5 Perturbative renormalization group

5.1 Expectation values in the Gaussian model

Can we treat the Landau–Ginzburg Hamiltonian as a perturbation to the Gaussian model? In particular, for zero magnetic field, we shall examine

$$
\begin{aligned}
\beta\mathcal{H} = \beta\mathcal{H}_0 + \mathcal{U} \equiv \int \mathrm{d}^d\mathbf{x}\left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots\right] \\
+ u\int \mathrm{d}^d\mathbf{x} m^4 + \cdots .
\end{aligned}
\tag{5.1}
$$

The *unperturbed* Gaussian Hamiltonian can be decomposed into independent Fourier modes, as

$$
\begin{aligned}
\beta\mathcal{H}_0 &= \frac{1}{V}\sum_{\mathbf{q}} \frac{t + Kq^2 + Lq^4 + \cdots}{2}|m(\mathbf{q})|^2 \\
&\equiv \int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\frac{t + Kq^2 + Lq^4 + \cdots}{2}|m(\mathbf{q})|^2 .
\end{aligned}
\tag{5.2}
$$

The *perturbative interaction* which mixes up the normal modes has the form

$$
\begin{aligned}
\mathcal{U} &= u\int \mathrm{d}^d\mathbf{x} m(\mathbf{x})^4 + \cdots \\
&= u\int \mathrm{d}^d\mathbf{x}\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}} \mathrm{e}^{-\mathrm{i}\mathbf{x}\cdot(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)} m_\alpha(\mathbf{q}_1)m_\alpha(\mathbf{q}_2)m_\beta(\mathbf{q}_3)m_\beta(\mathbf{q}_4) \\
&\quad + \cdots ,
\end{aligned}
\tag{5.3}
$$

where summation over α and β is implicit. The integral over $\mathbf{x}$ sets $\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4 = \mathbf{0}$, and

$$
\mathcal{U} = u\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}} m_\alpha(\mathbf{q}_1)m_\alpha(\mathbf{q}_2)m_\beta(\mathbf{q}_3)m_\beta(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) + \cdots . \tag{5.4}
$$

From the variance of the Gaussian weights, the two-point expectation values in a finite sized system with discretized modes are easily obtained as

$$
\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 = \frac{\delta_{\mathbf{q},-\mathbf{q}'}\,\delta_{\alpha,\beta}\,V}{t + Kq^2 + Lq^4 + \cdots}. \tag{5.5}
$$

In the limit of infinite size, the spectrum becomes continuous, and Eq. (5.5) goes over to

$$
\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 = \frac{\delta_{\alpha,\beta}(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{t + Kq^2 + Lq^4 + \cdots}. \tag{5.6}
$$

The subscript 0 is used to indicate that the expectation values are taken with respect to the unperturbed (Gaussian) Hamiltonian. Expectation values involving any product of m's can be obtained starting from the identity

$$\left\langle \exp\left[\sum_i a_i m_i\right]\right\rangle_0 = \exp\left[\sum_{i,j}\frac{a_i a_j}{2}\left\langle m_i m_j\right\rangle_0\right], \tag{5.7}$$

which is valid for any set of Gaussian distributed variables $\{m_i\}$. (This is easily seen by "completing the square.") Expanding both sides of the equation in powers of $\{a_i\}$ leads to

$$\begin{aligned}
&1 + a_i\langle m_i\rangle_0 + \frac{a_i a_j}{2}\langle m_i m_j\rangle_0 + \frac{a_i a_j a_k}{6}\langle m_i m_j m_k\rangle_0 + \frac{a_i a_j a_k a_l}{24}\langle m_i m_j m_k m_k\rangle_0 + \cdots = \\
&1 + \frac{a_i a_j}{2}\langle m_i m_j\rangle_0 + \frac{a_i a_j a_k a_l}{24}\big(\langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0 \\
&\quad + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0\big) + \cdots
\end{aligned} \tag{5.8}$$

Matching powers of $\{a_i\}$ on the two sides of the above equation gives

$$\left\langle \prod_{i=1}^{\ell} m_i\right\rangle_0 = \begin{cases} 0 & \text{for } \ell \text{ odd} \\ \text{sum over all pairwise contractions} & \text{for } \ell \text{ even.}\end{cases} \tag{5.9}$$

This result is known as *Wick's theorem*; and for example,

$$\langle m_i m_j m_k m_l\rangle_0 = \langle m_i m_j\rangle_0\langle m_k m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0 + \langle m_i m_k\rangle_0\langle m_j m_l\rangle_0.$$

