit of a free-field theory. If we want the continuum correlation to fall by $1/e$ over a distance of 1 fermi, we fix the two points a fermi apart in the continuum and overlay lattices of smaller and smaller sizes a. As $a \to 0$, the number of lattice sites within this 1 fermi separation keeps growing and the dimensionless correlation length has to keep growing at the same rate to keep the decay to $1/e$.

So, we are not going to rescale momenta as modes are eliminated. How about the field? In the Wilson approach the field gets rescaled even in free-field theory because k gets rescaled to $k' = sk$. We will not do that anymore. However, we will rescale by the factor Z introduced in connection with the renormalized quantities Γ_{R} and G_{R}. This Z was needed in perturbation theory to avert a blow-up of the k^2 term in Γ due to the loop correction. [Recall the appearance of Z in the two-loop diagram, Eq. (14.61)]. In the Wilsonian RG there will also be a correction to the k^2 term from loop diagrams (now integrated over the eliminated modes), and these will modify the coefficient of the k^2 term. We will bring in a Z to keep the coefficient of k^2 fixed at 1. The reason is not to cancel divergences, for there are none, but because *the strength of the interaction is measured relative to the free-field term*. For example, in a ϕ^4 theory if we rescale $\phi(x)$ by 5 this will boost the coefficients ϕ^2 and $(\nabla\phi)^2$ by 25 and that of the quartic term by 625. But it is still the same theory. For this reason, to compare apples to apples, one always rescales the k^2 coefficient to unity, *even if there are no infinities*.

Let us now begin the quest for the continuum theory.

Say we want a physical mass of 1 GeV or a correlation length of 1 fermi. First we pick a point t_0 on the RT where the dimensionless correlation length $\xi_0 = 2^0 = 1$, as indicated in Figure 14.5. We refer to the action at t_0 as $S(0)$.

No cut-off or lattice size has been associated with the point t_0, since everything is dimensionless in Wilson's approach. All momenta are measured in units of the cut-off, and the cut-off is unity at every stage in the RG. We now bring in *laboratory units* and assign to t_0 a momentum cut-off of $\Lambda_0 = 2^0 = 1$ GeV.

What is the mass corresponding to this ξ_0 in GeV? For this, we need to recall the connection between ξ and m:

$$G(r) \simeq e^{-mr} = \exp\left[-\frac{r}{a\xi}\right] = \exp\left[-\frac{r\Lambda}{\xi}\right], \tag{14.78}$$

which means that the mass is related to the cut-off and ξ as follows:

$$m = \frac{\Lambda}{\xi}. \tag{14.79}$$

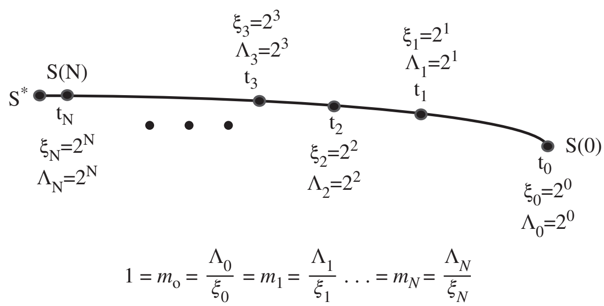

Figure 14.5 Points on the renormalized trajectory emanating from the fixed point S^*. To end up at the theory with cut-off $\Lambda_0 = 1$ GeV and action $S(0)$ after N RG steps of factor of 2 reduction of Λ, we must begin with the point labeled N, cut-off $\Lambda_N = 2^N$ GeV, $\xi_N = 2^N$ (dimensionless), and action $S(N)$. The sequence of points $S(N)$, $N \to \infty$ defines the continuum limit.

Thus, the mass corresponding to $S(0)$ is

$$m_0 = \frac{\Lambda_0}{\xi_0} = \frac{1\,\mathrm{GeV}}{1} = 1\,\mathrm{GeV}. \tag{14.80}$$

Imagine that we got to the point t_0 by performing N RG steps of size 2, starting with the point t_N where $\xi_N = 2^N$ and $\Lambda_N = 2^N$ GeV. *At every stage, the dimensionful mass is* 1 *GeV:*

$$m_N = \frac{\Lambda_N}{\xi_N} = 1. \tag{14.81}$$

Thus we have a sequence of actions, $S(n) : n = 0, 1, \ldots, N$, defined on smaller and smaller length scales or larger and larger momentum cut-offs, which produce the requisite physical mass. Not only is the mass fixed, the complete interaction is fixed to be $S(0)$. We have reverse-engineered it so that the theory at 1 GeV stays fixed at $S(0)$ while the underlying theory is defined on a sequence of actions $S(N)$ for which $\xi_N = 2^N$, and cut-off 2^N GeV, with $N \to \infty$. *We can make N as large as we like because ξ diverges as we approach S^*.*

We have managed to renormalize the theory by providing for each cut-off 2^N an action $S(t_N) \equiv S(N)$ that yields the theory $S(0)$ at low energies. This is the *continuum limit*.

This discussion also makes it obvious how to obtain a theory with a cut-off of 2 GeV: we just stop the RG one step earlier, at $S(1)$.

We can be more explicit about the continuum limit by invoking our presumed knowledge of ν. Near the fixed point we know that

$$\xi = t^{-\nu}. \tag{14.82}$$

This means that

$$2^N = t_N^{-\nu} \tag{14.83}$$

$$t_N = 2^{-N/\nu}, \tag{14.84}$$

which specifies the bare coupling or action $S(t_N) \equiv S(N)$ as a function of the cut-off $\Lambda_N = 2^N$ and the critical exponent ν. Just to be explicit: the bare action for cut-off $\Lambda = 2^N$ GeV is $S = S^* + 2^{-N/\nu} S_{\mathrm{rel}}$, where S_{rel} is the relevant eigenoperator (some linear combination of ϕ^2, ϕ^4, etc.) that moves us along the RT starting at S^*.

