重整化群

問題來源。① $\left(-\frac{\hbar^2}{2m}\nabla^2+g\delta(x)\right)\psi=E\psi$.

bond state.

if $g<0$, $E\to-\infty$.

我們作一假設，E_b 必是有限值.

② H原子 2S, 2P. 依照已有理論簡並.
或無限有大. (Lamb).
能量差.

理論(猶其是微擾)會給出無限大. 實驗卻是有限值.

與無窮大共存. 能標限定了認識.

1d. δ-potential. $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+g\delta(x)\right)\psi(x)=E\psi(x)=-|E|\psi(x)$. $E<0$. 只考慮束縛態.

標度分析. $E=(\frac{\hbar^2}{2m},g)$. $[E]=\text{energy}$

$[\frac{\hbar^2}{2m}\nabla^2]=\text{energy} \to [\frac{d^2}{dx^2}]=L^{-2}$ $\quad [\frac{\hbar^2}{2m}]=\text{Energy}\cdot L^2$.

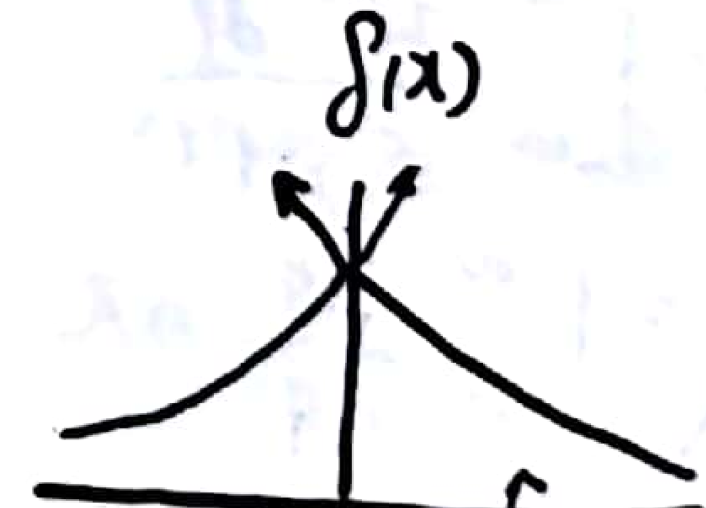
$[g\cdot\delta(x)]=\text{energy}$. $\quad \underline{[\delta(x)]=\frac{1}{L}} \to \int\delta(x)dx=1$.

$\therefore [g]=\text{energy}\cdot L$.

$\therefore E=(\frac{\hbar^2}{2m})^\alpha g^\beta \Rightarrow [E]=[E^\alpha][L^{2\alpha}][E^\beta][L^\beta]$

$\Leftrightarrow \begin{cases}2\alpha+\beta=0\\ \alpha+\beta=1\end{cases} \Rightarrow \begin{cases}\cancel{\alpha=1}\ \alpha=-1\\ \beta=2\end{cases}$

$E=\frac{2mg^2}{\hbar^2 4}$.



$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi=-|E|\psi$

右. $\psi_R=e^{-x\sqrt{\frac{2m|E|}{\hbar^2}}}\psi_0$

左. $\psi_L=e^{x\sqrt{\frac{2m|E|}{\hbar^2}}}\psi_0$

$-\frac{\hbar^2}{2m}(\psi_L'-\psi_R')=g\psi(0)$.

2d.

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+g\,\delta(\vec{x})\right)\psi(\vec{x})=E\psi(\vec{x}).$$

$\because \int \delta(\vec{x})\,dx\,dy=1.\quad \therefore [\delta(\vec{x})]=L^{-2}.$

$[\frac{\hbar^2}{2m}]\cdot[g]=[E]\cdot[L^2]$. 为了抵消[L]的影响、必要引入 $\lambda\sim[L^{-1}]$ 的参量、

$E=\frac{\hbar^2}{2m}\lambda^2 e^{-4\pi/(g/\frac{\hbar^2}{2m})}$ $\Rightarrow$ ①. g與λ無關. $E\to\infty$.

②. g與λ有關. m關係?

$g=g(\lambda)$. ← 描述g與λ關係 ← 反解.

重整化過程.

引入能標、理論已無力預言、只能解釋.

Wilson.

