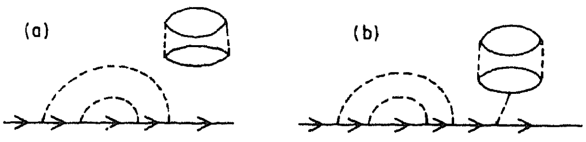

FIGURE 2.3

This drawing has the property that part of it is not topologically connected to the Green's function line $G^{(0)}(\mathbf{p}, t-t')$. Diagrams in which *all* parts are not connected are called *disconnected diagrams*. For example, in Fig. 2.3, part (a) is disconnected, while part (b) is connected. The disconnected parts, as in (2.95), provide just a multiplicative constant such as F_1 which multiplies the contribution from the connected parts.

2.6. VACUUM POLARIZATION GRAPHS

Now consider the factor which has been ignored to this point:

$$
{}_0\langle|S(\infty,-\infty)|\rangle_0 = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\cdots\int_{-\infty}^{\infty}dt_n \times {}_0\langle|T\hat{V}(t_1)\hat{V}(t_2)\cdots\hat{V}(t_n)|\rangle_0
$$

Again consider the electron–phonon interaction and evaluate the term $n=2$. The $n=1$ term vanishes as it did for the Green's function expansion,

$$
{}_0\langle|S|\rangle_0 = 1 + \frac{(-i)^2}{2!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\ {}_0\langle|T\hat{V}(t_1)\hat{V}(t_2)|\rangle_0 \tag{2.97}
$$

where

$$
\begin{aligned}
{}_0\langle|T\hat{V}(t_1)\hat{V}(t_2)|\rangle_0 &= \sum_{\mathbf{q}_1\mathbf{q}_2} M_{\mathbf{q}_1}M_{\mathbf{q}_2}\ {}_0\langle|T\hat{A}_{\mathbf{q}_1}(t_1)\hat{A}_{\mathbf{q}_2}(t_2)|\rangle_0 \\
&\times \sum_{\mathbf{k}_1\mathbf{k}_2ss'} {}_0\langle|T\hat{C}^{\dagger}_{\mathbf{k}_1+\mathbf{q}_1,s}(t_1)\hat{C}_{\mathbf{k}_1s}(t_1) \\
&\times \hat{C}^{\dagger}_{\mathbf{k}_2+\mathbf{q}_2,s'}(t_2)\hat{C}_{\mathbf{k}_2,s'}(t_2)|\rangle_0
\end{aligned}
$$

By using Wick's theorem,

$$
\begin{aligned}
&{}_0\langle|T\hat{A}_{\mathbf{q}_1}(t_1)\hat{A}_{\mathbf{q}_2}(t_2)|\rangle_0 = i\delta_{\mathbf{q}_1+\mathbf{q}_2}D^{(0)}(\mathbf{q}_1, t_1-t_2) \\
&{}_0\langle|T\hat{C}^{\dagger}_{\mathbf{k}_1+\mathbf{q}_1,s}(t_1)\hat{C}_{\mathbf{k}_1,s}(t_1)\hat{C}^{\dagger}_{\mathbf{k}_2-\mathbf{q}_1,s'}(t_2)\hat{C}_{\mathbf{k}_2,s'}(t_2)|\rangle_0 \\
&\quad = \delta_{\mathbf{q}_1}n_F(\xi_{\mathbf{k}_1})n_F(\xi_{\mathbf{k}_2}) + \delta_{\mathbf{k}_1=\mathbf{k}_2-\mathbf{q}_1}G^{(0)}(\mathbf{k}_1, t_1-t_2)G^{(0)}(\mathbf{k}_1+\mathbf{q}_1, t_2-t_1)
\end{aligned} \tag{2.98}
$$

The Feynman diagrams for the two terms in Eq. (2.98) are shown in Figs. 2.4(a) and 2.4(b). The (a) term is zero because there are no $\mathbf{q}=0$ phonons. The (b) term is nonzero and gives a contribution

$$
{}_0\langle|S(\infty,-\infty)|\rangle_0 = 1 + F_1 + \cdots \tag{2.99}
$$

(a) (b)

FIGURE 2.4

where F_1 is defined in (2.96). The constant F_1 appears whenever the closed bubble of Fig. 2.4(b) occurs, regardless of whether the term arises in the disconnected diagrams of $G(\mathbf{p}, t - t')$ or in the expansion of ${}_0\langle|S(\infty, -\infty|\rangle_0$.