We have managed to send the cut-off of the underlying theory to 2^N GeV with $N \to \infty$ holding fixed the action $S(0)$ for a theory with a cut-off of 1 GeV, but we need more. We need to ensure that not only does the low-energy action have a limit $S(0)$, as $\Lambda_N \to \infty$, but so do all the M-point correlation functions $G(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_M)$ defined by

$$\langle \phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\cdots\phi(\boldsymbol{k}_M)\rangle = (2\pi)^d \delta\left(\sum_i \boldsymbol{k}_i\right) G(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_M). \tag{14.85}$$

Since we measure momentum in fixed laboratory units, the surviving momenta and fields $\phi(k)$ in the $\Lambda_0 = 1$ GeV theory are a subset of the momenta and fields in the underlying $\Lambda_N = 2^N$ GeV theory.

This may suggest that

$$G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(N)) = G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(0)). \tag{14.86}$$

However, Eq. (14.86) is incorrect. The reason is that the fields that appear in $S(0)$ are different from the ones we began with in $S(N)$, because we rescale the field to keep the coefficient of the k^2 term fixed in the presence of higher-loop corrections.

So, at every RG step we define a renormalized ϕ_{R} as follows:

$$\phi_{\mathrm{R}}(\boldsymbol{k}) = Z^{-\frac{1}{2}}\phi(\boldsymbol{k}), \tag{14.87}$$

and write S in terms of that field. If there are N steps in the RG the same equation would hold, with Z being the product of the Z's from each step. *So, the fields entering* $S(0)$ *are rescaled versions of the original fields entering* $S(N)$.

This means that, for the M-point correlation,

$$G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(N)) = Z(N)^{\frac{M}{2}}G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(0)), \tag{14.88}$$

where $Z(N)$ is the net renormalization factor after N RG steps starting with cut-off 2^N.

Look at the $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(N))$ on the left-hand side. This is the correlation function of a theory with a growing cut-off. The coupling is chosen as a function of cut-off that grows like 2^N. If G is finite as $N\to\infty$, we have successfully renormalized. The equation above expresses G as the product of two factors. The second factor is a correlation function evaluated in a theory with action $S(0)$ which remains fixed as $N\to\infty$ by construction. It has a finite non-zero mass and a finite cut-off, and is thus free of ultraviolet and infrared divergences. So we are good there. But, this need not be true of the Z-factor in front, because it is the result of (product over) Z's from N steps, with $N\to\infty$. Let us take the Z factor to the left-hand side:

$$Z(N)^{-\frac{M}{2}}G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(N)) = G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(0)). \tag{14.89}$$

The left-hand side is now finite as $N\to\infty$, namely $G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(0))$. In other words, *the correlation functions of the renormalized fields are finite and cut-off independent as the cut-off approaches* ∞. This is the continuum limit.

In this approach it is obvious how, by choosing just one coupling (the initial value t_N of the distance from the fixed point along the RT) as a function of the cut-off ($\Lambda = 2^N$), we have an expression for finite correlation functions computed in terms of the finite renormalized interaction $S(0)$. Renormalizability is not a miracle if we start with an RG fixed point with a relevant coupling (or couplings) and proceed as above.

14.7.1 Possible Concerns

You may have some objections or concerns at this point.

What about $t<0$? Is there not a flow to the left of S^*? There is, and it defines another continuum theory. In the magnetic case the two sides would correspond to the ordered and disordered phases. However, the rest of the discussion would be similar. (There are some cases, like Yang–Mills theory, where the fixed point is at the origin and the region of negative coupling is unphysical [9, 10].)

You may object that we have found a smooth limit for the correlation of the renormalized fields, whereas our goal was to find the correlations of the original fields. Have we not found a nice answer to the wrong question? No. As mentioned earlier (without proof), the physical results of a field theory – masses, scattering amplitudes, and so on – are unaffected by such a k- and x-independent rescaling of the fields. So what we have provided in the end are finite answers to all physical questions pertaining to the low-energy physics in the continuum.

Another very reasonable objection is that the preceding diagram and discussion hide one important complexity. Even though the flow along the RT is one-dimensional, it takes place in an infinite-dimensional space of all possible couplings. As we approach the fixed point S^* along the RT, we have to choose the couplings of an infinite number of terms like the ϕ^2, ϕ^4, ϕ^6, $\phi^2(\nabla\phi)^2$, and so on of the short-distance interaction. This seems impractical. It also seems to have nothing to do with standard renormalization, where we vary one or two couplings to banish cut-off dependence.

14.7.2 Renormalization with Only Relevant and Marginal Couplings

We resolve this by bringing in the irrelevant directions and seeing what they do to the preceding analysis. Look at Figure 14.6.

Besides the RT, I show one irrelevant trajectory that flows into the fixed point. This is a stand-in for the *entire multidimensional critical surface*, which includes every critical system of this class. Somewhere in the big $\boldsymbol{K}$ space is an axis describing a simple coupling, which I call r_0. It could be the nearest-neighbor coupling K of an Ising model or some combination of the elementary couplings $r_0\phi^2$ and $u_0\phi^4$ of a scalar field theory which can be varied to attain criticality. We will see how to define the continuum limit by taking a sequence of points on the r_0 axis.

Though the interaction is simple, we can hit criticality by varying its strength. The critical point, where the r_0 axis meets the critical surface, is indicated by r^*.

Now, r^* is a *critical* point while S^* is a *fixed point*. The two differ by irrelevant terms. This means that the correlation functions at r^* will not have the scaling forms of S^* in general. To see the ultimate scaling forms associated with the fixed point S^*, we do not have to renormalize: if we evaluate the correlation functions at r^* in the limit $k \to 0$ or $r \to \infty$, they will exhibit these laws. For example, at the Ising critical point, $G(k) \simeq 1/k^{2-\eta}$ will result as $k \to 0$, or $G(r) \simeq 1/r^{\frac{1}{4}}$ will follow as $r \to \infty$, despite being formulated on a lattice with just the symmetry of a square.

Of course, we can understand this in terms of the RG. If we limit ourselves to $k \to 0$, we are permitted to trade our initial theory with a large Λ for one with $\Lambda \simeq k$, which is related by RG flow to S^*.

