總結. ① 無處不在的無窮大 ② 場、粒子 ③ 場的連續化產生發散 與物理密切相關

場. $\phi(x)$. $\begin{cases} x\text{ 點波函數} \\ x\text{ 點振動幅度} \end{cases}$

自由粒子 或 諧振子. $e^{-\int_a^b \mathcal{L}(t)dt}$.

$$\int D\phi\, e^{-S[\phi]} = \int \underbrace{Dx_1 Dx_2 \cdots Dx_N}_{\mathbb{R}^{\otimes N}}\, e^{\cdots}$$

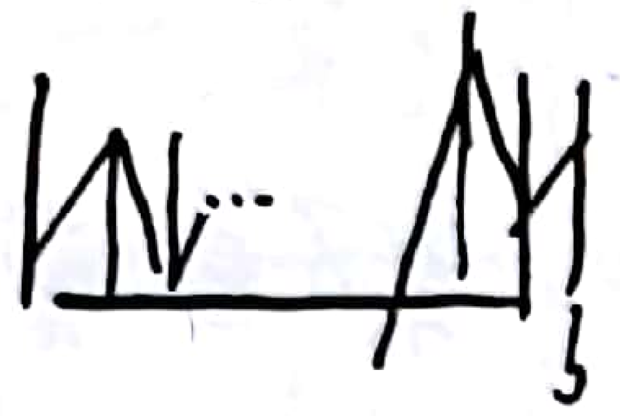
$$\lim_{n\to\infty}\int Dx_1\cdots Dx_n\, e^{-S(x)}.$$

or $\int D\phi\, e^{-S[\phi]}$. 無窮維積分

對其計算. ① 實空間 ② k空間

Jacob, 求令 Jacob=1.

$Dx_i = Dx_{t_i} = DX(t_i)$.



對於多粒子. $\prod_{\alpha=1}^{m}\int Dx_{m1}\, Dx_{m2}\cdots Dx_{mn}\, e^{-\int \mathcal{L}\, dt}$.

-o-o-o- ..
m

↓ 對於無窮多粒子.

D_m $x_{mi} \to \phi(x,t)$.

$$\int D\phi(\vec{x})\, e^{-S[\phi]} \sim \lim_{n\to\infty}\int D\phi(\vec{x}_1)\, D\phi(\vec{x}_2)\cdots D\phi(\vec{x}_n)\, e^{-S[\phi]}.$$

$\vec{x}_i$ 是參數坐標. 積分發生在參數空間. $\phi(\vec{x}_i)\in\mathbb{R}$. 在 $\vec{x}_i$ 處振幅.

$H=\frac{p^2}{2m}+\frac{m\omega^2}{2}x^2$ $\quad [x,p]=i.$ $\quad$ path integral.

↓ ↓ ↓

$H=\omega a^\dagger a.$ $\quad [a,a^\dagger]=1.$ $\quad$ 相干態.

↓ ↓

$H=\frac{1}{2}(\partial_\mu\phi)^2+\frac{m^2}{2}\phi^2(x)+\frac{\lambda}{4!}\phi^4(x)$. → QFT.

發散 { RG. $g_\lambda\sim\lambda$.

粒子物理 4d

相變、凝聚態.

凝聚態、低維

(D=1, D=2 會發生很多有趣的事).

無BEC → 無長程有序. (甚至無法形成晶格)

(BKT, Wagner-Mermin; Ising Model. Hall effect. 高溫超導.

計算問題。

波動方程、 $\mathcal{L}=(\partial_t\phi)^2-V^2(\partial_x\phi)^2$. (1+1) dimensional.

(Boson)、

$$\mathcal{L}=(\partial_t\phi)^2-v^2\{(\partial_x\phi)^2+(\partial_y\phi)^2\}-\underline{g\phi^4}\quad (2+1)\text{ dimensional}$$

K-G eq. $\mathcal{L}=(\partial_t\phi)^2-(\partial_x\phi)^2-(\partial_y\phi)^2-(\partial_z\phi)^2=(\partial_\mu\phi)^2-\underline{g\phi^4}$ (3+1) dimensional.