Reference. Su-Long Nyeo. Am J. Phy (1994)
Dalumette. A hint of renormalization (2003)

$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\lambda\delta(\vec{x})\psi=E\psi.\quad \lambda>0.$ ~~LET~~ $E<0.$

FT: $\begin{cases} f(\vec{x})=\sum_k f(k)e^{i\vec{k}\vec{x}}=\left(\frac{1}{2\pi}\right)^d\int f(k)e^{i\vec{k}\vec{x}} \\ f(\vec{k})=\int f(\vec{x})e^{-i\vec{k}\vec{x}}d\vec{x}. \end{cases}$

$\delta(x)$: $\int e^{i\vec{k}\cdot\vec{x}}d\vec{k}=2\pi\delta(x).$

$\int e^{i\vec{k}\vec{x}}dx=2\pi\delta(k).$

$\delta(x)=\frac{1}{2\pi}\int e^{ikx}dk=\sum_k e^{ikx}.$

$\int_0^{+\infty}e^{bx}dx=\frac{e^{bx}}{b}\Big|_0^{+\infty}$

$\varepsilon=-\frac{1}{b}$

$\int_{-\infty}^{\infty}\frac{2\varepsilon}{\varepsilon^2+k^2}dk$

$=\int_{-\infty}^{\infty}\frac{2\varepsilon^2\,dq}{\varepsilon^2+q^2\varepsilon^2}$

$=\int_{-\infty}^{\infty}\frac{dq}{1+q^2}=2\pi$

$\int_0^{+\infty}e^{ikx-\varepsilon x}dx+\int_{-\infty}^{0}e^{ikx+\varepsilon x}dx$

$=\frac{1}{\varepsilon-ik}+\frac{1}{\varepsilon+ik}=\frac{2\varepsilon}{\varepsilon^2+k^2}.$

$\psi(x)=\sum_k e^{ikx}\psi(k)$ 代入方程.

$\delta(x)=\sum_q e^{iqx}$

$\sum_k\frac{\hbar^2}{2m}k^2\psi(k)$ ~~$+\sum_{qk}e^{i(q+k)\cdot x}\psi(k)$~~

$-\lambda\sum_{qk}e^{i(q+k)\cdot x}\psi(k)=$ ~~E_k~~ $E\sum_k e^{ikx}\psi(k).$

$$\psi(k)\frac{\hbar^2}{2m}k^2-\lambda\sum_q\psi(k-q)=E\psi(k).$$

$$\left(\because\ \psi(0)=\sum_q e^{iq\cdot 0}\psi(q)=\sum\psi(q).\quad\therefore\right)$$

$$\psi(k)\frac{\hbar^2}{2m}k^2-\lambda\psi(0)=E\psi(k)=-B\psi(k)$$

$$\Rightarrow\ \psi(k)\left(\frac{\hbar^2}{2m}k^2+B\right)=\lambda\psi(0).$$

$$\psi(0)=\sum_k\psi(k)=\sum_k\frac{\lambda}{\frac{\hbar^2}{2m}k^2+B}\psi(0)$$

$$\Rightarrow\ \sum_k\frac{\lambda}{\frac{\hbar^2}{2m}k^2+B}=1.$$ (吸引勢有解，排斥勢無解).

對 $d=1,2,3$ 均成立

但求解有些問題

$d=1$. 有確定性的解. $\sum_k\frac{\lambda}{\frac{\hbar^2}{2m}k^2+B}=1.$ 定義 $\lambda=\lambda_0\frac{\hbar^2}{2m}$, $B=\frac{\hbar^2}{2m}B_0$

$$\Rightarrow\ \sum_k\frac{\lambda_0}{k^2+B_0}=1$$

$$\Rightarrow\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda_0}{k^2+B_0}dk=1\ \sim\int\frac{1}{k^2}dk$$

$$\Rightarrow\ E=-\frac{2mg^2}{4\hbar^2}$$

$d=2$. $\left(\frac{1}{2\pi}\right)^2\int d^2k\,\frac{\lambda_0}{k^2+B_0}=1.$

$$\Rightarrow\ 1=\underbrace{\frac{1}{2\pi}\int_0^\Lambda\frac{\lambda_0 k}{k^2+B_0}dk}.\ \sim\int_0^\Lambda\frac{\lambda_0 k}{k^2+B_0}dk+\underbrace{\int_\Lambda^\infty\frac{\lambda_0 k}{k^2+B_0}dk}_{\sim\ \lambda_0\int_\Lambda^\infty\frac{k}{k^2}dk\ \sim\ \ln(\infty)-\ln(\Lambda)}$$

發散

給定 B_0. 其結果一定發散 (若 λ 為常數).

但 B_0 又是實驗給出的結果.

$$\Rightarrow\ \frac{1}{\lambda_0}=\frac{1}{2\pi}\int_0^\Lambda\frac{k}{k^2+B_0}dk=\frac{1}{4\pi}\ln(k^2+B_0)\Big|_0^\Lambda=\frac{1}{4\pi}\ln\left(1+\frac{\Lambda^2}{B_0}\right)$$

$$\Rightarrow\ e^{\frac{4\pi}{\lambda_0}}=\frac{\Lambda^2+B_0}{B_0}\approx\frac{\Lambda^2}{B_0}\qquad(\Lambda\gg B_0).$$

$$\therefore\ B_0=\Lambda^2e^{-\frac{4\pi}{\lambda_0}}\quad\Rightarrow\ \underbrace{E=-\frac{\hbar^2}{2m}\Lambda^2e^{-\frac{4\pi}{\lambda}\left(\frac{\hbar^2}{2m}\right)}}.$$

該等式. 左邊跟 λ 無關. 右邊是 λ 的函數. 如何理解該結果?