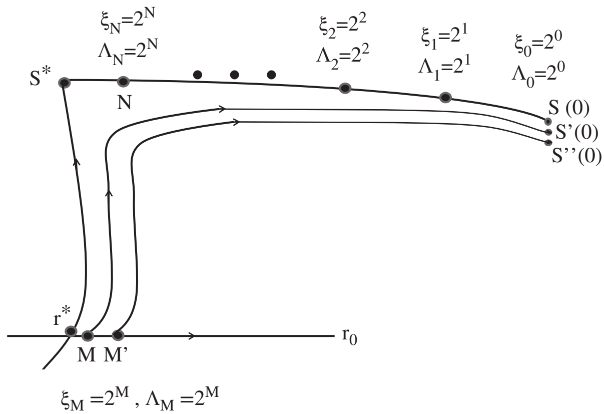

Figure 14.6 Flow with one relevant direction (the RT) and one irrelevant direction, which is a stand-in for the entire critical surface. The axis labeling the simple coupling r_0 (which could stand for $r_0\phi^2$) cuts the critical surface at r^*. Look at the points on the trajectory emanating from the point M on the r_0 axis. At point M, $\Lambda_M = 2^M$ and $\xi_M = 2^M$. We will end up at the theory with cut-off $\Lambda_0 = 1$ GeV and action $S'(0)$ after M RG steps of factor of 2 reduction of Λ. The sequence of points $S(M)$, $M \to \infty$ defines the continuum limit defined using just a single simple relevant coupling like r_0. If we start at M' we will reach $S''(0)$ (equivalent in the infrared to $S(0)$ and $S'(0)$) after $M-1$ steps. This is how one renormalizes in quantum field theory, by choosing simple couplings as a function of cut-off. The coupling M' corresponds to $\Lambda = 2^{M-1}$.

To define the continuum theory starting on this axis corresponding to a simple coupling, we pick a point M such that after M RG steps (of powers of 2) we arrive at the point $S'(0)$ that differs from $S(0)$, the theory generated from S^*, by a tiny amount *in the irrelevant direction*. The tiny irrelevant component will vanish asymptotically, and even when it is non-zero will make negligible corrections in the infrared. This result is inevitable given the irrelevance of the difference between r^* and S^*. We can go to the continuum limit by starting closer and closer to the critical surface (raising M) and reaching the target $S'(0)$ after more and more steps. As $M \to \infty$, our destination $S'(0)$ will coalesce with $S(0)$, which lies on the RT.

As a concrete example, consider Figure 13.4. Look at the dotted line parallel to the r_0 axis that comes straight down and crosses the critical line joining the Gaussian and WF fixed points. By starting closer and closer to the critical point where the dotted line crosses the critical line, we can renormalize the continuum theory based on the WF fixed point. The flow will initially flow toward the WF fixed point, and eventually will run alongside the RT. We can arrange to reach a fixed destination on the RT (the analog of $S(0)$) by starting at the appropriate distance from the critical line. You can also vary u_0 at fixed (negative) r_0 to approach the critical line with the same effect.

Now we can see the answer to a common question: how does a field theorist manage to compensate for a change in cut-off by renormalizing (i.e., varying with Λ) one or two couplings, whereas in Wilson's scheme, it takes a change in an infinite number of couplings? In other words, when we flow along the RT, i.e., vary one parameter t, we are actually varying an infinite number of elementary couplings in $\boldsymbol{K}$-space. How can a field theorist achieve the same result varying one or two couplings? The answer is that the field theorist does not really compensate for all the changes a changing cut-off produces. This is simply impossible. Whereas in Wilson's approach all correlation functions right up to the cut-off are preserved under the RG, in the field theory, *only correlations in the limit* $k/\Lambda \to 0$ *are preserved.*

Let us dig a little deeper into this. Suppose we begin at the point M, where $\Lambda = 2^M$, and reach the point $S'(0)$ in the figure after M RG steps of size 2. Say we ask what bare coupling with a cut-off 2^{M-1} will reproduce the answers of M with $\Lambda = 2^M$. It does not exist in general. Suppose, however, that we ask only about correlations in the infrared limit, $k/\Lambda \to 0$. Now we may trade the initial couplings for those on the RG trajectory. The point M flows to $S'(0)$ after M steps, i.e., when $\Lambda = 1$. The difference between $S(0)$ and the $S'(0)$ are technically and literally irrelevant in the infrared limit. If we start on the r_0 axis at M', at a suitably chosen point a little to the right of M, we can, after $M-1$ steps, reach the point $S''(0)$ that agrees with $S(0)$ and $S'(0)$ up to irrelevant corrections. It follows that if we reduce the cut-off by 2 we must change M to M', and if we increase the cut-off by 2 we must change M' to M. In other words, for each cut-off 2^M there is a point on the r_0 axis that has the same long-distance physics as the point M does with $\Lambda = 2^M$. This is how one renormalizes in QFT.

In QFT, one does not apologize for considering only the limit $k/\Lambda \to 0$ because there, Λ is an artifact that must be sent to ∞ at the end. So, $k/\Lambda \to 0 \; \forall \; k$.

Suppose I add a tiny irrelevant coupling, say $w_6\phi^6$, to the simple interaction of the starting point M. (Imagine the point is shifted slightly out of the page by w_6.) After M steps, the representative point again has to end up close to the RT. It may now end up slightly to the left or right of $S'(0)$ (ignore the component outside the page, which must have shrunk under the RG). Say it is to the right. This is what would have happened had we started with no w_6 but with a slightly bigger r_0 (a little to the right of M). A similar thing is true if the end point with w_6 in the mix is to the left of S'_0. In either case, *the effect of an irrelevant perturbation is equivalent to a different choice of the initial relevant coupling.*

It is understood above that w_6 is finite in units of the cut-off, and hence is very small in laboratory units, scaling as Λ^{-2} in $d = 4$. Had it been of order μ^{-2}, where μ is some fixed mass, it would not have been possible to absorb its effects by renormalization because it could correspond to an infinite perturbation in the natural units, namely Λ. But this is what field theorist tend to do in declaring it a non-renormalizable theory.