The terms in the series for ${}_0\langle|S(\infty, -\infty|\rangle_0$ are called *vacuum polarization terms*. Some terms for $n = 4$, where there are two phonon lines, are shown in Fig. 2.5. Each of these diagrams or terms represents a constant F_j which one can evaluate by doing the required time and wave vector integrals. The constant ${}_0\langle|S(\infty, -\infty|\rangle_0$ could be evaluated by computing all the F_j and then summing them ($F_0 = 1$):

$$ {}_0\langle|S(\infty, -\infty|\rangle_0 = \sum_{j=0}^{\infty} F_j \tag{2.100}$$

This procedure is unnecessary because of a cancellation theorem.

The next theorem also simplifies the calculation of the Green's function expansion (2.76). This theorem is that the vacuum polarization diagrams exactly cancel the disconnected diagrams in the expansion for $G(\mathbf{p}, t - t')$. The net result is that in calculating $G(\mathbf{p}, t - t')$ one needs only to evaluate the connected diagrams. The other contributions, from the disconnected diagrams and from ${}_0\langle|S(\infty, -\infty|\rangle_0$, exactly cancel one another.

This theorem will not be proven, but only explained. Call $G_c(\mathbf{p}, t - t')$ the summation of all connected diagrams, and the basic theorem is that

$$ {}_0\langle|T\hat{C}_{\mathbf{p}}(t)\hat{C}_{\mathbf{p}}^{\dagger}(t')S(\infty, -\infty|\rangle_0 = G_c(\mathbf{p}, t - t')_0\langle|S(\infty, -\infty|\rangle_0 $$

The Green's function (2.43) is just the summation of all the connected diagrams:

$$ G(\mathbf{p}, t - t') = G_c(\mathbf{p}, t - t') \tag{2.101}$$

The proof of this theorem is just a counting problem. One must convince oneself that each connected diagram has, in higher-order terms in the S-matrix expansion, all disconnected parts which exactly add up to ${}_0\langle|S(\infty, -\infty|\rangle_0$. For example, the self-energy diagram in Fig. 2.2(a) has in higher order the vacuum polarization terms shown in Fig. 2.6. The summation of all these terms, to all orders, is just the factor ${}_0\langle|S(\infty, -\infty|\rangle_0$. The important point is that each disconnected part is just a constant factor F_j. This theorem is very convenient, since it states that one can just ignore the disconnected diagrams. They do not need to be calculated. It is just as well, since when they are evaluated they often turn out to be infinity. In fact, it is easy

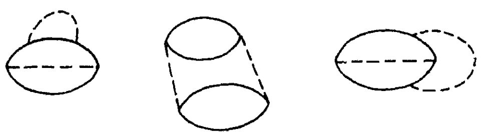

FIGURE 2.5 Vacuum polarization graphs

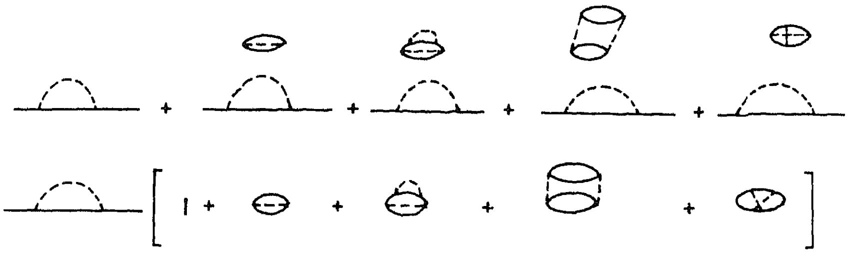

FIGURE 2.6

to show that they are infinity. The disconnected term F_1 is defined in (2.96). The integrand is only a function $f(t_1 - t_2)$ of $t_1 - t_2$. Change integration variables to

$$\tau = t_1 - t_2$$

$$s = \tfrac{1}{2}(t_1 + t_2)$$

and Eq. (2.96) becomes

$$F_1 = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\tau f(\tau) \tag{2.102}$$

The important point is that there is no dependence on s, so

$$\int_{-\infty}^{\infty} ds = \infty \tag{2.103}$$

One can show that each disconnected part has an "extra" time integral and has the same infinity.