5.2 Expectation values in perturbation theory

In the presence of an interaction $\mathcal{U}$, the expectation value of any operator $\mathcal{O}$ is computed perturbatively as

$$\begin{aligned}
\langle\mathcal{O}\rangle &= \frac{\int \mathcal{D}\vec{m}\,\mathcal{O}\mathrm{e}^{-\beta\mathcal{H}_0-\mathcal{U}}}{\int \mathcal{D}\vec{m}\,\mathrm{e}^{-\beta\mathcal{H}_0-\mathcal{U}}} = \frac{\int \mathcal{D}\vec{m}\,\mathrm{e}^{-\beta\mathcal{H}_0}\mathcal{O}[1-\mathcal{U}+\mathcal{U}^2/2-\cdots]}{\int \mathcal{D}\vec{m}\,\mathrm{e}^{-\beta\mathcal{H}_0}[1-\mathcal{U}+\mathcal{U}^2/2-\cdots]} \\
&= \frac{Z_0[\langle\mathcal{O}\rangle_0 - \langle\mathcal{O}\mathcal{U}\rangle_0 + \langle\mathcal{O}\mathcal{U}^2\rangle_0/2-\cdots]}{Z_0[1-\langle\mathcal{U}\rangle_0+\langle\mathcal{U}^2\rangle_0/2-\cdots]}.
\end{aligned} \tag{5.10}$$

Inverting the denominator by an expansion in powers of $\mathcal{U}$ gives

$$\begin{aligned}
\langle\mathcal{O}\rangle &= \left[\langle\mathcal{O}\rangle_0 - \langle\mathcal{O}\mathcal{U}\rangle_0 + \frac{1}{2}\langle\mathcal{O}\mathcal{U}^2\rangle_0 - \cdots\right]\left[1+\langle\mathcal{U}\rangle_0 + \langle\mathcal{U}\rangle_0^2 - \frac{1}{2}\langle\mathcal{U}^2\rangle_0 - \cdots\right] \\
&= \langle\mathcal{O}\rangle_0 - (\langle\mathcal{O}\mathcal{U}\rangle_0 - \langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0) + \frac{1}{2}\big(\langle\mathcal{O}\mathcal{U}^2\rangle_0 - 2\langle\mathcal{O}\mathcal{U}\rangle_0\langle\mathcal{U}\rangle_0 \\
&\quad + 2\langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0^2 - \langle\mathcal{O}\rangle_0\langle\mathcal{U}^2\rangle_0\big) + \cdots \\
&\equiv \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle\mathcal{O}\mathcal{U}^n\rangle_0^c.
\end{aligned} \tag{5.11}$$

The *connected averages* (cumulants) are defined as the combination of unperturbed expectation values appearing at various orders in the expansion. Their significance will become apparent in diagrammatic representations, and from the following example.

Let us calculate the two-point correlation function of the Landau–Ginzburg model to first order in the parameter u. (In view of their expected irrelevance, we shall ignore higher order interactions, and also only keep the lowest order Gaussian terms.) Substituting Eq. (5.4) into Eq. (5.11) yields

$$\begin{aligned}\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = {} & \langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0 - u\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}} \\ & \times[\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')m_i(\mathbf{q}_1)m_i(\mathbf{q}_2)m_j(\mathbf{q}_3)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0 \\ & -\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle_0\langle m_i(\mathbf{q}_1)m_i(\mathbf{q}_2)m_j(\mathbf{q}_3)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0] \\ & +\mathcal{O}(u^2). \end{aligned} \tag{5.12}$$

To calculate $\langle \mathcal{O}\mathcal{U}\rangle_0$ we need the unperturbed expectation value of the product of six m's. This can be evaluated using Eq. (5.9) as the sum of all pair-wise contractions, 15 in all. Three contractions are obtained by first pairing m_α to m_β, and then the remaining four m's in $\mathcal{U}$. Clearly these contractions cancel exactly with corresponding ones in $\langle\mathcal{O}\rangle_0\langle\mathcal{U}\rangle_0$. The only surviving terms involve contractions that connect $\mathcal{O}$ to $\mathcal{U}$. This cancellation persists at all orders, and $\langle\mathcal{O}\mathcal{U}^n\rangle_0^c$ contains only terms in which all $n+1$ operators are connected by contractions. The remaining 12 pairings in $\langle\mathcal{O}\mathcal{U}\rangle_0$ fall into two classes:

(1) Four pairings involve contracting m_α and m_β to m's with the same index, e.g.