14.8 Theory with Two Parameters

Consider next the Gaussian fixed point in $d < 4$ when it has *two* relevant directions. Look at the flow in Figure 14.7. A generic point near the fixed point (the origin) will run away

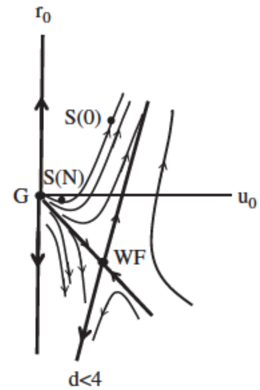

Figure 14.7 The situation in $d < 4$ when the Gaussian fixed point has two relevant directions. One can arrange to end up with a continuum theory with action $S(0)$, containing *two* free parameters, by starting closer and closer to the origin on the RT that passes through the point $S(0)$. Two bare couplings will have to be tuned, based on two relevant exponents that describe their growth under RG.

along the curves shown. We can make any point on any of those flow lines our destination $S(0)$ describing the continuum theory with 1 GeV cut-off, and arrange to get there after N RG steps by a suitable choice of initial coordinates $S(N)$ close to the origin. The flow away from the fixed point will now be controlled by two eigenvalues. We get a two-parameter family of continuum theories here. (The discussion near the fixed point is in terms of the simple interactions $r_0\phi^2$ and $u_0\phi^4$ because $\boldsymbol{K}^*$ is at the origin.)

In the relevant space of the Gaussian fixed point, there is a line connecting it to the Wilson–Fisher fixed point WF. If you pick a generic point on that line you will flow to WF and end up with its exponents. (In laboratory units, such a starting point will have a very large dimensionful coupling $\lambda = \Lambda^{4-d}u_0$.) But there is a way to fight that flow to WF: start closer and closer to G in such a way that after N steps you reach a fixed destination on the line. That would be a continuum theory that is massless but has one free parameter. (This is not a *natural* theory because the bare coupling is unnaturally small, being of order μ^{4-d}, where μ is some fixed mass, rather than of order Λ^{4-d}.)

14.8.1 Triviality of ϕ_4^4

Finally, consider the ϕ_4^4 theory based on the Gaussian fixed point that has one relevant coupling (mass or r_0) and interaction u_0 which is marginal at tree level but flows logarithmically slowly to the Gaussian fixed point at one loop. For this reason, this is not a suitable fixed point for constructing an interacting theory in the continuum. But suppose we try anyway. Since we can always make a theory massive, let us focus on getting an interacting field theory. So we begin with a point on the marginally attractive direction depicted in Figure 14.8.

Figure 14.8 The flow of coupling in ϕ_4^4. The origin is marginally attractive. To end up at some $u(1)$ in a theory with, say, a 1 GeV cut-off, we need to begin at *larger* bare values $u_0(\Lambda)$, which in fact diverge as $\Lambda \to \infty$. It has been shown numerically by Wilson that any $u(\Lambda)$, including $u(\Lambda) = \infty$, flows to the origin, rendering the continuum theory trivial. The only way to define an interacting ϕ_4^4 is to construct one based on a strong coupling fixed point u^*. If u^* is the bare coupling, it will not move under the RG and define an interacting massless theory. We can also arrange to end up with u at a fixed distance to the left or right of u^* by starting out closer and closer to it in a way determined by the relevant eigenvalue at u^*.

We can parametrize this point by u_0 and assume r_0 is adjusted to put us on the critical line. Say we want a final coupling $u(1)$ in a 1 GeV cut-off theory. Let this target coupling be in the weak coupling regime where we have established marginal irrelevance. To get to $u(1)$ in the long-distance theory, we need to begin with a *larger* bare value $u(\Lambda)$ because the coupling is irrelevant. In perturbation theory, the desired bare value grows without limit as $\Lambda \to \infty$ (see Eq. (14.51)), rendering perturbation theory meaningless. The only legitimate way to construct an interacting ϕ_4^4 theory is to base it on a fixed point u^*, if we can find one. If we begin there at the bare level, we will stay there under the RG and define a massless interacting theory. We could also begin slightly to its left so as to end at our target value $u(1)$ after N steps, starting closer and closer to u^* as $N \to \infty$. This would be an interacting theory. (We can also get a theory with a fixed u to the right of u^*.) So now we are back to relevant flow coming out of the strong coupling fixed point u^*. However, Wilson has verified by thorough numerical analysis that such a fixed point does not exist anywhere on the u axis, including the point at infinity. The general consensus now is that ϕ_4^4 is trivial, i.e., non-interacting.

14.9 The Callan–Symanzik Equation

I will now provide a very brief introduction to this equation due to Callan [11] and Symanzik [12]. It is used extensively in quantum field theory as well as critical phenomena. It is mostly used to study the behavior of correlation functions in some extreme kinematical region: large momenta to describe asymptotic freedom in QCD [9, 10] or small momenta to describe critical phenomena. In the latter case it is the only practical way to deal with higher orders in the ε expansion.

14.9.1 Basis for the Callan–Symanzik Equation

Recall that in the Wilson approach, by construction, a theory with a cut-off Λ and couplings $\boldsymbol{K}(1)$ is equivalent to a theory with a cut-off Λ/s and couplings $\boldsymbol{K}(s)$ as long we ask questions below the new cut-off Λ/s. (We use laboratory units in which the cut-off shrinks by a factor s and momenta are not rescaled.) Correlation functions are, however, not invariant under this change of cut-off due to the change in the scale of the field to keep the k^2 term fixed after every iteration. The original ϕ we started with is related to the ϕ_R that appears in the theory with the new cut-off as

$$\phi(\boldsymbol{k}) = Z^{\frac{1}{2}}\phi_R(\boldsymbol{k}). \tag{14.90}$$

Consequently,

$$Z(N)^{-\frac{M}{2}} G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(N)) = G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,S(0)), \tag{14.91}$$

where the action $S(0)$ and the corresponding coupling $\boldsymbol{K}(0)$ are reached after N RG steps of cut-off reduction by 2.

The Callan–Symanzik equation is derived in quantum field theory from a similar relation which, however, holds only in the limit $\Lambda \to \infty$, or more precisely $k/\Lambda \to 0$, where k is any fixed momentum. The reason for the restriction is that a cut-off change can be compensated by changing a handful of (marginal and relevant) couplings only in this limit, in which irrelevant corrections vanish as positive powers of k/Λ. The Callan–Symanzik equation is not limited to the study of correlation functions as $\Lambda \to \infty$ in QFT. We can also use it in critical phenomena where Λ is some finite number $\Lambda \simeq 1/a$, provided we want to study the limit $k/\Lambda \to 0$, i.e., at distances far greater than the lattice size a. All that is required in both cases is that $k/\Lambda \to 0$.