$A^2=A^iA_i=g^{ij}A_iA_j$. $g=(+,-,-,-)$. interaction 冷原子、BEC應用。

Boson場相互作用、接觸勢 $g\phi^4$

※凝聚態場中、對時間偏導的是線性項。

如 Schrödinger eq. $\mathcal{L}=\psi^\dagger i\partial_t\psi-\psi^\dagger H\psi$.

$$\mathcal{L}=\frac{1}{2}(\partial_t\phi)^2-\frac{1}{2}V^2(\partial_x\phi)^2=T-V.\qquad \mathcal{H}=\frac{1}{2}(\partial_t\phi)^2+\frac{1}{2}V^2(\partial_x\phi)^2=T+V.$$

$$\pi=\partial_t\phi \Leftrightarrow \pi=\frac{\partial\mathcal{L}}{\partial\dot\phi}$$

$$\therefore\mathcal{H}=\pi\dot\phi-\mathcal{L}=(\partial_t\phi)^2-\frac{1}{2}(\partial_t\phi)^2+\frac{V^2}{2}(\partial_x\phi)^2.$$

$$Z=\int D\phi\, e^{i\int dt L}=\int D\phi\, e^{i\int d\vec{x}\,\mathcal{L}(\partial_\mu\phi,\phi)}$$

$$=\int D\phi(x_1)\cdots D\phi(x_N)\, e^{i\int d\vec{x}\,\mathcal{L}(\underline{\partial_\mu\phi,\phi})}$$ 有相互作用.

$$=\cancel{\int\phi}\int D\phi\, e^{i\int d\vec{x}\frac{1}{2}((\partial_\mu\phi)^2-g\phi^4)}$$

$$=\int D\phi\, e^{\underbrace{-i\int d\mu\left(\phi\left(\frac{\partial_t^2-v^2\partial_x^2}{2}\right)\phi+g\phi^4\right)}_{\text{二次型積分}}}$$

$$\int e^{-ax^2}dx.$$

$$(\partial_\mu\phi)^2=\partial_\mu\phi\cdot\partial_\mu\phi=\underbrace{\partial_\mu(\phi\partial_\mu\phi)}_{0}-\phi(\partial_\mu^2)\phi$$

(全微分 $\int_{-\infty}^{\infty}dx\,(\partial_x f)=0$)

有相互作用、Jacobi會極其複雜。

需用 Feynman diagram

wick 收縮　linked cluster 定理。

二次型積分.

$\int p(x)dx = 1$. $p(x) \propto e^{-Ax^2}$. 高斯積分. 概率分佈. 隨機方法. 重要思想能

$$\int dx\, e^{-\frac{A}{2}x^2} = \sqrt{\frac{2\pi}{A}} \Rightarrow \int dx_1 dx_2\, e^{-\frac{1}{2}(\lambda_1 x_1^2 + \lambda_2 x_2^2)} = \frac{(\sqrt{2\pi})^2}{\sqrt{(\lambda_1\lambda_2)}}.$$

推廣. $$\int dx_1 dx_2 \cdots dx_N\, e^{\underbrace{-\frac{1}{2}(\lambda_i x_i^2)}_{\lambda_i x_i^2 \leftrightarrow x^TAx = (x_1 \cdots x_N)\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_N\end{pmatrix}\begin{pmatrix}x_1\\ \vdots \\ x_N\end{pmatrix}}} = \frac{(\sqrt{2\pi})^N}{\sqrt{(\lambda_1\lambda_2\cdots\lambda_N)}} \Leftrightarrow \int dx_1 \cdots dx_N\, e^{-\frac{1}{2}x^TAx} = \frac{(\sqrt{2\pi})^N}{\sqrt{\det A}}$$

$$\Downarrow$$

(令 $Dx_i = \sqrt{2\pi}\, dx_i$

那时 $\int Dx\, e^{-\frac{1}{2}x^TAx} = \frac{1}{\sqrt{\det A}}$.)

對於任意 $A = (A_{ij})_{N\times N}$. $A_{ij} \in \mathbb{R}$. $A^T = A$.

$$\int Dx\, e^{-\frac{1}{2}x^TAx} = \frac{1}{\sqrt{\det A}}.$$

$\det A = \det(O^T\lambda O)$

$= \det(\lambda)$

(實矩陣對角化 $O(N)$.

$\because X = OY$. 實現 $O^TAO = \mathrm{diag}(\lambda_i)$.

則有 $\det(O) = 1$. 線性變換.

$\therefore dx_1 \cdots dx_N = J\, dy_1 \cdots dy_N$.