$E = -\frac{\hbar^2}{2m}\Lambda^2 e^{-4\pi/\lambda}\cdot\left(\frac{\hbar^2}{2m}\right).$

與λ有關參數需互相抵消. ∴ $\lambda=\lambda(\Lambda)$.

重整化的目的：調整參數與λ關係、從而獲得有限的觀察值.

$\underline{\ln(-E)} = \ln\left(\frac{\hbar^2}{2m}\right) + \underline{2\ln\Lambda - 4\pi\frac{\hbar^2}{2m}\frac{1}{\lambda}}.$

Λ無關

$\frac{d\ln(-E)}{d\Lambda}=0.$ $\quad 0=\frac{2}{\Lambda}d\Lambda + 4\pi\frac{\hbar^2}{2m}\frac{1}{\lambda^2}d\lambda.$

① 直接求λ與Λ關係

② 求其微分形式.

$4\pi\frac{\hbar^2}{2m}\frac{1}{\lambda^2}d\lambda = -\frac{2}{\Lambda}d\Lambda.$

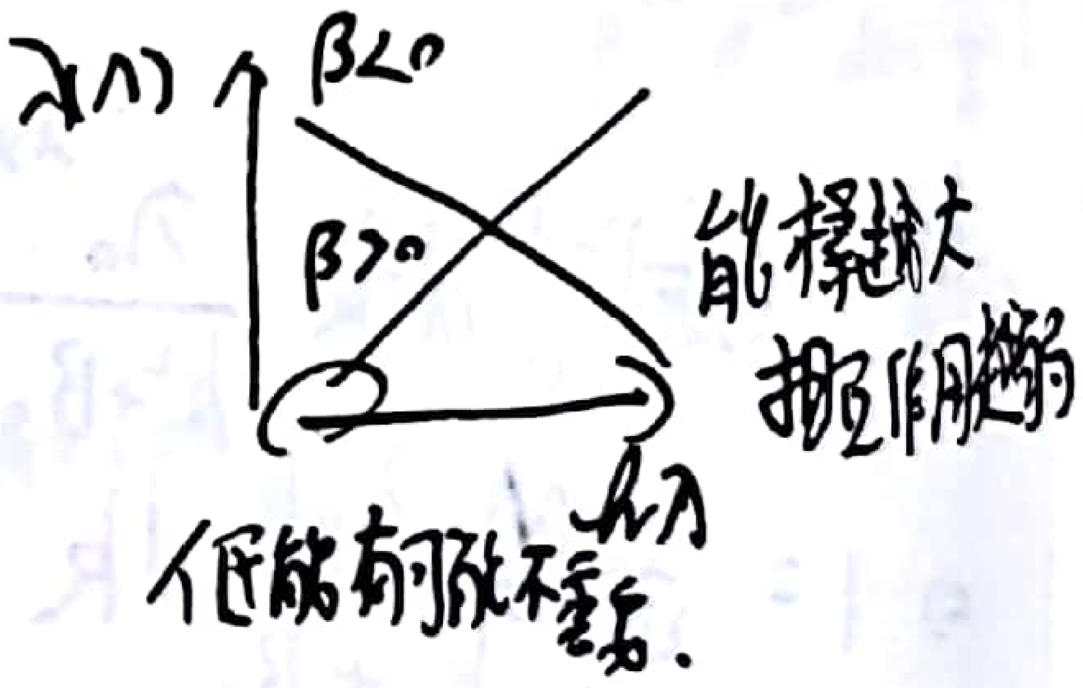
$\cancel{\Rightarrow \Lambda\frac{d\lambda}{d\Lambda} = \frac{4m\lambda^2}{4\pi\hbar^2} = \frac{m}{\pi\hbar^2}\lambda^2 < 0}$

$\Rightarrow \Lambda\frac{d\lambda}{d\Lambda} = -\frac{4m\lambda^2}{4\pi\hbar^2} = -\frac{m\lambda^2}{\pi\hbar^2} < 0.$

$\beta(\lambda) = \left(\frac{d\lambda}{d\Lambda}\right)\Lambda.$

$\beta(\lambda) = \Lambda\frac{d\lambda}{d\Lambda} = \frac{d\lambda}{d\ln\Lambda} = \beta.$

$\lambda = \lambda_0 + \beta\ln\Lambda$



$\frac{1}{\lambda_\Lambda} = \frac{1}{4\pi}\int_0^\Lambda \frac{k\,dk}{k^2+B_0}$ 重整化條件. 實驗觀測情況.