We begin with the central claim of renormalization theory that the correlations of

$$\phi_R = Z^{-\frac{1}{2}}\phi, \tag{14.92}$$

expressed in terms of the renormalized mass and coupling,

$$G_R(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,m,\lambda) = \lim_{\Lambda\to\infty} Z^{-M/2}(\lambda_0,\Lambda/m_0)G(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M,m_0(\Lambda),\lambda_0(\Lambda),\Lambda), \tag{14.93}$$

are finite and independent of Λ.

For a theory with a mass m we have seen that the renormalized *inverse* propagator Γ and four-point amplitude $\Gamma_R(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4)$ can be made to obey

$$\Gamma_R(0) = m^2, \tag{14.94}$$

$$\left.\frac{d\Gamma_R(k)}{dk^2}\right|_{k=0} = 1, \tag{14.95}$$

$$\Gamma_R(0,0,0,0) = \lambda. \tag{14.96}$$

We are going to study a critical (massless) theory in what follows. Although we can impose

$$\Gamma_R(0) = 0 \tag{14.97}$$

to reflect zero mass, we cannot impose Eqs. (14.95) and (14.96). This is because in a massless theory both these quantities have infrared divergences at $k=0$. These are physical, just like the diverging Coulomb cross section. So we pick some point $k=\mu>0$ where these quantities can be finite, and demand that

$$\left.\frac{d\Gamma_{\mathrm{R}}(k)}{dk^2}\right|_{k=\mu}=1, \tag{14.98}$$

$$\Gamma_{\mathrm{R}}(\mu,\mu,\mu,\mu)=\lambda=\mu^{\varepsilon}u_{\mathrm{R}}. \tag{14.99}$$

This calls for some explanation.

First, μ is arbitrary, and any choice of μ can be used to specify a theory. If you change μ you will have to change λ accordingly if you want to describe the same theory.

Next, we are working in $d=4-\varepsilon$ dimensions, where λ has dimension ε. It is expressed as the product of a dimensionless parameter u_{R} and the factor μ^{ε}, which restores the right engineering dimension.

Finally, $\Gamma(\mu,\mu,\mu,\mu)$ is a schematic: it stands for a symmetric way to choose the momenta all of the scale μ:

$$\boldsymbol{k}_i\cdot\boldsymbol{k}_j=\frac{\mu^2}{3}(4\delta_{ij}-1). \tag{14.100}$$

We will not need this expression from now on.

It is to be noted that the theory is not renormalizable in $d=4-\varepsilon$ due to the power-law infrared divergences that arise. However, if we expand everything in a double series in u and ε, the infinities (which will be logarithmic) can be tamed order by order, i.e., renormalized away. This double expansion will be understood from now on.

14.9.2 Massless $M=2$ Correlations in $d=4-\varepsilon$

I will illustrate the Callan–Symanzik approach with the case $M=2$, that is, two-point correlations, and study just the critical (massless) case in $d=4-\varepsilon$. Consider the system at point P in Figure 14.9 lying on the critical line joining the Gaussian and WF fixed points. It has a cut-off Λ and a coordinate $u(\Lambda)\equiv u$. We are interested in $\Gamma(k,u,\Lambda)$ in the limit $k/\Lambda\to 0$. We cannot use simple perturbation theory, even if u is small, because the expansion parameter will turn out to be $u\ln\frac{\Lambda}{k}$. The trick is to move the cut-off to a value of the order of k, thereby avoiding large logarithms, and work with the coupling $u(k)$ rather than $u=u(\Lambda)$. It is during this cut-off reduction that the coupling will flow from $u(\Lambda)$ to $u(k)$. We expect that $u(k)\to u^*$, the WF fixed point, as $k\to 0$.

It is convenient to work with the *inverse* propagator $\Gamma=G^{-1}$, which obeys

$$\Gamma_{\mathrm{R}}(k,u_{\mathrm{R}},\mu)=\lim_{\Lambda\to\infty}\Big[Z^1(u(\Lambda),\Lambda/\mu)\Gamma(k,u(\Lambda),\Lambda)\Big]. \tag{14.101}$$

The key to the Callan–Symanzik equation approach is the observation that since the left-hand side is independent of Λ (in the limit $\Lambda\to\infty$), so must be the right-hand side,

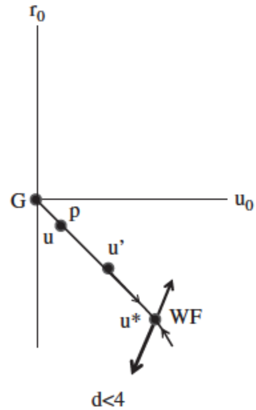

Figure 14.9 We want $\Gamma(k,u(\Lambda),\Lambda)$ as $k \to 0$ at a point P with coupling $u(\Lambda) = u$. The RG flow takes us to the WF fixed point u^* via the point u'.

which means

$$\lim_{\Lambda\to\infty} \Lambda \frac{d}{d\Lambda}\left[Z(u(\Lambda),\Lambda/\mu)\Gamma(k,u(\Lambda),\Lambda)\right] = 0. \tag{14.102}$$

Writing out the explicit and implicit Λ derivatives, we find that

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(u,\Lambda/\mu)\frac{\partial}{\partial u} - \gamma(u,\Lambda/\mu)\right]\Gamma(k,u(\Lambda),\Lambda) = 0, \tag{14.103}$$

where

$$\beta(u,\Lambda/\mu) = \Lambda \frac{\partial u(\Lambda)}{\partial\Lambda}\bigg|_{\mu,u_R}, \tag{14.104}$$

$$\gamma(u,\Lambda/\mu) = -\Lambda \frac{\partial \ln Z(u(\Lambda),\Lambda/\mu)}{\partial\Lambda}\bigg|_{\mu,u_R}. \tag{14.105}$$

Next, we argue that since μ does not enter Γ, it cannot enter the dimensionless functions γ or β, which must therefore be functions only of $u(\Lambda)$. Thus we arrive at the *Callan–Symanzik equation*:

