11

CHARACTERISTIC CLASSES

Given a fibre F, a structure group G and a base space M, we may construct many fibre bundles over M, depending on the choice of the transition functions. Natural questions we may ask ourselves are how many bundles there are over M with given F and G, and how much they differ from a trivial bundle $M \times F$. For example, we observed in section 10.5 that an SU(2) bundle over S^4 is classified by the homotopy group $\pi_3(\mathrm{SU}(2)) \cong \mathbb{Z}$. The number $n \in \mathbb{Z}$ tells us how the transition functions twist the local pieces of the bundle when glued together. We have also observed that this homotopy group is evaluated by integrating $\mathrm{tr}\,\mathcal{F}^2 \in H^4(S^4)$ over S^4, see theorem 10.7.

Characteristic classes are subsets of the cohomology classes of the base space and measure the *non-triviality* or *twisting* of a bundle. In this sense, they are *obstructions* which prevent a bundle from being a trivial bundle. Most of the characteristic classes are given by the de Rham cohomology classes. Besides their importance in classifications of fibre bundles, characteristic classes play central roles in index theorems.

Here we follow Alvalez-Gaumé and Ginsparg (1984), Eguchi *et al* (1980), Gilkey (1995) and Wells (1980). See Bott and Tu (1982), Milnor and Stasheff (1974) for more mathematical expositions.

11.1 Invariant polynomials and the Chern–Weil homomorphism

We give here a brief summary of the de Rham cohomology group (see chapter 6 for details). Let M be an m-dimensional manifold. An r-form $\omega \in \Omega^r(M)$ is *closed* if $\mathrm{d}\omega = 0$ and *exact* if $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{r-1}(M)$. The set of closed r-forms is denoted by $Z^r(M)$ and the set of exact r-forms by $B^r(M)$. Since $\mathrm{d}^2 = 0$, it follows that $Z^r(M) \supset B^r(M)$. We define the rth de Rham cohomology group $H^r(M)$ by

$$H^r(M) \equiv Z^r(M)/B^r(M).$$

In $H^r(M)$, two closed r-forms ω_1 and ω_2 are identified if $\omega_1 - \omega_2 = \mathrm{d}\eta$ for some $\eta \in \Omega^{r-1}(M)$. Let M be an m-dimensional manifold. The formal sum

$$H^*(M) \equiv H^0(M) \oplus H^1(M) \oplus \cdots \oplus H^m(M)$$

is the cohomology ring with the product $\wedge : H^*(M) \times H^*(M) \to H^*(M)$ induced by $\wedge : H^p(M) \times H^q(M) \to H^{p+q}(M)$. Let $f : M \to N$ be a smooth map. The pullback $f^* : \Omega^r(N) \to \Omega^r(M)$ naturally induces a linear map $f^* : H^r(N) \to H^r(M)$ since f^* commutes with the exterior derivative: $f^*\,\mathrm{d}\omega = \mathrm{d}f^*\omega$. The pullback f^* preserves the algebraic structure of the cohomology ring since $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

11.1.1 Invariant polynomials

Let $M(k, \mathbb{C})$ be the set of complex $k \times k$ matrices. Let $S^r(M(k, \mathbb{C}))$ denote the vector space of symmetric r-linear $\mathbb{C}$-valued functions on $M(k, \mathbb{C})$. In other words, a map

$$\tilde{P} : \overset{r}{\otimes}\, M(k, \mathbb{C}) \to \mathbb{C}$$

is an element of $S^r(M(k, \mathbb{C}))$ if it satisfies, in addition to linearity in each entry, the symmetry

$$\begin{aligned}\tilde{P}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_r) \\ = \tilde{P}(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_r) \qquad 1 \le i, j \le r\end{aligned} \tag{11.1}$$

where $a_p \in \mathrm{GL}(k, \mathbb{C})$. Let

$$S^*(M(k, \mathbb{C})) \equiv \bigoplus_{r=0}^{\infty} S^r(M(k, \mathbb{C}))$$

denote the formal sum of symmetric multilinear $\mathbb{C}$-valued functions. We define a product of $\tilde{P} \in S^p(M(k, \mathbb{C}))$ and $\tilde{Q} \in S^q(M(k, \mathbb{C}))$ by

$$\begin{aligned}\tilde{P}\tilde{Q}(X_1, \ldots, X_{p+q}) \\ = \frac{1}{(p+q)!} \sum_P \tilde{P}(X_{P(1)}, \ldots, X_{P(p)}) \tilde{Q}(X_{P(p+1)}, \ldots, X_{P(p+q)})\end{aligned} \tag{11.2}$$

where P is the permutation of $(1, \ldots, p+q)$. $S^*(M(k, \mathbb{C}))$ is an algebra with this multiplication.