假設能標已知. $\frac{1}{4\pi}\int_0^\mu \frac{k\,dk}{k^2+B_0} = \frac{1}{\lambda_r}$ $\mu<\Lambda$.

$\frac{1}{\lambda_\Lambda}-\frac{1}{\lambda_r} = \frac{1}{4\pi}\int_\mu^\Lambda \frac{k\,dk}{k^2+B_0} = \frac{1}{4\pi}\int_\mu^\Lambda \frac{k\,dk}{k^2} = \frac{1}{4\pi}\ln\frac{\Lambda}{\mu}$

$\lambda \gg \mu \gg B_0$

∴ $\frac{1}{\lambda_\Lambda} = \underset{\text{實驗觀測}}{\frac{1}{\lambda_r}} + \frac{1}{4\pi}\ln\left(\frac{\Lambda}{\mu}\right)_\infty.$

隱含能標下的測量.

3d.

$$\frac{1}{\lambda_0} = \left(\frac{1}{2\pi}\right)^3 4\pi \int_0^\Lambda \frac{k^2 dk}{k^2+B_0} = \frac{1}{2\pi^2}\int_0^\Lambda \frac{k^2 dk}{k^2+B_0} \sim \Lambda.$$

HW. 討論3d情況.

$$\frac{1}{\lambda_0} = a\Lambda + b\ln\Lambda + \text{finite}.$$

① 微擾的、凝聚態注重低能物理、$\beta<0$ 項更為重要、能標越小,相互作用越大.

$\Lambda\to\infty$. $\lambda_0(\Lambda)\to 0$. 漸近自由 (QCD) $\beta<0$.

處理發散 ① ~~cutf~~ cutoff. ② 維度正規化. 等價.

$d=2-\epsilon$. $\epsilon\to 0$.

$d=2$.

$$\frac{1}{\lambda} = \frac{1}{4\pi}\int \frac{d^dk}{k^2+B_0} = \cancel{\frac{1}{4\pi}\Omega_d\int_0^\infty \frac{k^{d-1}dk}{k^2+B_0}}$$

$$= \frac{1}{4\pi}\underbrace{\Omega_d}_{立體角}\int_0^\infty \frac{k^{d-1}dk}{k^2+B_0} = \frac{1}{4\pi}\Omega_d\int_0^\infty \frac{k^{1-\epsilon}}{k^2+B_0}dk.$$

$$\epsilon>0. \sim \int_0^{+\infty}\frac{k^{1-\epsilon}}{k^2}dk \propto \int_0^{+\infty}\frac{1}{k^{1+\epsilon}}dk \to \text{finite}$$

$$\int_0^\Lambda \frac{d^dk}{k^2+B_0} = \int_0^\infty \frac{d^{d-\epsilon}k}{k^2+B_0}$$

$$= \underbrace{\Omega_2\int_0^\Lambda \frac{k\,dk}{k^2+B_0}}_{\ln\Lambda\ 發散} \qquad = \Omega_{2-\epsilon}\int_0^\infty \frac{k^{1-\epsilon}dk}{k^2+B_0}$$

建立ϵ與Λ關係. 討論ϵ與討論Λ 等價.

$\int_0^\infty \frac{k^{1-\epsilon}}{k^2+B}dk$ 發散、

$$= \text{finite} + \int_c^\infty \frac{k^{1-\epsilon}}{k^2}dk$$

$$= \text{finite} + \int_c^\infty \frac{dk}{k^{1+\epsilon}} \sim b/\epsilon + \text{finite}$$

$\ln\Lambda + \text{finite}$.

$\therefore \Lambda\to\infty,\ \epsilon\to 0,\quad \epsilon \propto \frac{1}{\ln\Lambda}.$

概念. ①重整化、令觀測值與能標無關，不得不引入另外一個發散抵消入的參數.

②. β函數. 耦合常數 $\beta<0$. 漸近自由. 能標越小. 相互作用越大.

$\beta>0$. 能標小. 相互作用小. 可忽略.

③. 正規化. $\varepsilon \Leftrightarrow \Lambda$. $\Lambda\to\infty \Leftrightarrow \varepsilon\to 0$.

HW. 討論 2d $\delta(x)$ 在
很度正規化方法下的
結果.

以下模型. ϕ^4 理論(核心)

linear σ model

non linear σ model.

XY Model.

O(N) model.

S-G model

Fermi Gas Model.

B-H model

Ginzburg-Landau Model

Kondo Model.

發散指認，聚態無處不在.