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(u)\frac{\partial}{\partial u} - \gamma(u)\right]\Gamma(k,u(\Lambda),\Lambda) = 0. \tag{14.106}$$

The solution, derived by the method of characteristics, is

$$\Gamma(k,u(\Lambda_1),\Lambda_1) = \exp\left[\int_{\ln\Lambda_2}^{\ln\Lambda_1} \gamma(u(\ln\Lambda))d\ln\Lambda\right]\Gamma(k,u(\Lambda_2),\Lambda_2). \tag{14.107}$$

The solution is readily understood in Wilson's picture. The correlation function with cut-off Λ_2 is the same as that with Λ_1, provided we use the renormalized coupling in going from Λ_1 to Λ_2 and account for the field rescaling factor Z. Imagine doing the mode elimination in stages. Each stage will contribute a factor to Z, and the final Z will be a product of the Z's in each step depending on the coupling u at that stage. We reason as follows:

$$\Gamma(\Lambda_1)Z(\Lambda_1) = \Gamma(\Lambda_2)Z(\Lambda_2) = \Gamma_{\mathrm{R}} \tag{14.108}$$

$$\Gamma(\Lambda_1) = \frac{Z(\Lambda_2)}{Z(\Lambda_1)}\Gamma(\Lambda_2) \tag{14.109}$$

$$= e^{(\ln Z(\Lambda_2) - \ln Z(\Lambda_1))}\Gamma(\Lambda_2) \tag{14.110}$$

$$= \exp\left[\int_{\ln\Lambda_1}^{\ln\Lambda_2} \frac{d\ln Z}{d\ln\Lambda} d\ln\Lambda\right]\Gamma(\Lambda_2) \tag{14.111}$$

$$= \exp\left[\int_{\ln\Lambda_2}^{\ln\Lambda_1} \gamma(u(\ln\Lambda)) d\ln\Lambda\right]\Gamma(\Lambda_2), \text{ with} \tag{14.112}$$

$$\gamma = -\frac{d\ln Z}{d\ln\Lambda}. \tag{14.113}$$

We verify that the solution Eq. (14.107) satisfies Eq. (14.106) by taking $\Lambda_1 \frac{\partial}{\partial\Lambda_1}$ of both sides:

$$\Lambda_1 \frac{\partial\Gamma(k,u(\Lambda_1),\Lambda_1)}{\partial\Lambda_1} + \beta(u(\Lambda_1))\frac{\partial\Gamma(k,u(\Lambda_1),\Lambda_1)}{\partial u(\Lambda_1)} = \gamma(u(\ln\Lambda_1))\Gamma(k,u(\Lambda_1),\Lambda_1). \tag{14.114}$$

Sometimes Eq. (14.107) is written in terms of an integral over the running coupling $u(\Lambda)$:

$$\Gamma(k,u(\Lambda_1),\Lambda_1) = \exp\left[\int_{u_2\equiv u(\Lambda_2)}^{u_1\equiv u(\Lambda_1)} \gamma(u)\frac{du}{\beta(u)}\right]\Gamma(k,u(\Lambda_2),\Lambda_2). \tag{14.115}$$

This version comes in handy if the integral over u is dominated by a zero of the β-function. We will have occasion to use it.

14.9.3 Computing the β-Function

The first step in using the Callan–Symanzik equation is the computation of β, which we will do to one loop. We begin with the renormalization condition,

$$u_{\mathrm{R}}\mu^{\varepsilon} = \Lambda^{\varepsilon}\left[u(\Lambda) - \frac{3u^2(\Lambda)}{16\pi^2}\ln\frac{\Lambda}{\mu}\right], \tag{14.116}$$

where the right-hand side was encountered earlier for the case $d = 4$ where $\varepsilon = 0$. Now we have to introduce the Λ^{ε} in front as part of the definition of the coupling. Setting to zero the $\ln \Lambda$-derivative of both sides (at fixed μ and u_R), we have (keeping only terms of order εu and u^2),

$$0 = \varepsilon u(\Lambda) + \underbrace{\frac{du(\Lambda)}{d\ln \Lambda}}_{\beta(u)} - \frac{3u^2(\Lambda)}{16\pi^2}. \tag{14.117}$$

(We anticipate that β will be of order εu or u^2, and do not take the $\ln \Lambda$-derivative of the $3u^2$ term, for that would lead to a term of order u^3 or $u^2\varepsilon$.) The result is

$$\beta(u) = -\varepsilon u + \frac{3u^2}{16\pi^2}. \tag{14.118}$$

The way β is defined, as Λ increases (more relevant to QFT), u flows toward the origin, while if Λ decreases (more relevant to us), it flows away and hits a zero at

$$u^* = \frac{16\varepsilon\pi^2}{3}. \tag{14.119}$$

That is,

$$\beta(u^*) = 0. \tag{14.120}$$

This is the WF fixed point. For future use, note that the slope of the β-function at the fixed point is

$$\omega = \left. \frac{d\beta(u)}{du} \right|_{u^*} = -\varepsilon + \frac{6u^*}{16\pi^2} = \varepsilon. \tag{14.121}$$

This irrelevant exponent $\omega = \varepsilon$ determines how quickly we approach the fixed point as we lower the cut-off. Here are the details.

14.9.4 Flow of $u - u^*$

Let us write a variable cut-off as

$$\Lambda(s) = \frac{\Lambda}{s}, \qquad s > 1. \tag{14.122}$$

It follows that

$$\frac{d}{d\ln \Lambda} = -\frac{d}{d\ln s}. \tag{14.123}$$

The coupling

$$u(s) \equiv u(\Lambda/s) \tag{14.124}$$

flows as follows:

$$\frac{du(s)}{d\ln s} = -\frac{du(\Lambda)}{d\ln \Lambda} = \varepsilon u(s) - \frac{3u_s^2}{16\pi^2} \equiv \bar{\beta}(u) = -\beta(u). \tag{14.125}$$

Integrating the flow of the coupling as a function of s, starting from $u(1) = u$, gives

$$\int_{u(1)=u}^{u(s)} \frac{du'}{\bar{\beta}(u')} = \ln s. \tag{14.126}$$

Now we expand $\bar{\beta}$ near the fixed point:

$$\bar{\beta}(u') = \bar{\beta}(u^*) - \omega(u' - u^*) = 0 - \omega(u' - u^*) = (-\omega)(u' - u^*). \tag{14.127}$$

(The minus in front of ω reflects the switch from β to $\bar{\beta} = -\beta$.) Substituting this into the previous equation, we get

$$\int_{u(1)=u}^{u(s)} \frac{du'}{(-\omega)(u' - u^*)} = \ln s, \tag{14.128}$$

with the solution

$$u(s) - u^* = (u - u^*)s^{-\omega} = (u - u^*)s^{-\varepsilon}. \tag{14.129}$$

That is, *the initial deviation from the fixed point* $(u - u^*)$ *shrinks by a factor* $s^{-\varepsilon} = s^{-\omega}$ *under the RG transformation* $\Lambda \to \Lambda/s$. Equation (14.129) will be recalled shortly.