Let G be a matrix group and $\mathfrak{g}$ its Lie algebra. In practice, we take $G = \mathrm{GL}(k, \mathbb{C}), \mathrm{U}(k)$ or $\mathrm{SU}(k)$. The Lie algebra $\mathfrak{g}$ is a subspace of $M(k, \mathbb{C})$ and we may consider the restrictions $S^r(\mathfrak{g})$ and $S^*(\mathfrak{g}) \equiv \bigoplus_{r \ge 0} S^r(\mathfrak{g})$. $\tilde{P} \in S^r(\mathfrak{g})$ is said to be invariant if, for any $g \in G$ and $A_i \in \mathfrak{g}$, $\tilde{P}$ satisfies

$$\tilde{P}(\mathrm{Ad}_g A_1, \ldots, \mathrm{Ad}_g A_r) = \tilde{P}(A_1, \ldots, A_r) \tag{11.3}$$

where $\mathrm{Ad}_g A_i = g^{-1} A_i g$. For example,

$$\begin{aligned}\tilde{P}(A_1, A_2, \ldots, A_r) &= \mathrm{str}(A_1, A_2, \ldots, A_r) \\ &\equiv \frac{1}{r!} \sum_P \mathrm{tr}(A_{P(1)}, A_{P(2)}, \ldots, A_{P(r)})\end{aligned} \tag{11.4}$$

is symmetric, r-linear and invariant, where 'str' stands for the **symmetrized trace** and is defined by the last equality. The set of G-invariant members of $S^r(\mathfrak{g})$ is denoted by $I^r(G)$. Note that $\mathfrak{g}_1 = \mathfrak{g}_2$ does not necessarily imply $I^r(G_1) = I^r(G_2)$. The product defined by (11.2) naturally induces a multiplication

$$I^p(G) \otimes I^q(G) \to I^{p+q}(G). \tag{11.5}$$

The sum $I^*(G) \equiv \bigotimes_{r\geq 0} I^r(G)$ is an algebra with this product.

Take $\tilde{P} \in I^r(G)$. The shorthand notation for the diagonal combination is

$$P(A) \equiv \tilde{P}(\underbrace{A, A, \ldots, A}_{r}) \qquad A \in \mathfrak{g}. \tag{11.6}$$

Clearly, P is a polynomial of degree r and called an **invariant polynomial**. P is also Ad G-invariant,

$$P(\mathrm{Ad}_g A) = P(g^{-1}Ag) = P(A) \qquad A \in \mathfrak{g}, g \in G. \tag{11.7}$$

For example, $\mathrm{tr}(A^r)$ is an invariant polynomial obtained from (11.4). In general, an invariant polynomial may be written in terms of a sum of products of $P_r \equiv \mathrm{tr}(A^r)$.

Conversely, any invariant polynomial P defines an invariant and symmetric r-linear form $\tilde{P}$ by expanding $P(t_1 A_1 + \cdots + t_r A_r)$ as a polynomial in t_i. Then $1/r!$ times the coefficient of $t_1 t_2 \cdots t_r$ is invariant and symmetric by construction and is called the **polarization** of P. Take $P(A) \equiv \mathrm{tr}(A^3)$, for example. Following the previous prescription, we expand $\mathrm{tr}(t_1 A_1 + t_2 A_2 + t_3 A_3)^3$ in powers of t_1, t_2 and t_3. The coefficient of $t_1 t_2 t_3$ is

$$\begin{aligned} &\mathrm{tr}(A_1A_2A_3 + A_1A_3A_2 + A_2A_1A_3 + A_2A_3A_1 + A_3A_1A_2 + A_3A_2A_1) \\ &\quad = 3\,\mathrm{tr}(A_1A_2A_3 + A_2A_1A_3) \end{aligned}$$

where the cyclicity of the trace has been used. The polarization is

$$\tilde{P}(A_1, A_2, A_3) = \tfrac{1}{2}\mathrm{tr}(A_1A_2A_3 + A_2A_1A_3) = \mathrm{str}(A_1, A_2, A_3).$$

In the previous chapter, we introduced the local gauge potential $\mathcal{A} = \mathcal{A}_\mu\, \mathrm{d}x^\mu$ and the field strength $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ on a principal bundle. We have shown that these geometrical objects describe the associated vector bundles as well. Since the set of connections $\{\mathcal{A}_i\}$ describes the twisting of a fibre bundle, the non-triviality of a principal bundle is equally shared by its associated bundle. In fact, if (10.57) is employed as a definition of the local connection in a vector bundle, it can be defined even without reference to the principal bundle with which it is originally associated. Later, we encounter situations in which use of vector bundles is essential (the Whitney sum bundle, the splitting principle and so on).