$$\begin{aligned}&\langle m_\alpha(\mathbf{q})\, m_i(\mathbf{q}_1)\rangle_0\langle m_\beta(\mathbf{q}')\, m_i(\mathbf{q}_2)\rangle_0\langle m_j(\mathbf{q}_3)\, m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0 \\ &\quad = \frac{\delta_{\alpha i}\,\delta_{\beta i}\,\delta_{jj}\,(2\pi)^{3d}\,\delta^d(\mathbf{q}+\mathbf{q}_1)\,\delta^d(\mathbf{q}'+\mathbf{q}_2)\,\delta^d(\mathbf{q}_1+\mathbf{q}_2)}{(t+Kq^2)\,(t+Kq'^2)\,(t+Kq_3^2)},\end{aligned} \tag{5.13}$$

where we have used Eq. (5.6). After summing over i and j, and integrating over $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$, these terms make a contribution

$$-4u\frac{n\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{(t+Kq^2)^2}\int\frac{\mathrm{d}^d\mathbf{q}_3}{(2\pi)^d}\frac{1}{t+Kq_3^2}. \tag{5.14}$$

(2) Eight pairings involve contracting m_α and m_β to m's with different indices, e.g.

$$\begin{aligned}&\langle m_\alpha(\mathbf{q})m_i(\mathbf{q}_1)\rangle_0\langle m_\beta(\mathbf{q}')m_j(\mathbf{q}_3)\rangle_0\langle m_i(\mathbf{q}_2)m_j(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\rangle_0 \\ &\quad = \frac{\delta_{\alpha i}\delta_{\beta j}\delta_{ij}(2\pi)^{3d}\delta^d(\mathbf{q}+\mathbf{q}_1)\delta^d(\mathbf{q}'+\mathbf{q}_3)\delta^d(\mathbf{q}_1+\mathbf{q}_3)}{(t+Kq^2)(t+Kq'^2)(t+Kq_2^2)}.\end{aligned} \tag{5.15}$$

Summing over all indices, and integrating over the momenta leads to an overall contribution of

$$-8u\frac{\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{(t+Kq^2)^2}\int\frac{\mathrm{d}^d\mathbf{q}_2}{(2\pi)^d}\frac{1}{t+Kq_2^2}. \tag{5.16}$$

Adding up both contributions, we obtain

$$\langle m_\alpha(\mathbf{q})\, m_\beta(\mathbf{q}')\rangle = \frac{\delta_{\alpha\beta}\,(2\pi)^d\,\delta^d\,(\mathbf{q}+\mathbf{q}')}{t+Kq^2}\left[1-\frac{4u(n+2)}{t+Kq^2}\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}+\mathcal{O}(u^2)\right]. \tag{5.17}$$

5.3 Diagrammatic representation of perturbation theory

The calculations become more involved at higher orders in perturbation theory. A diagrammatic representation can be introduced to help keep track of all possible contractions. To calculate the ℓ-point expectation value $\langle\prod_{i=1}^{\ell} m_{\alpha_i}(\mathbf{q}_i)\rangle$, at pth order in u, proceed according to the following rules:

(1) Draw ℓ *external points* labeled by $(\mathbf{q}_i, \alpha_i)$ corresponding to the coordinates of the required correlation function. Draw p *vertices* with four legs each, labeled by *internal* momenta and indices, e.g. $\{(\mathbf{k}_1, i), (\mathbf{k}_2, i), (\mathbf{k}_3, j), (\mathbf{k}_4, j)\}$. Since the four legs are not equivalent, the four point vertex is indicated by two solid branches joined by a dotted line. (The extension to higher order interactions is straightforward.)

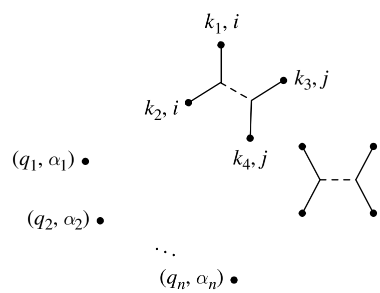
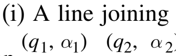
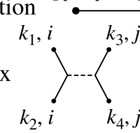


Fig. 5.1 Elements of the diagrammatic representation of perturbation theory.

(2) Each point of the graph now corresponds to one factor of $m_{\alpha_i}(\mathbf{q}_i)$, and the unperturbed average of the product is computed by Wick's theorem. This is implemented by joining all external and internal points *pair-wise*, by lines connecting one point to another, in all topologically distinct ways; see (5) below.

(3) The algebraic value of each such graph is obtained as follows: (i) A line joining a pair of points represents the two point average;[1] e.g. a connection 

, corresponds to $\delta_{\alpha_1\alpha_2}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2)/(t+Kq_1^2)$; (ii) A vertex 

stands for a term $u(2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$ (the delta-function insures that momentum is conserved).

[1] Because of its original formulation in quantum field theory, the line joining two points is usually called a *propagator*. In this context, the line represents the world-line of a particle in time, while the perturbation $\mathcal{U}$ is an "interaction" between particles. For the same reason, the Fourier index is called a "momentum".