14.9.5 Computing γ

The function γ begins at two loops. Armed with the two-loop result

$$Z = 1 + \frac{u^2}{6(4\pi)^4} \ln \frac{\mu}{\Lambda} + \cdots, \tag{14.130}$$

we find

$$\gamma = -\frac{d \ln Z}{d \ln \Lambda} = \frac{u^2}{6(4\pi)^4}. \tag{14.131}$$

At the fixed point

$$u^* = \frac{16\pi^2 \varepsilon}{3}, \tag{14.132}$$

we have

$$\gamma(u^*) \equiv \gamma^* = \frac{\varepsilon^2}{54}. \tag{14.133}$$

For later use, note that

$$\gamma' = \left.\frac{d\gamma}{du}\right|_{u^*} = \frac{\varepsilon}{144\pi^2}. \tag{14.134}$$

14.9.6 Computing $\Gamma(k,u,\Lambda)$

Now we are ready to confront the correlation function $\Gamma(k,u,\Lambda)$, which is the two-point function on the critical line shown in Figure 14.9. We want to know its behavior as a function of k as $k \to 0$. We expect it to be controlled by the WF fixed point.

The equation obeyed by $\Gamma(k,u,\Lambda)$ is

$$\left[-\frac{\partial}{\partial \ln s} - \bar{\beta}(u(s))\frac{\partial}{\partial u} - \gamma(u(s))\right]\Gamma(k,u(s),\Lambda/s) = 0. \tag{14.135}$$

Suppose we are at the fixed point, where $\bar{\beta} = 0$ and

$$\gamma = \gamma(u^*) \equiv \gamma^*. \tag{14.136}$$

The equation to solve is

$$\frac{\partial\Gamma(k,u^*,\Lambda/s)}{\partial \ln s} = -\gamma(u^*)\Gamma(k,u^*,\Lambda/s), \tag{14.137}$$

with an obvious solution

$$\Gamma(k,u^*,\Lambda) = s^{\gamma^*}\Gamma(k,u^*,\Lambda/s). \tag{14.138}$$

By dimensional analysis,

$$\Gamma(k,u^*,\Lambda/s) = k^2 f\left(\frac{k}{\Lambda/s}\right) = k^2 f\left(\frac{ks}{\Lambda}\right). \tag{14.139}$$

Substituting this into Eq. (14.138), we arrive at

$$\Gamma(k,u^*,\Lambda) = s^{\gamma^*}k^2 f\left(\frac{ks}{\Lambda}\right). \tag{14.140}$$

Now we choose

$$s = \frac{\Lambda}{k}, \tag{14.141}$$

which just means

$$\frac{\Lambda}{s} = k, \tag{14.142}$$

i.e., the new cut-off equals the momentum of interest. With this choice,

$$\Gamma(k,u^*,\Lambda) = \left(\frac{\Lambda}{k}\right)^{\gamma^*} k^2 f(1) \simeq k^{2-\gamma^*}. \tag{14.143}$$

Comparing to the standard form

$$\Gamma(k) \simeq k^{2-\eta}, \tag{14.144}$$

we find that

$$\eta = \gamma(u^*) \equiv \gamma^*. \tag{14.145}$$

In case you wondered how $\Gamma(k)$ can go as $k^{2-\eta}$ when it has engineering dimension 2, the answer is given above: $k^{-\eta}$ is really $\left(\frac{k}{\Lambda}\right)^{-\eta}$.

Finally, we ask how subleading corrections to the fixed point behavior arise if we start at some $u \neq u^*$ with a k that is approaching zero. For this, we return to the solution to the Callan–Symanzik equation

$$\Gamma(k, u(\Lambda_1), \Lambda_1) = \exp\left[\int_{\ln\Lambda_2}^{\ln\Lambda_1} \gamma(u(\ln\Lambda')) d\ln\Lambda'\right] \Gamma(k, u(\Lambda_2), \Lambda_2). \tag{14.146}$$

Let

$$\Lambda_1 = \Lambda, \tag{14.147}$$
$$\Lambda_2 = \Lambda/s, \tag{14.148}$$
$$u(\Lambda/s) \equiv u(s), \tag{14.149}$$
$$u(\Lambda) \equiv u(1). \tag{14.150}$$

Then

$$\Gamma(k, u(1), \Lambda) = \exp\left[\int_s^1 \gamma(u'(s'))\frac{-ds'}{s'}\right] \Gamma(k, u(s), \Lambda/s). \tag{14.151}$$

Corrections are going to arise from both the exponential factor and $\Gamma(k, u(s), \Lambda/s))$, due to the fact that at any non-zero $\frac{\Lambda}{s} = k$, the coupling $u(s)$ is close to, but not equal to, u^*, which is reached only asymptotically.

