$E = f(\eta, g_n)$ Am. J. Phys.

有限值 發散、 有限、否則 $f(\ )$ 發散。

相變情況下、也會發散（相變點、關聯長度）

Path Integral.

$[x,p]=i.$ Schrödinger eq $P_{ab}(t) = \int_a^b Dx Dp\, e^{i\int_{t_0}^{t} L\,dt}$ ———— Advanced QM.

$\Downarrow$

$[a, a^+]=1.$ $H = \omega a^+ a.$ $\rightarrow$ QFT.

$\Downarrow$

$H = \omega_j a_i^+ a_j + U(a_j a_i^+) \rightarrow$

$\Downarrow$

$H = \int dx\,(\quad) \rightarrow$

場量子化/波函數量子化/正則量子化/二次量子化。

Dirac. (1927) $\psi = \sum_n C_n \varphi_n.$

$$\begin{cases} i\dot{C}_n = \dfrac{\partial E}{\partial C_n^*} \\ i\dot{C}_n^* = -\dfrac{\partial E}{\partial C_n} \end{cases} \longleftrightarrow \begin{cases} N = C_n^* C_n \\ \theta. \end{cases} \sim \quad C_n = \sqrt{N_n}\, e^{i\theta_n}.$$

Nagosa.

一維聲子模型。 Simon's book. N-Jrme's book.

$$H = \frac{1}{2} m \dot{x}_n^2 + k(x_n - x_{n+1})^2 = \underbrace{\frac{1}{2} m \dot{x}_n^2 + 2k x_n^2}_{\text{諧振子}} - \underbrace{2k\, x_n x_{n+1}}_{\text{相互作用}}$$

考慮方法一. $\hbar\omega(a_n^+ a_n + \frac{1}{2}) - 2k\, x_n x_{n+1}.$ $a_n(t) = a_n e^{-i\omega_n t}$

(一次量子化)、 泊松括號表述

$$\begin{cases} [a_n, a_m] = 0 \\ [a_m^+, a_m^+] = 0 \\ [a_n, a_m^+] = \delta_{nm} \end{cases}$$

$\cancel{x_n = (q) a_n (q^*) a_n^+}$.

$x_n = q a_n + q^* a_n^+.$

$P_n = m \dot{x}_n = \cancel{-i\omega_n t} - i\omega_n m\,[q a_n - q^* a_n^+]$

$\therefore \cancel{[x_n, x_m] = i\hbar \delta_{nm}}$

$$\therefore \begin{cases} [x_n, P_n] = i\hbar \delta_{nm} \\ [x_n, x_m] = 0 \\ [P_n, P_m] = 0 \end{cases}$$

$\sum_n f(n) = \int dx\, f(x)$

離散. 場.

$x_n = x(n) \overset{\text{def}}{=} \phi(n).$

1. 位移.
$\phi(x)$: 在 x 點振動幅度
$\phi^\dagger(x)$. 在 x 點動量.

2. $\begin{cases} [x_n, p_m] = i\hbar\delta_{nm} \\ [x_n, x_m] = 0. \\ [p_n, p_m] = 0. \end{cases}$ $\overset{\text{連續化}}{\longleftrightarrow}$ $\begin{cases} [\phi(x), \phi^\dagger(y)] = i\hbar\, \delta(x-y) \\ [\phi(x), \phi(y)] = 0. \\ [\phi^\dagger(x), \phi^\dagger(y)] = 0. \end{cases}$

求解:

$\cancel{H = \hbar\sum_n \omega(a_n^\dagger a_n + \tfrac{1}{2}) - \sum_n 2k x_n x_{n+1}}$

$\mathcal{L} = \frac{1}{2} m \dot{x}_n^2 - \left(\frac{1}{2} m\omega^2 x_n^2 - k x_n x_{n+1}\right)$

$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{a}_n} = \left(\frac{\partial \mathcal{L}}{\partial a_n}\right) \overset{\text{連續化}}{\longleftrightarrow} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}\right) = \frac{\partial \mathcal{L}}{\partial \phi(x)}.$

$\cancel{m\ddot{a}_n = -m\omega^2 a_n}$

$m\ddot{x}_n = -m\omega^2 x_n + 2k\underbrace{(x_{n+1} + x_{n-1})}_{\downarrow\ \text{連續化}}.$

$\begin{cases} x_{n+1} = x_n + x_n' a + \frac{1}{2} x_n'' a^2 \\ x_{n-1} = x_n - x_n' a + \frac{1}{2} x_n'' a^2 \end{cases}$

$\Rightarrow x_{n+1} + x_{n-1} = 2x_n + x_n'' a^2.$

$\therefore m\ddot{x}_n = -m\omega^2 x_n + 2k(2x_n + a^2 x_n'').$

↓ 連續化.

