Comments on Nonlinear Wave Equations as Models for Elementary Particles

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tion of motion for $\langle q^2\rangle$ and we obtain

$$i\hbar(\partial/\partial t)\langle q^2\rangle + \tfrac{1}{2}\,\mathrm{tr}_a\, q^2[R, p^2] = N\gamma\langle\mu\rangle\,\mathrm{tr}_a\, q^2[p, R],$$

$$i\hbar(\partial/\partial t)\langle q^2\rangle - i\hbar\langle qp + pq\rangle = N\gamma\langle\mu\rangle\,\mathrm{tr}_a\, q^2[p, R],$$

which becomes

$$(\partial/\partial t)\langle q^2\rangle - \langle qp + pq\rangle = 2N\gamma\langle\mu\rangle\langle q\rangle. \qquad \text{(A4)}$$

In Sec. IV we need the equation of motion for $\langle pq\rangle$ which can be obtained from $\langle qp + pq\rangle$ since

$$\langle qp + pq\rangle = i\hbar + 2\langle qp\rangle.$$

We find the equation of motion for $\langle pq + pq\rangle$ by multiplying Eq. (3.10) by $(pq + qp)$ and taking the trace. The result is

$$(\partial/\partial t)\langle qp + pq\rangle - 2(\langle p^2\rangle - \Omega^2\langle q^2\rangle) = 2N\gamma\langle\mu\rangle\langle p\rangle. \qquad \text{(A5)}$$

Equations (A3), (A4), and (A5) constitute three first-order, linear inhomogeneous equations for the three second moments of the electromagnetic field, $\langle p^2\rangle$, $\langle q^2\rangle$, and $\langle pq\rangle$. In general, if we go to nth order in $\gamma N\gamma$ we find a set of first-order inhomogeneous linear equations for the nth moments. The inhomogeneities depend on moments of lower order than the nth.

It is important to note that even if all the second moments are zero initially they will grow to nonzero values because the inhomogeneous terms depend on $\langle p\rangle$, $\langle q\rangle$, and $\langle\mu\rangle$.

The eigenfrequencies of the homogeneous equations for $\langle q^2\rangle$, $\langle p^2\rangle$, and $\langle pq\rangle$ are 0, $\pm 2i\Omega$. Since these frequencies are prominent in the inhomogeneous terms, the second moments are strongly coupled to the SCFA quantities $\langle p\rangle$, $\langle q\rangle$, and $\langle\mu\rangle$.

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Comments on Nonlinear Wave Equations as Models for Elementary Particles

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It is shown that for a wide class of nonlinear wave equations there exist no stable time-independent solutions of finite energy. The possibility is considered whether elementary particles might be oscillating solutions of some nonlinear wave equation, in which the wavefunction is periodic in the time though the energy remains localized.

1. INTRODUCTION

In an attempt to find a model for *extended* elementary particles, as opposed to singular *point* particles, Enz[1] has recently considered the nonlinear equation

$$\nabla^2\theta - (1/c^2)(\partial^2\theta/\partial t^2) = \tfrac{1}{2}\sin 2\theta, \qquad (1)$$

which is derived from the variation principle

$$\delta\int\left[\frac{1}{c^2}\left(\frac{\partial\theta}{\partial t}\right)^2 - (\nabla\theta)^2 - \sin^2\theta\right]d^3\mathbf{r}\,dt = 0. \qquad (2)$$

$\theta(\mathbf{r}, t)$ is a c-number wavefunction which is required to be free of singularities for all $\mathbf{r}$ and t. In the one-dimensional case (∇^2 replaced by $\partial^2/\partial x^2$) Enz showed that (1) has time-independent solutions where the energy is localized about a point on the x axis; if we further require that the solution be stable with respect to small deformations then only certain discrete energy values are permitted. In addition these one-dimensional solutions possess certain symmetry and topological properties which Enz suggests might correspond in the three-dimensional case to such discrete quantum numbers as charge or parity.