It has been shown that the one-particle Green's functions consist of just connected diagrams:

$$G(\mathbf{p}, t - t') = -i \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n$$

$$\times\, {}_0\langle | T\hat{C}_{\mathbf{p}\sigma}(t_1)\hat{C}^{\dagger}_{\mathbf{p}\sigma}(t')\hat{V}(t_1) \cdots \hat{V}(t_n)|\rangle_0 \text{ (connected)}$$

The next step is to get rid of the $1/n!$ factor. It is eliminated because there are just $n!$ terms exactly alike in each bracket of the nth term in the expansion.

Considering only different terms gives the result

$$G(\mathbf{p}, t - t') = -i \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n {}_0\langle | T\hat{C}_{\mathbf{p}\sigma}(t)\hat{C}^{\dagger}_{\mathbf{p}\sigma}(t')$$

$$\times\, \hat{V}(t_1) \cdots \hat{V}(t_n)|\rangle_0 \text{ (different connected)} \tag{2.104}$$

The obvious question is then: how can you tell when terms are different? Usually one can tell by inspection, although sometimes terms must be examined carefully. In Fig. 2.2, terms (a) and (b) are the same and provide the $2 = 2!$ necessary for this $n = 2$ term. Similarly, (c) and (e) are identical. For example, (a) and (b) differ only in the variables t_1, t_2 and $\mathbf{q}_1$. But these are dummy variables of integration and so may be relabeled, respectively, to t_2, t_1 and $-\mathbf{q}_1$. Then the (a) term is obviously the same as (b).

The next terms in the electron–phonon expansion have $n = 4$, which are diagrams with two phonons. Here each different connected diagram is found $4! = 24$ times.

2.7. DYSON'S EQUATION

The Green's function of energy is defined by taking the usual Fourier transform with respect to the time variable:

$$G(\mathbf{p}, E) = \int_{-\infty}^{\infty} dt e^{iE(t-t')} G(\mathbf{p}, t - t') \tag{2.105}$$

This time integral has already been evaluated for the unperturbed Green's function with the following results. For a single particle in a band, the result from (2.59) is

$$G^{(0)}(\mathbf{p}, E) = \frac{1}{E - \varepsilon_{\mathbf{p}} + i\delta}$$

For a fermion in a degenerate electron gas, the result (2.64) is

$$G^{(0)}(\mathbf{p}, E) = \frac{1}{E - \varepsilon_{\mathbf{p}} + i\delta_{\mathbf{p}}}$$

The Fourier transform in time is applied to each term in the S-matrix summation:

$$G(\mathbf{p}, E) = -i \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt e^{iE(t-t')} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n$$
$$\times {}_0\langle|T\hat{C}_{\mathbf{p}\sigma}(t)\hat{C}^{\dagger}_{\mathbf{p}\sigma}(t')\hat{V}(t_1)\cdots\hat{V}(t_n)|\rangle_0 \text{ (different connected)} \tag{2.106}$$

To see what sort of terms develop, consider the example of the electron–phonon interaction. The first two terms [Figs. 2.2(c) and 2.2(e) are zero] are $G^{(0)}$ plus the self-energy term in Fig. 2.2(a):

$$G(\mathbf{p}, E) = G^{(0)}(\mathbf{p}, E) + (-i)^2 \sum_{\mathbf{q}} |M_{\mathbf{q}}|^2 \int_{-\infty}^{\infty} dt e^{iE(t-t')} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2$$
$$\times G^{(0)}(\mathbf{p}, t - t_1) G^{(0)}(\mathbf{p} - \mathbf{q}, t_1 - t_2)$$
$$\times G^{(0)}(\mathbf{p}, t_2 - t') D^{(0)}(\mathbf{q}, t_1 - t_2) \tag{2.107}$$

The phonon Green's function of energy is defined the same way:

$$D(\mathbf{q}, \omega) = \int_{-\infty}^{\infty} dt e^{it\omega} D(\mathbf{q}, t)$$
$$D(\mathbf{q}, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-it\omega} D(\mathbf{q}, \omega) \tag{2.108}$$

Using the unperturbed phonon Green's function in (2.107),

$$D^{(0)}(\mathbf{q}, t_1 - t_2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t_1 - t_2)} D^{(0)}(\mathbf{q}, \omega) \tag{2.109}$$