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Discrete Mass, Elementary Length, and a Topological Invariant as a Consequence of a Relativistic Invariant Variational Principle

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A nonlinear, Lorentz-invariant action principle is introduced. This variation principle, which deals with an "internal" angular variable θ, is obtained from analogy considerations of moving Bloch walls in magnetic crystals and of elementary particles. In the one-dimensional case the resulting Euler equations can be solved by elementary functions, describing, in the rest system, a spatially extended energy distribution. The total energy, and thus the mass, has a discrete character. An elementary length appears as a parameter of this energy distribution, i.e., as a parameter of the structure. The moving structure undergoes the appropriate Lorentz contraction. An invariant of topological nature comparable to the invariant of the Möbius strip is furthermore inherent in this structure. This invariant has the symmetry properties of the elementary charge. A solution of the three-dimensional case including an internal rotation seems possible.

THE concept of a point-like elementary particle being the subject of dynamics and nothing else is, at present, widely, but not generally, accepted. According to this concept the discrete mass and charge of the particle, as well as its stability, must be considered as empirical facts. The alternative view of a particle extending in space[1] is becoming more attractive. This view will also be held in this paper. In the following, an attempt will be made to find an explanation for the outstanding fact of the discreteness of the charge and the mass, as well as for the stability of the particle.

Classical dynamics is not adequate for a deeper insight into the particles, as this theory describes the motion of *already existing* mass points in space and time. It is important to realize that the same applies to quantum mechanics, which deals with the dynamical properties of already given, *point-like particles*. From this it is concluded that quantum theory in its present form cannot contribute to the solution of the problem of the elementary particle. Moreover, a *nonlinear* theory is certainly required for a proper description of a stable particle, because a linear theory would allow for superpositions, which is contrary to the idea of an integral stable particle.

In classical as well as in quantum mechanics there is a divergence problem associated with point-like particles.

In the latter theory this forms an even more severe problem.[2] This divergence problem is an indication that the idea of an elementary particle extending in space should be favored, i.e., a particle with a structure.

In this paper such a structure, described in the rest system, is considered to be the result of a variational principle of the energy density in space. This means that the stable elementary particle is the entity of lowest energy with respect to any variation of the parameters involved. The energy principle is, thus, considered to be even *more* fundamental than the existence of elementary particles: It governs their constitution.

An indication of how such a theory may be built up is given by considerations based on an analogy between the properties of elementary particles and those of Bloch walls in magnetic crystals. It is believed that this far-reaching analogy, which includes statistical and dynamical aspects, is deep enough to serve as the basis for a model of a stable elementary particle. This analogy will be described elsewhere in more detail.[3]

The Bloch wall is a region in a crystal of nonzero width separating domains with different orientation of the magnetization. Inside the wall, the magnetization gradually changes direction, e.g., in an uniaxial crystal from, say, a direction parallel to the c axis to an antiparallel one. A Bloch wall is able to move, and a kinetic energy and a mass[4,5] can be ascribed to it. The stability of the Bloch wall is due to a minimization of two energy contributions, magnetic anisotropy energy and exchange energy.

In close analogy to the magnetic case, "anisotropy energy" and "exchange energy" are now introduced in space. The anisotropy energy density is given by

$$F_K = K \sin^2\theta\,, \tag{1}$$

where K[erg cm^{-3}] is a constant and θ is an angular variable depending on the coordinates x, y, z, and t. θ describes the rotation of a unit vector n in an auxiliary two-dimensional space u,v. θ is not an angle in physical space and, therefore, is not subjected to coordinate transformations. The introduction of this extra variable θ might be compared with the introduction of the spin variable in early quantum mechanics. Internal variables not playing a symmetric role, neither among each other, nor with respect to space-time, have often been introduced in elementary-particle physics.

The expression corresponding to the exchange energy in the magnetic case is extended by introducing a time-

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A268**, 57 (1962).
[2] P. A. M. Dirac, Proc. Roy. Soc. (London) **A209**, 291 (1951).

[3] U. Enz (to be published).
[4] W. Döring, Z. Naturforsch. **39**, 373 (1948).
[5] C. P. Bean and R. W. Deblois, Bull. Am. Phys. Soc. **4**, 53 (1959).

like term and may be written as

$$F_A=A\left[\left(\frac{\partial\theta}{\partial x}\right)^2+\left(\frac{\partial\theta}{\partial y}\right)^2+\left(\frac{\partial\theta}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial\theta}{\partial t}\right)^2\right]. \quad (2)$$

The generalization to four coordinates assures the Lorentz invariance of the theory. A is a constant with the dimensions of energy per unit length. The following variational principle forms the base of the theory:

$$\delta\int(F_A+K\sin^2\theta)\,dx\,dy\,dz\,dt=0\,. \quad (3)$$

This equation defines in the rest system the functions θ minimizing the total energy. In the general case it represents an action principle. Equation (3) leads to the Euler equation,

$$\Delta\theta-\frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2}=\frac{K}{2A}\sin2\theta\,. \quad (4)$$