These suggestive results of Enz for the one-dimensional case then lead us to consider the following problem: Can (1) or some similar nonlinear equation have stable, time-independent, localized solutions in three dimensions? If such solutions exist then it would be an attractive hypothesis that the allowed energies correspond to the rest energies of elementary particles.

The answer given to the above question by this

[1] U. Enz, Phys. Rev. 131, 1392 (1963). We have taken Enz's constants K and A both equal to 1, which amounts to a suitable choice of units of length and energy.

paper is *no*. The equation

$$\nabla^2\theta - (1/c^2)(\partial^2\theta/\partial t^2) = \tfrac{1}{2}f'(\theta), \quad (3)$$

derived from the variation principle

$$\delta \int \left[\frac{1}{c^2}\left(\frac{\partial\theta}{\partial t}\right)^2 - (\nabla\theta)^2 - f(\theta)\right] d^3\mathbf{r}\, dt = 0, \quad (4)$$

will be proved to have no stable, time-independent, localized solutions for any $f(\theta)$. In particular, Enz's equation (1) with $f(\theta) = \sin^2\theta$ has no such solutions. By "localized" solution we shall mean one where $\int (\nabla\theta)^2 d^3\mathbf{r}$ and $\int f(\theta)\, d^3\mathbf{r}$ converge when the integrals are taken over all space.

2. PROOF

If θ is a function of $\mathbf{r}$ only, we can replace (4) by $\delta E = 0$ with the energy E given by

$$E = \int [(\nabla\theta)^2 + f(\theta)]\, d^3\mathbf{r}.$$

A necessary condition for the solution to be stable is that the second-order variation $\delta^2 E \geq 0$. Suppose $\theta(\mathbf{r})$ is a localized solution of $\delta E = 0$. Define $\theta_\lambda(\mathbf{r}) = \theta(\lambda\mathbf{r})$ where λ is an arbitrary constant, and write $I_1 = \int (\nabla\theta)^2 d^3\mathbf{r}$, $I_2 = \int f(\theta)\, d^3\mathbf{r}$.
Then

$$E_\lambda = \int [(\nabla\theta_\lambda)^2 + f(\theta_\lambda)]\, d^3\mathbf{r} = I_1/\lambda + I_2/\lambda^3$$

on changing the variable of integration from $\mathbf{r}$ to $\lambda\mathbf{r}$; whence

$$(dE_\lambda/d\lambda)_{\lambda=1} = -I_1 - 3I_2,$$
$$(d^2E_\lambda/d\lambda^2)_{\lambda=1} = 2I_1 + 12I_2.$$

Since θ_λ is a solution of $\delta E = 0$ for $\lambda = 1$, we must have

$$(dE_\lambda/d\lambda)_{\lambda=1} = 0, \qquad I_2 = -\tfrac{1}{3}I_1,$$
$$(d^2E_\lambda/d\lambda^2)_{\lambda=1} = -2I_1 < 0.$$

That is, $\delta^2 E < 0$ for a variation corresponding to a uniform stretching of the "particle." Hence the solution $\theta(\mathbf{r})$ is unstable, proving the theorem.

In the above proof no restriction was placed on the sign of $f(\theta)$. In Enz's equation (1) we have $f(\theta) = \sin^2\theta \geq 0$, which means that the energy density has the desirable feature of being everywhere positive. However it is interesting to note that if $f(\theta) \geq 0$ then $\delta E = 0$ has no nontrivial localized solutions *at all*, either stable or unstable. For in this case both I_1 and I_2 are necessarily nonnegative so that $I_1 + 3I_2 = 0$ has only the trivial solution $I_1 = I_2 = 0$, giving $\theta = 0$. (Our result here is not applicable to the one-dimensional case where Enz does obtain stable solutions. In one dimension we obtain $E_\lambda = \lambda I_1 + I_2/\lambda$ yielding $I_1 = I_2$ on differentiation, which gives no contradiction.)