$m\ddot{\phi}_n = -m\omega^2 \phi_n + 2k\phi_n + 2ka^2 \frac{\partial^2}{\partial n^2}\phi_n$

↓

$m\ddot{\phi}(x,t) = -m\omega^2 \phi(x,t) + 2k\phi(x,t) + 2k\frac{\partial^2}{\partial x^2}\phi(x,t).$

$= 2k\frac{d^2}{dx^2}\phi(x,t).$ 聲波方程、 $\longrightarrow m\frac{\partial^2}{\partial t^2}\phi = 2k\frac{\partial^2}{\partial x^2}\phi.$

Path Integral

$\int Dx\, e^{i\int \mathcal{L}(x,\dot{x})dt}$

$\|$

$\cancel{\lim_{n\to\infty} \int Dx_1 Dx_2 \cdots Dx}$

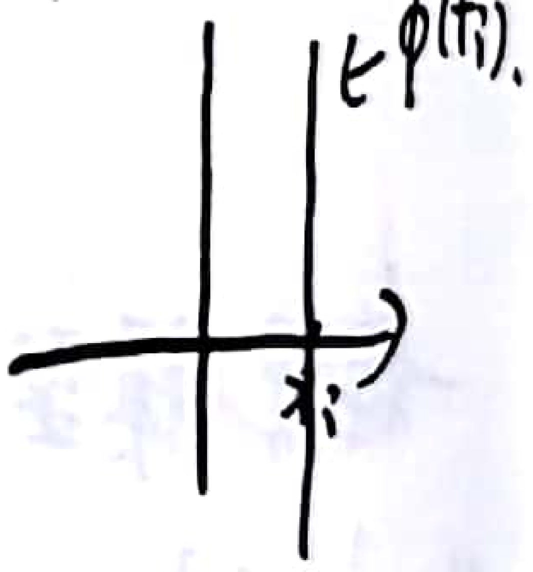
$\lim_{n\to\infty} \int Dx_1 Dx_2 \cdots Dx_n\, e^{i\int dt(\cdots)}$

↓ 離散連續化

$x_i = x(t_i) = \phi(t_i)$

$\|$

$\int D\phi\, e^{i\int \mathcal{L}(\phi,\dot{\phi})dt}.$



事實: 聲波 → 聲子 → $\epsilon_k = v|k|$.

(離散系數)

求在 x 點的振動大小的方差? $\langle x^2 \rangle = \langle (a + a^\dagger)^2 \rangle = \langle aa \rangle + \langle a^\dagger a^\dagger \rangle + \cancel{\langle a a^\dagger \rangle} + \langle a^\dagger a \rangle + \langle a a^\dagger \rangle$

$\propto (2n+1)$

$\langle \phi^\dagger(x)\phi(x)\rangle \quad \left(\phi(x) = \frac{1}{\sqrt{V}} \sum_k e^{ikx} \phi_k \right)$

$= \frac{1}{V} \langle \phi_k^\dagger \phi_{k'} \rangle e^{ix(k-k')} = \sum_k \frac{1}{V} \langle \phi_k^\dagger \phi_k \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{e^{\beta v|k|}-1} dk.$

出現問題、

$\int_0^\pi \frac{dk}{e^{\beta v k}-1} \sim \int_0^\pi \frac{dk}{\beta v k}$ 發散 (紅外發散)、

但 $\int_0^\pi \frac{d^2k}{e^{\beta v k}-1} \to \int \frac{d^2k}{k}$ 有限。

線性色散關係會導致紅外發散、不穩定 (理論上出現發散)。 (Wager-Mermin Theorem some 2D system 長程無序、

($E_k = v|k|$ 行為 Goldstone 保證了發散)

低維物理系統、出現奇異行為。

(維度低、Bose-Einstein 凝聚不會出現)。
(1D, 2D)

⇓

系统不穩定、

~~1d 平方色散 線性色散關係~~

2d $E_k = v|k|^2$ 也不穩定

解決 Thouless, Kosterlitz.

Path Integral.

$\langle q_n, t_n | q_0, t_0 \rangle = \cancel{N \int [Dq] e^{i\int_{t_0}^{t} S[q]dt}} = N\int [dq] e^{i\int_{t_0}^{t} \mathcal{L}[q]dt}$

$= \cancel{\int [Dq}$

$= \int Dq\, e^{i\int_{t_0}^{t} \mathcal{L}dt}.$

$= \lim_{n\to\infty} \int Dq_1 Dq_2 \cdots Dq_{n-1}\, e^{iS[q]}$

$\left(\begin{array}{l} \int dq\, e^{-Aq^2+Jq} \quad \int Dq\, e^{-Aq^2+Jq} = e^{\frac{J^2}{4A}} \\ \int Dq\, e^{-Aq^2} = 1. \end{array} \right)$

D_q 與 d_q 之間差一常數。

類比。

↓

PI

$\mathcal{L} = \frac{1}{2}m\dot{x}^2(t) - V(x).$

單粒子運動

多粒子.

$\int Dx_1 \cdots Dx_{n-1}\, e^{iS[x]}$

↓

∞粒子 $\int DX(t)\, e^{iS[x]}$

$\int D\phi(x,t)\, e^{i\int dx dt \mathcal{L}}$

QFT.

$\frac{1}{2}m\dot{\phi}^2(t) - V(\phi(t))$

$\int D\phi\, e^{iS[\phi]}$

$= \int D\phi(x,t)\, e^{i\int dx dt \mathcal{L}(\phi, \partial\phi)}$

$\int D\phi(x_1) \cdots D\phi(x_n)\, e^{iS[\phi]}$