Let $P(M, \mathbb{C})$ be a principal bundle. We extend the domain of invariant polynomials from $\mathfrak{g}$ to $\mathfrak{g}$-valued p-forms on M. For $A_i\eta_i$ ($A_i \in \mathfrak{g}, \eta \in \Omega^{p_i}(M)$; $1 \le i \le r$), we define

$$\tilde{P}(A_1\eta_1, \ldots, A_r\eta_r) \equiv \eta_1 \wedge \ldots \wedge \eta_r \tilde{P}(A_1, \ldots, A_r). \tag{11.8}$$

For example, corresponding to (11.4), we have

$$\mathrm{str}(A_1\eta_1, \ldots, A_r\eta_r) = \eta_1 \wedge \ldots \wedge \eta_r \,\mathrm{str}(A_1, \ldots, A_r).$$

The diagonal combination is

$$P(A\eta) \equiv \underbrace{\eta \wedge \ldots \wedge \eta}_{r} P(A). \tag{11.9}$$

The action $\tilde{P}$ or P on general elements is given by the r-linearity. In particular, we are interested in the invariant polynomial of the form $P(\mathcal{F})$ in the following. The importance of invariant polynomials resides in the following fundamental theorem.

Theorem 11.1. (**Chern–Weil theorem**) Let P be an invariant polynomial. Then $P(\mathcal{F})$ satisfies

(a) $\mathrm{d}P(\mathcal{F}) = 0$.
(b) Let $\mathcal{F}$ and $\mathcal{F}'$ be curvature two-forms corresponding to different connections $\mathcal{A}$ and $\mathcal{A}'$. Then the difference $P(\mathcal{F}') - P(\mathcal{F})$ is exact.

Proof. (a) It is sufficient to prove that $\mathrm{d}P(\mathcal{F}) = 0$ for an invariant polynomial $P_r(\mathcal{F})$ which is homogeneous of degree r, since any invariant polynomial can be decomposed into homogeneous polynomials. First consider the identity,

$$\tilde{P}_r(g_t^{-1}X_1g_t, \ldots, g_t^{-1}X_rg_t) = \tilde{P}_r(X_1, \ldots, X_r)$$

where $g_t \equiv \exp(tX)$ and $X, X_i \in \mathfrak{g}$. By putting $t = 0$ after differentiation with respect to t, we obtain

$$\sum_{i=1}^{r} \tilde{P}_r(X_1, \ldots, [X_i, X], \ldots, X_r) = 0. \tag{11.10}$$

Next, let A be a $\mathfrak{g}$-valued p-form and Ω_i be a $\mathfrak{g}$-valued p_i-form ($1 \le i \le r$). Without loss of generality, we may take $A = X\eta$ and $\Omega_i = X_i\eta_i$ where $X, X_i \in \mathfrak{g}$ and η (η_i) is a p-form (p_i-form). Define

$$\begin{aligned}[\Omega_i, A] &\equiv \eta_i \wedge \eta[X_i, X] \\ &= X_iX(\eta_i \wedge \eta) - (-1)^{pp_i}XX_i(\eta \wedge \eta_i). \end{aligned} \tag{11.11}$$

Let us note that

$$\begin{aligned}
&\tilde{P}_r(\Omega_1, \ldots, [\Omega_i, A], \ldots, \Omega_r) \\
&\quad = \eta_1 \wedge \ldots \wedge \eta_i \wedge \eta \wedge \ldots \wedge \eta_r \tilde{P}_r(X_1, \ldots, X_i X, \ldots, X_r) \\
&\qquad - (-1)^{p \cdot p_i} \eta_1 \wedge \ldots \wedge \eta \wedge \eta_i \wedge \ldots \\
&\qquad \ldots \wedge \eta_r \tilde{P}_r(X_1, \ldots, X X_i, \ldots, X_r) \\
&\quad = \eta \wedge \eta_1 \wedge \ldots \wedge \eta_r (-1)^{p(p_1+\cdots+p_i)} \\
&\qquad \times \tilde{P}_r(X_1, \ldots, [X_i, X], \ldots, X_r).
\end{aligned}$$

From this and (11.10), we find

$$\sum_{i=1}^{r} (-1)^{p(p_1+\cdots+p_i)} \tilde{P}_r(\Omega_1, \ldots, [\Omega_i, A], \ldots, \Omega_r) = 0. \tag{11.12}$$

