The functional determinant

Jordan Bell
jordan.bell@gmail.com
Department of Mathematics, University of Toronto

April 22, 2015

1 Gaussians

Let $A \in \mathscr{B}(\mathbb{R}^n)$ have positive spectrum. Because A is positive, it has a unique positive square root $\sqrt{A}$, which also has positive spectrum. Then, using the fact that $\int_{\mathbb{R}} \exp(-x^2)dx = \sqrt{\pi}$,

$$\begin{aligned}
\int_{\mathbb{R}^n} \exp(-\pi \langle x, Ax \rangle)dx &= \int_{\mathbb{R}^n} \exp\left(-\pi \left\langle x, \sqrt{A}\sqrt{A}x \right\rangle\right) dx \\
&= \int_{\mathbb{R}^n} \exp\left(-\left\langle \sqrt{\pi}\sqrt{A}x, \sqrt{\pi}\sqrt{A}x \right\rangle\right) dx \\
&= \int_{\mathbb{R}^n} \exp(-\langle x, x \rangle)|\det(\sqrt{\pi}\sqrt{A})^{-1}|dx \\
&= \frac{1}{\det(\sqrt{\pi}I)} \frac{1}{\det(\sqrt{A})} \int_{\mathbb{R}^n} \exp(-\langle x, x \rangle)dx \\
&= \pi^{-d/2}(\det A)^{-1/2} \int_{\mathbb{R}^n} \exp(-\langle x, x \rangle)dx \\
&= (\det A)^{-1/2};
\end{aligned}$$

$\frac{1}{\det(\sqrt{A})} = (\det A)^{-1/2}$ because $A = \sqrt{A}\sqrt{A}$ and $\det(\sqrt{A}) > 0$.

2 Zeta functions

Suppose that $A \in \mathscr{B}(\mathbb{R}^n)$ has positive spectrum: $0 < \lambda_1 \leq \cdots \leq \lambda_n$. Define

$$\zeta_A(s) = \sum_{k=1}^n \frac{1}{\lambda_k^s} = \sum_{k=1}^n \exp(-s \log \lambda_k), \qquad s \in \mathbb{C}.$$

As

$$\zeta_A'(s) = \sum_{k=1}^n - \log(\lambda_k) \exp(-s \log \lambda_k) = \sum_{k=1}^n - \frac{\log \lambda_k}{\lambda_k^s},$$

we have

$$-\zeta_A'(0) = \sum_{k=1}^{n} \log \lambda_k,$$

hence

$$\exp(-\zeta_A'(0)) = \prod_{k=1}^{n} \lambda_k = \det A.$$

3 Further reading

Leon A. Takhtajan, *Quantum Mechanics for Mathematicians*, p. 262.

Nicole Berline, Ezra Getzler and Michèle Vergne, *Heat Kernels and Dirac Operators*, p. 296.

Jürgen Jost, *Geometry and Physics*, p. 101.

Eberhard Zeidler, *Quantum Field Theory II: Quantum Electrodynamics*, p. 570.

John Baez, Week 127, `http://math.ucr.edu/home/baez/week127.html`

Klaus Kirsten, *Functional determinants in higher dimensions using contour integrals*, `http://arxiv.org/abs/1005.2595`, and *Basic zeta functions and some applications in physics*, `http://arxiv.org/abs/1005.2389`

H. Kumagai, *The determinant of the Laplacian on the n-sphere*, Acta Arith. **91** (1999), no. 3, 199-208.

Predrag Cvitanović, *Spectral determinants*, `http://chaosbook.org/chapters/det.pdf`

Steven Rosenberg, *The Laplacian on a Riemannian Manifold: An Introduction to Analysis on Manifolds*, Chapter 5.