Consider first the exponential factor. Expanding γ near u^* as

$$\gamma(u') = \gamma^* + \gamma'(u' - u^*) + \cdots, \tag{14.152}$$
$$\gamma' = \frac{\varepsilon}{144\pi^2} \quad \text{[Eq. (14.134)]}, \tag{14.153}$$

we have, in the exponent,

$$\begin{aligned}\int_s^1 \gamma(u'(s'))\frac{-ds'}{s'} &= \int_1^s (\gamma^* + \gamma'(u(s') - u^*))\frac{ds'}{s'} \\ &= \gamma^* \ln s + \gamma'(u - u^*)\int_1^s (s')^{-\omega}\frac{ds'}{s'} \\ &= \gamma^* \ln s + \frac{\gamma'}{\omega}(u - u^*)(1 - s^{-\omega}).\end{aligned} \tag{14.154}$$

Thus the exponential factor becomes

$$\exp[\cdots] = s^{\gamma^*}\left(1+\frac{\gamma'}{\omega}(u-u^*)(1-s^{-\omega})\cdots\right). \tag{14.155}$$

Next, consider

$$\begin{aligned}\Gamma(k,u(s),\Lambda/s)) &= \Gamma(k,u^*+u(s)-u^*,\Lambda/s)\\ &= k^2 f\left[\frac{ks}{\Lambda},u^*+(u(s)-u^*)\right]\\ &= k^2 f\left[\frac{ks}{\Lambda},u^*+(u-u^*)s^{-\omega}\right].\end{aligned} \tag{14.156}$$

If we now set

$$s = \frac{\Lambda}{k} \tag{14.157}$$

and recall that $\omega = \varepsilon$, we find, upon putting the two factors in Eqs. (14.155) and (14.156) together, an irrelevant correction of the form $\left(\frac{k}{\Lambda}\right)^{\varepsilon}$:

$$\Gamma(k,u(\Lambda),\Lambda) = k^2\left(\frac{\Lambda}{k}\right)^{\gamma^*}\left(a+c\left(\frac{k}{\Lambda}\right)^{\varepsilon}\right), \tag{14.158}$$

where a and c are some constants.

14.9.7 Variations of the Theme

The preceding introduction was aimed at giving you an idea of how the Callan–Symanzik machine works by focusing on Γ, corresponding to two-particle correlations, and only for the critical case. There are so many possible extensions and variations.

The first variation is to go to the non-critical theory, where, in addition to the marginal coupling u, we have a relevant coupling, denoted by t, which as usual measures deviation from criticality. It multiplies the operator ϕ^2, whose presence calls for additional renormalization. The final result will be quite similar: as $k/\Lambda \to 0$, the flow will first approach the fixed point and then follow the renormalized trajectory.

Next, we can go from correlations of two fields to M fields and work with $\Gamma(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_M)$.

Finally, let us go back to the relation between bare and renormalized Γ's:

$$\Gamma_{\mathrm{R}}(k,u_{\mathrm{R}}(\mu),\mu) = \lim_{\Lambda\to\infty} Z(u(\Lambda),\Lambda/\mu)\Gamma(k,u(\Lambda),\Lambda). \tag{14.159}$$

We got the Callan–Symanzik equation by saying that since the Γ_{R} on the left-hand side had no knowledge of Λ, i.e., was cut-off independent, we could set the $\ln\Lambda$-derivative of the right-hand side to zero. This equation describes how the bare couplings and correlations have to change with the cut-off to keep fixed some renormalized quantities directly related to experiment.

Instead, we could argue that since Γ does not know about μ, the $\ln\mu$-derivative of the left-hand side must equal the same derivative acting on just the Z on the right-hand side (which has been expressed as $Z(u_R(\mu), \Lambda/\mu)$). The resulting equation,

$$\left[\frac{\partial}{\partial \ln\mu} + \beta\frac{\partial}{\partial u_R} - \gamma\right]\Gamma_R(k, u_R(\mu), \mu) = 0, \tag{14.160}$$

where

$$\beta(u_R) = \left.\frac{\partial u_R}{\partial \ln\mu}\right|_{u(\Lambda),\Lambda}, \tag{14.161}$$

$$\gamma(u_R) = \left.\frac{\partial \ln Z}{\partial \ln\mu}\right|_{u(\Lambda),\Lambda}, \tag{14.162}$$

dictates how the renormalized coupling and correlations must change with μ in order to represent the same underlying bare theory. (Again, the dimensionless functions β and γ cannot depend on μ/Λ because they are determined by Γ_R, which does not know about Λ.)

We can use either approach to get critical exponents, flows, and Green's functions (because Γ and Γ_R differ by Z, which is momentum and position independent), but there are cultural preferences. In statistical mechanics, the bare correlations are physically significant and describe underlying entities like spins. The cut-off is real and given by $\Lambda \simeq 1/a$. To particle physicists, the cut-off is an artifact, and the bare Green's functions and couplings are crutches to be banished as soon as possible so that they can work with experimentally measurable, finite, renormalized quantities defined on the scale μ. They prefer the second version based on Γ_R.

References and Further Reading

[1] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, Dover (2005). Gives a more thorough treatment of QFT and renormalization.

[2] M. Le Bellac, *Quantum and Statistical Field Theory*, Oxford University Press (1992).

[3] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press (1996).

[4] D. J. Amit, *Field Theory, Renormalization Group and Critical Phenomena*, World Scientific (1984).

[5] C. Itzykson and J. M. Drouffe, *Statistical Field Theory*, vol. I, Cambridge University Press (1989).

[6] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, Addison-Wesley (1992).

[7] K. G. Wilson, Reviews of Modern Physics, **47**, 773 (1975). The first few pages of this paper on RG and the Kondo problem are the best reference for renormalzing QFT.

[8] K. G. Wilson and J. R. Kogut, Physics Reports, **12**, 74 (1974). Provides a discussion of the triviality of ϕ_4^4.

[9] D. J. Gross and F. Wilczek, Physical Review Letters, **30**, 1343 (1973).

[10] H. D. Politzer, Physical Review Letters, **30**, 1346 (1973). In these two Nobel Prize winning works, these authors showed that quantum chromodynamics, the gauge theory of quarks and gluons, was *asymptotically free*, i.e., the coupling vanished

at very short distances or very large momenta and grew at long distances or small momenta. This allowed us to understand why quarks seemed free inside the nucleon in deep inelastic scattering and yet were confined at long distances. The β-function of this theory has a zero at the origin and the coupling grows as we move toward long distances. In all other theories like ϕ^4 or quantum electrodynamics, the behavior is exactly the opposite.

[11] C. Callan, Physical Review D, **2**, 1541 (1970).

[12] K. Symanzik, Communications in Mathematical Physics, **18**, 227 (1970).