ample of a translationally invariant tight-binding model[35] which realizes the QH effect without Landau levels. We discuss the procedure of dimensional reduction, from which all topological effects of the $(1+1)$-dimensional insulators can be obtained. This section serves as a simple pedagogical example for the more complex case of the TRI insulators presented in Sec. III and IV.

A. First Chern number and topological response function in (2+1)-dimensions

In general, the tight-binding Hamiltonian of a $(2+1)$-dimensional band insulator can be expressed as

$$H = \sum_{m,n;\alpha,\beta} c^\dagger_{m\alpha} h^{\alpha\beta}_{mn} c_{n\beta} \tag{1}$$

with m,n as the lattice sites and $\alpha,\beta=1,2,\ldots,N$ as the band indices for a N-band system. With translation symmetry $h^{\alpha\beta}_{mn}=h^{\alpha\beta}(\vec{r}_m-\vec{r}_n)$, the Hamiltonian can be diagonalized in a Bloch wave function basis,

$$H = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} h^{\alpha\beta}(\mathbf{k}) c_{\mathbf{k}\beta}. \tag{2}$$

The minimal coupling to an external electromagnetic field is given by $h^{\alpha\beta}_{mn}\to h^{\alpha\beta}_{mn}e^{iA_{mn}}$, where A_{mn} is a gauge potential defined on a lattice link with sites m,n at the end. To linear order, the Hamiltonian coupled to the electromagnetic field is obtained as

$$H \simeq \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} h(\mathbf{k}) c_{\mathbf{k}} + \sum_{\mathbf{k},\mathbf{q}} A^i(-\mathbf{q}) c^\dagger_{\mathbf{k}+\mathbf{q}/2} \frac{\partial h(\mathbf{k})}{\partial k_i} c_{\mathbf{k}-\mathbf{q}/2}$$

with the band indices omitted. The dc response of the system to external field $A^i(\mathbf{q})$ can be obtained by the standard Kubo formula,

$$\sigma_{ij} = \lim_{\omega\to 0} \frac{i}{\omega} Q_{ij}(\omega + i\delta),$$

$$Q_{ij}(i\nu_m) = \frac{1}{\Omega\beta}\sum_{\mathbf{k},n} \mathrm{tr}\{J_i(\mathbf{k})G[\mathbf{k},i(\omega_n+\nu_m)]\cdot J_j(\mathbf{k})G(\mathbf{k},i\omega_n)\}, \tag{3}$$

with the dc $J_i(\mathbf{k})=\partial h(\mathbf{k})/\partial k_i$, $i,j=x,y$, Green's function $G(\mathbf{k},i\omega_n)=[i\omega_n-h(\mathbf{k})]^{-1}$, and Ω as the area of the system. When the system is a band insulator with M fully occupied bands, the longitudinal conductance vanishes, i.e., $\sigma_{xx}=0$, as expected, while σ_{xy} has the form shown in Ref. 5,

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi}\int dk_x \int dk_y f_{xy}(\mathbf{k}), \tag{4}$$

with

$$f_{xy}(\mathbf{k}) = \frac{\partial a_y(\mathbf{k})}{\partial k_x} - \frac{\partial a_x(\mathbf{k})}{\partial k_y},$$

$$a_i(\mathbf{k}) = -i\sum_{\alpha\in\mathrm{occ}} \langle \alpha\mathbf{k}|\frac{\partial}{\partial k_i}|\alpha\mathbf{k}\rangle, \quad i=x,y.$$

Physically, $a_i(\mathbf{k})$ is the $U(1)$ component of Berry's phase gauge field (adiabatic connection) in momentum space. The quantization of the first Chern number,

$$C_1 = \frac{1}{2\pi}\int dk_x \int dk_y f_{xy}(\mathbf{k}) \in \mathbb{Z}, \tag{5}$$

is satisfied for any continuous states $|\alpha\mathbf{k}\rangle$ defined on the BZ.

Due to charge conservation, the QH response $j_i=\sigma_H\varepsilon^{ij}E_j$ also induces another response equation,

$$j_i = \sigma_H \epsilon^{ij} E_j \Rightarrow \frac{\partial \rho}{\partial t} = -\nabla\cdot\mathbf{j} = -\sigma_H \nabla\times\mathbf{E} = \sigma_H\frac{\partial B}{\partial t} \tag{6}$$

$$\Rightarrow \rho(B) - \rho_0 = \sigma_H B, \tag{7}$$

where $\rho_0=\rho(B=0)$ is the charge density in the ground state. Equations (6) and (7) can be combined together in a covariant way,

$$j^\mu = \frac{C_1}{2\pi}\epsilon^{\mu\nu\tau}\partial_\nu A_\tau, \tag{8}$$

where $\mu,\nu,\tau=0,1,2$ are temporal and spatial indices. Here and below we will take the units $e=\hbar=1$ so that $e^2/h=1/2\pi$.

The response [Eq. (8)] can be described by the topological Chern-Simons field theory of the external field A_μ,

$$S_{\mathrm{eff}} = \frac{C_1}{4\pi}\int d^2x \int dt A_\mu \epsilon^{\mu\nu\tau}\partial_\nu A_\tau, \tag{9}$$

in the sense that $\delta S_{\mathrm{eff}}/\delta A_\mu=j^\mu$ recovers the response [Eq. (8)]. Such an effective action is topologically invariant, in agreement with the topological nature of the first Chern number. All topological responses of the QH state are contained in the Chern-Simons theory.[8]

B. Example: Two band models

To make the physical picture clearer, the simplest case of a two-band model can be studied as an example.[35] The Hamiltonian of a two-band model can be generally written as

$$h(\mathbf{k}) = \sum_{a=1}^{3} d_a(\mathbf{k})\sigma^a + \epsilon(\mathbf{k})\mathbb{I}, \tag{10}$$

where $\mathbb{I}$ is the 2×2 identity matrix and σ^a are the three Pauli matrices. Here we assume that the σ^a represent a spin or pseudospin degree of freedom. If it is a real spin then the σ^a are thus odd under time reversal. If the $d_a(\mathbf{k})$ are odd in $\mathbf{k}$ then the Hamiltonian is time-reversal invariant. However, if any of the d_a contain a constant term then the model has explicit time-reversal symmetry breaking. If the σ^a is a pseudospin then one has to be more careful. Since, in this case, $\mathcal{T}^2=1$ then only σ^y is odd under time reversal (because it is imaginary) while σ^x,σ^z are even. The identity matrix is even under time reversal and $\varepsilon(\mathbf{k})$ must be even in $\mathbf{k}$ to preserve