~~$\therefore Dx = Dy$~~. $\therefore DX = DY$.)

$$J\left(\int DY e^{-\frac{1}{2}(\lambda_i y_i^2)}\right) = \frac{1}{\sqrt{\lambda_1 \cdots \lambda_N}}.$$

$(J=1)$

$$\therefore \int Dx\, e^{-\frac{1}{2}x^TAx} = e^{-\ln\sqrt{\det A}} = e^{-\frac{1}{2}\underbrace{\ln\det A}_{\ln(\det A) = \ln(\mathrm{diag}(\lambda_1, \lambda_2 \cdots \lambda_N))}} = e^{-\frac{1}{2}\mathrm{Tr}(\ln A)}$$

$$\int dx\, e^{-Ax^2} = \sqrt{\frac{2\pi}{A}}$$

$$\int dx\, e^{-Ax^2 + Jx} = \int dx\, e^{-A(x^2 - \frac{J}{A}x + \frac{J^2}{4A^2} \cancel{+} - \frac{J^2}{4A^2})}$$

$$= \int dx\, e^{-A(x - \frac{J}{2A})^2 + \frac{J^2}{4A}}$$

$$= \int dx\, e$$

$$= \sqrt{\frac{2\pi}{A}}\, e^{\frac{J^2}{4A}}$$

可表出相互作用.

~~如 $\int dx\, e^{-Ax^2}$~~

多变量情形、

$\int Dx\, e^{-\frac{1}{2}x^TAx + J^Tx}$ $\qquad J x = J^T x = \sum_i J_i x_i$.

平移 $x = y + q$.

$-\frac{1}{2}(y^T + q^T)A(y+q) + J^Tx + J^Tq$

$= -\frac{1}{2}y^TAy - \frac{1}{2}\underbrace{y^TAq}_{= q^TAy} - \frac{1}{2}\underline{q^TAy} - \frac{1}{2}q^TAq + \underline{J^Ty} + J^Tq$.

q^TAy. $\qquad q = A^{-1}J$.

$\because y^TAq = y_iA_{ij}q_j = q_jA_{ji}y_i = q^TAy$.

$= \frac{1}{\sqrt{\det A}} e^{J^Tq - \frac{1}{2}q^TAq}$.

$= \frac{1}{\sqrt{\det A}} e^{J^TAJ - \frac{1}{2}J^{-1}(A^{-1})^TAA^{-1}J}$.

$= \frac{1}{\sqrt{\det A}} e^{\frac{1}{2}J^TA^{-1}J}$.

$$\int Dx\, x_ix_j e^{-\frac{1}{2}x^TAx}$$

$$= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\int Dx\, e^{-\frac{1}{2}x^TAx + J^Tx}\Big|_{J=0}$$

$$= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\frac{1}{\sqrt{\det A}} e^{\frac{1}{2}J^TA^{-1}J}\Big|_{J=0}$$

$$= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\frac{1}{\sqrt{\det A}}\left(1 + \frac{1}{2}J^TA^{-1}J + \cdots\right)\Big|_{J=0}$$

$$= \frac{(A^{-1})_{ij}}{\sqrt{\det A}}.$$

$$\begin{cases} z = x+iy \\ \bar{z} = x-iy \end{cases}$$

複數情況.

~~$\int dz\, d\bar{z}\, A$~~ $\int dz\, d\bar{z}\, e^{-\frac{1}{2}A\bar{z}z} \simeq \int Dz\, D\bar{z}\, e^{-\frac{1}{2}A\bar{z}z} \sim \int dx\, dy\, e^{-\frac{1}{2}A(x^2+y^2)}$

$= \left(\frac{1}{\sqrt{\det A}}\right)^2 = \frac{1}{\det A}$.

複數情況變為 2份實數的copy.

$$\int D\bar{z}\, Dz\, e^{-\frac{1}{2}\bar{z}Az + \bar{z}J + \bar{J}z}.$$

(縮寫表示 $AB = A^T B$).

$\Rightarrow \begin{cases} z = w + q \\ \bar{z} = \bar{w} + \bar{q} \end{cases}$

HW: 文小朋 2-13, 2-14.

$$\Rightarrow \frac{1}{\det A} e^{?\cdot \bar{J}A^{-1}J}$$

傅氏變換 (動量空間下處理).

傅氏變換下 Jacobi.

$\int D\phi_1 D\phi_2 \cdots D\phi_n\, e^{-S[\phi]}$. $\qquad \phi_i = \phi(x_i) = \frac{1}{\sqrt{V}} \sum_k e^{ik\cdot x_i} \phi_k$. 線性變換.

$\Downarrow$

$\int D\phi(k_1) \cdots D\phi(k_n)\, J e^{-S[\phi]}$. $\qquad \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} = U \begin{pmatrix} \phi(k_1) \\ \vdots \\ \phi(k_N) \end{pmatrix}$

線性變換與場無關. $\qquad$ U只與 x, k 有關, 與場無關

$U_{ij} = \frac{1}{\sqrt{V}} e^{ik_i\cdot x_j}$ $\qquad (UU^+)_{ij} = U_{ik}(U^+)_{kj}$

$= U_{ik} U^*_{jk}$

$= \frac{1}{V}\sum_l e^{ik_i x_l} e^{-ik_j\cdot x_l}$

$= \frac{1}{V}\sum_l e^{i(k_i - k_j)x_l}$

$= \delta_{ij}$.