We can easily extend the above proof to certain cases where we have a complex, multicomponent wavefunction ψ^A rather than the real scalar function θ; the superscript A denotes some tensor or spinor index. For example we can carry through the above proof for wave equations derived from the variation principle

$$\delta \int [\sum_{\substack{AB\\ \iota\kappa}} c_{AB}\, g^{\iota\kappa}(\partial\psi^{*A}/\partial x^\iota)(\partial\psi^B/\partial x^\kappa) - f(\psi^{*A}, \psi^B)]\, d^4x = 0,$$

where c_{AB} is an arbitrary positive definite Hermitian matrix, and $g^{\iota\kappa}$ the usual metric tensor ($\iota, \kappa = 0, 1, 2, 3$). If c_{AB} is not definite, or if the coefficients of $(\partial\psi^{*A}/\partial x^\iota)(\partial\psi^B/\partial x^\kappa)$ are not of the simple product form $c_{AB}g^{\iota\kappa}$, then the condition for stability is no longer $\delta^2 E \geq 0$ and the proof fails.

3. DISCUSSION

We are thus faced with the disconcerting fact that no equation of type (4) has any time-independent solutions which could reasonably be interpreted as elementary particles. Some possible ways out of this difficulty are:

(a) We could take a Lagragian in which the derivatives occur in higher powers than the second. For example, with the form $[(\nabla\theta)^2 - (1/c^2)(\partial\theta/\partial t)^2]^n$ the nonexistence proof of Sec. 2 fails for $n > \frac{3}{2}$. Such a Lagrangian, however, leads to a very complicated differential equation.

(b) We could consider first-order spinor equations, such as

$$\delta \int [i\psi^\dagger(\partial\psi/\partial t + c\boldsymbol{\alpha}\cdot\nabla\psi) - f(\psi^\dagger, \psi)]\, d^3\mathbf{r}\; dt = 0, \quad (5)$$

where ψ is a Dirac 4-component spinor and $\psi^\dagger$ its Hermitian conjugate, $\boldsymbol{\alpha}$ is the usual Dirac matrix, and $f(\psi^\dagger, \psi)$ is an arbitrary Lorentz-invariant function. With a first-order equation of this type, the condition for stability is no longer $\delta^2 E \geq 0$, but is now very complicated, and the author has been unable to prove or disprove the existence of stable time-independent solutions of (5) for general functions $f(\psi^\dagger, \psi)$.

(c) Quantization of the field equations by replacing the wavefunction by an operator satisfying some postulated commutation relations. Quantized equations of type (5) have been investigated extensively by Heisenberg *et al.*,[2] who find particle-like solutions.

(d) Elementary particles might correspond to stable, localized solutions which are *periodic* in time, rather than time-independent.

We shall confine ourselves here to a consideration of Possibility (d), that elementary particles are oscillating localized concentrations of energy. We know experimentally[3] that a particle of momentum $\mathbf{p}$ has an associated de Broglie[4] wavevector $\mathbf{k} = \mathbf{p}/\hbar$; relativistic invariance then suggests that a particle of mass m at rest should have a de Broglie frequency $\omega = mc^2/\hbar$. If elementary particles correspond to stable periodic solutions of some nonlinear wave equation, then we could possibly identify the frequency of this oscillation with the de Broglie frequency.

A particularly simple form of periodic solution is one where the structure rotates at a constant angular velocity ω about a fixed direction, say the Z axis; i.e., the wavefunction is a function of x', y', z', where

$$x' = x\cos\omega t + y\sin\omega t,$$
$$y' = -x\sin\omega t + y\cos\omega t,$$
$$z' = z.$$

Then the variation principle (4) is equivalent to

$$\delta\int\left[(\nabla'\theta)^2 - \frac{\omega^2}{c^2}|L\theta|^2 + f(\theta)\right]d^3\mathbf{r}' = 0, \qquad (6)$$

where

$$L = -i[x'(\partial/\partial y') - y'(\partial/\partial x')].$$

However the condition for stability of solutions is now very complicated, and the author has been unable to demonstrate either the existence or nonexistence of stable solutions of Eq. (6).

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