Let us first consider the two-dimensional case where θ depends on x and t only, so that (4) reduces to

$$\frac{\partial^2\theta}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2}=\frac{K}{2A}\sin2\theta\,. \quad (5)$$

In the case $\partial^2\theta/\partial t^2=0$, this represents the well-known equation[4] governing the structure of the Bloch wall. Such an equation has also been used by Finkelstein and Misner.[6] Solutions of this equation in the rest system corresponding to a minimum of the energy are $\theta=n\pi$, $n=0,1,2,\cdots$, and

$$\sin\theta=\pm[\cosh(K/A)^{1/2}(x+x_a)]^{-1}. \quad (6)$$

The trivial solutions correspond to a vanishing energy density in whole space, i.e., to the vacuum. The nontrivial solutions represent a rotation of the vector n from 0° to 180°, the signs indicating the sense of rotation. One sign corresponds to a left-handed screw, the other to a right-handed one. The general time-independent solution of (5), consisting of elliptical functions, does not represent a true minimum of (3). A length

$$x_0=\int_{-\infty}^{+\infty}\sin\theta dx=\pi\left(\frac{A}{K}\right)^{1/2} \quad (7)$$

may be defined denoting the region on the x axis where the rotation takes place and where the energy density differs essentially from zero. The length x_0 is a parameter of the structure (6) and can be considered as an *elementary length*. An energy surface density

$$E_s=4(AK)^{1/2} \quad (8)$$

is obtained by introducing (6) into (3). A mass

$$m_s=E_s/c^2=\frac{4}{c^2}(AK)^{1/2}, \quad (9)$$

localized in the region x_0, may be attributed to the structure (6). The discreteness of this mass is a consequence of the fact that (6) is deduced from a variational principle.

An invariant of topological nature is inherent in the structure (6). This invariant is closely connected to the twist of the Möbius strip and may be described as the total angle of rotation $\pm\pi$ of θ.[6] The invariant $+\pi$ corresponds to a structure with a right-handed screw sense and $-\pi$ to one with a left-handed screw sense. This invariant $\pm\pi$, which is of great importance in the present model, has the symmetry of the electric charge. A spatial mirror symmetry exists between two structures with different sign of the invariant.

Thus, we arrive at the following picture of a one-dimensional particle: The energy density in space is zero except in a region of the order of magnitude x_0, where a discrete amount of energy and, thus, mass is concentrated. This amount of mass (9) is free to move along the x axis, as the solutions (6) of the variational principle are invariant with respect to a transformation $x\rightarrow x+x_a$.

The invariance of (5) with respect to the special Lorentz transformation assures the conservation of momentum and energy. It is, therefore, possible to define the proper relativistic dynamics in the x-t plane. From (5) one obtains the following relations for the moving particle:

$$\sin\theta=\pm\left[\cosh\frac{x-vt}{x_v}\right]^{-1};\quad x_v=x_0(1-\beta^2)^{1/2}, \quad (10)$$

and from this

$$m(v)=\frac{4}{c^2}(AK)^{1/2}(1-\beta^2)^{-1/2}, \quad (11)$$

where $\beta=v/c$.

The present model is invariant with respect to the transformation CP, if a right-handed structure (6) having the invariant $+\pi$ is compared with the electron and a left-handed one having the invariant $-\pi$ is compared with the positron. Mirroring then transforms a particle into its antiparticle. Two particles with different invariants attract and finally annihilate each other, whereas two structures with the same invariant repel each other. This corresponds to the annihilation of a positron-electron pair. It should be noted, however, that in this stage of the theory no attempt is made to identify a certain particle with the present model. A comparison of the invariant $\pm\pi$ with the baryon number is also possible. The essential features of the present model are

[6] D. Finkelstein and C. W. Misner, Ann. Phys. (N. Y.) **6**, 230 (1959).

TABLE I. Elements of the model of a one-dimensional particle.

K	"Anisotropy" of the auxiliary space u, v
A	"Exchange constant"
θ	Hypothetical angular variable
x_0	Elementary length (7)
E_s	Energy of the wall per cm^2 (8)
m_s	Mass per cm^2 (9)
$\pm\pi$	Invariant inherent in (6) of a topological nature (Möbius strip) having the symmetry properties of, e.g., the electric charge

given in Table I. It should be realized, however, that concepts such as electromagnetic field and spin do not appear in this one-dimensional case. The fact that a rather high number of essential properties of a stable particle is obtained from the two simple assumptions (1) and (2) makes us believe that along these lines of thinking a three-dimensional model including field and spin may be obtained. This generalization starts from Eq. (4). Solutions are planar "walls" equivalent to (6) with an arbitrary orientation in space. These solutions have an additional invariance with respect to a rotation in space. It is interesting to note that spherically symmetrical, time-independent solutions of (4) representing a minimum energy do not exist. The generalized model includes a random rotation of the "wall" with a mean angular velocity ω. By this rotation, a second elementary length $r_c = c/\omega$ having a statistical character is introduced. x_0 will be identified with the classical electron diameter and r_c with the Compton wavelength. The invariant of topological nature is expected to be retained in the three-dimensional case.

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