$\Leftrightarrow UU^+ = 1$.

$\det U = e^{i\theta}$.

$\int D\phi_1 \cdots D\phi_N\, e^{i\int dt\, \mathcal{L}}$

$= \int D\phi_1 \cdots D\phi_N\, e^{\frac{i}{2}\int d\vec{x}\, ((\partial_t\phi)^2 - v^2(\partial_x\phi)^2)}$

配分函數計算. Wick rotation $t = i\tau$.

$Z = \int D\phi_1 \cdots D\phi_N\, e^{-\frac{1}{2}\int_0^\beta d\tau\, dx\, ((\partial_\tau\phi)^2 + v^2 \underbrace{(\partial_x\phi)^2}_{\text{矩陣化 } \phi_i A_{ij} \phi_j})}$.

$\Leftrightarrow \int D\phi(k_1)\, D\phi(k_2) \cdots D\phi(k_N)\, e^{-\sum_k \phi_k (\cdots) \phi_k}$

指數上帶變為以某種方式對k求和.

(Jacobi = 1). $\qquad$ 系統的平移對稱性. $\qquad$ 從而可解 安家积分變為簡单積分.

$\phi(k)$性质. $\phi(x)\in\mathbb{R}$. $\underset{\mathbb{C}}{\underline{\phi(k)}}=\frac{1}{\sqrt{V}}\sum_x \underset{\mathbb{C}}{\underline{e^{i\vec{k}\cdot\vec{x}}}}\underset{\mathbb{R}}{\underline{\phi(x)}}$ $\phi^*(-k)=\phi(k)$

$$\int D\phi(k_1)\cdots D\phi(k_n)=\prod_k\int D\phi(k)\bar{\phi}(k)$$

$$\int dz\,d\bar{z}\,e^{-\frac{1}{2}Az\bar{z}}$$

$\int d\tau\,dx(\partial_\tau\phi)^2$ $\longleftrightarrow$ $\begin{matrix}x=(x,\tau)\\ k=(k,\omega)\end{matrix}$ $\phi(x)=\sum_k\phi(k)e^{ikx}\frac{1}{\sqrt{V}}$.

$=\sum_{kk'}\phi(k)\phi(k')\,\omega\omega'\,\cancel{\frac{1}{V}\int e^{ikx}e^{ik'x}dx\,d\tau}$

~~唯一不为0的解. $k'=-k$~~.

$\frac{1}{V}\int e^{ikx}e^{ik'x}dx\,d\tau=\sum_k\omega^2\bar{\phi}(k)\phi(k)$.

唯一不为0的解 $k'=-k$

(类似. $\int d\tau\,dx(\partial_x\phi)^2=\sum_k k^2\bar{\phi}_k\phi_k$)

(变为第子行列式. 变为动量空间 去计算.

$\therefore\int D\phi_1\cdots D\phi_N\,e^{-\frac{1}{2}\int d\tau\,dx((\partial_\tau\phi)^2+v^2(\partial_x\phi)^2)}$

$=\prod_k\left(\int D\bar{\phi}(k)D\phi(k)\,e^{-\frac{1}{2}(\omega^2+k^2)\bar{\phi}_k\phi_k}\right)$

传播子的计算.

$$\langle x\rangle=\frac{\int p(x)x\,dx}{\int p(x)\,dx}$$

$\langle\phi(x)\phi(y)\rangle=\sum_{kk'}\frac{1}{V}e^{i\vec{k}\cdot\vec{x}+i\vec{k}'\cdot\vec{y}}\langle\phi(k)\phi(k')\rangle$

$=\sum_{kk'}e^{ikx+ik'y}\dfrac{\int\phi(k)\phi(k')e^{-S}D\phi D\bar{\phi}}{\int e^{-S}D\phi D\bar{\phi}}$

(由于 $S=\sum_k(\omega^2+k^2)\bar{\phi}_k\phi_k$.
唯一有贡献 $-k=k'$)

$\therefore$ $=\cancel{\frac{1}{V}\sum_k e^{ik}}$

$=\frac{1}{V}\sum_k e^{ik(x-y)}\dfrac{\int\bar{\phi}(k)\phi(k)e^{-\frac{1}{2}\sum_q(\omega^2+q^2)\bar{\phi}(q)\phi(q)}D\bar{\phi}(q)D\phi(q)}{\int e^{-\frac{1}{2}\sum_q(\omega^2+q^2)\bar{\phi}(q)\phi(q)}D\bar{\phi}(q)D\phi(q)\cdots}$

$=\frac{1}{\omega^2+k^2}$