Next, consider the derivative,

$$\begin{aligned}
\mathrm{d}\tilde{P}_r(\Omega_1, \ldots, \Omega_r) &= \mathrm{d}(\eta_1 \wedge \ldots \wedge \eta_r) \tilde{P}_r(X_1, \ldots, X_r) \\
&= \sum_{i=1}^{r} (-1)^{(p_1+\cdots+p_{i-1})} (\eta_1 \wedge \ldots \wedge \mathrm{d}\eta_i \wedge \ldots \wedge \eta_r) \\
&\quad \times \tilde{P}_r(X_1, \ldots, X_i, \ldots, X_r) \\
&= \sum_{i=1}^{r} (-1)^{(p_1+\cdots+p_{i-1})} \tilde{P}_r(\Omega_1, \ldots, \mathrm{d}\Omega_i, \ldots, \Omega_r).
\end{aligned} \tag{11.13}$$

Let $A = \mathcal{A}$ and $\Omega_i = \mathcal{F}$ in (11.12) and (11.13) for which $p = 1$ and $p_i = 2$. By adding 0 of the form (11.12) to (11.13) we have

$$\begin{aligned}
&\mathrm{d}\tilde{P}_r(\mathcal{F}, \ldots, \mathcal{F}) \\
&\quad = \sum_{i=1}^{r} [\tilde{P}_r(\mathcal{F}, \ldots, \mathrm{d}\mathcal{F}, \ldots, \mathcal{F}) + \tilde{P}_r(\mathcal{F}, \ldots, [\mathcal{A}, \mathcal{F}], \ldots, \mathcal{F})] \\
&\quad = \sum_{i=1}^{r} \tilde{P}_r(\mathcal{F}, \ldots, \mathcal{D}\mathcal{F}, \ldots, \mathcal{F}) = 0
\end{aligned} \tag{11.14}$$

since $\mathcal{D}\mathcal{F} = \mathrm{d}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0$ (the Bianchi identity). We have proved

$$\mathrm{d}P_r(\mathcal{F}) = \mathrm{d}\tilde{P}_r(\mathcal{F}, \ldots, \mathcal{F}) = 0.$$

(b) Let $\mathcal{A}$ and $\mathcal{A}'$ be two connections on E and let $\mathcal{F}$ and $\mathcal{F}'$ be the respective field strengths. Define an interpolating gauge potential $\mathcal{A}_t$, by

$$\mathcal{A}_t \equiv \mathcal{A} + t\theta \qquad \theta \equiv (\mathcal{A}' - \mathcal{A}) \qquad 0 \le t \le 1 \tag{11.15}$$

so that $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_1 = \mathcal{A}'$. The corresponding field strength is

$$\mathcal{F}_t \equiv \mathrm{d}\mathcal{A}_t + \mathcal{A}_t \wedge \mathcal{A}_t = \mathcal{F} + t\mathcal{D}\theta + t^2\theta^2 \tag{11.16}$$

where $\mathcal{D}\theta = \mathrm{d}\theta + [\mathcal{A}, \theta] = \mathrm{d}\theta + \mathcal{A} \wedge \theta + \theta \wedge \mathcal{A}$. We first note that

$$\begin{aligned} P_r(\mathcal{F}') - P_r(\mathcal{F}) = P_r(\mathcal{F}_1) - P_r(\mathcal{F}_0) &= \int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} P_r(\mathcal{F}_t) \\ &= r \int_0^1 \mathrm{d}t\, \tilde{P}_r\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_t, \mathcal{F}_t, \ldots, \mathcal{F}_t\right). \end{aligned} \tag{11.17}$$

From (11.16), we find that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} P_r(\mathcal{F}_t) &= r\tilde{P}_r(\mathcal{D}\theta + 2t\theta^2, \mathcal{F}_t, \ldots, \mathcal{F}_t) \\ &= r\tilde{P}_r(\mathcal{D}\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t) + 2rt\tilde{P}_r(\theta^2, \mathcal{F}_t, \ldots, \mathcal{F}_t). \end{aligned} \tag{11.18}$$

Note also that

$$\mathcal{D}\mathcal{F}_t = \mathrm{d}\mathcal{F}_t + [\mathcal{A}, \mathcal{F}_t] = -[\mathcal{A}_t, \mathcal{F}_t] + [\mathcal{A}, \mathcal{F}_t] = t[\mathcal{F}_t, \theta]$$

where use has been made of the Bianchi identity $\mathcal{D}_t\mathcal{F}_t = \mathrm{d}\mathcal{F}_t + [\mathcal{A}_t, \mathcal{F}_t] = 0$. [$\mathcal{D}$ is the covariant derivative with respect to $\mathcal{A}$ while $\mathcal{D}_t$ is that with respect to $\mathcal{A}_t$.] It then follows that

$$\begin{aligned} &\mathrm{d}[\tilde{P}_r(\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t)] \\ &\quad = \tilde{P}_r(\mathrm{d}\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t) - (r-1)\tilde{P}_r(\theta, \mathrm{d}\mathcal{F}_t, \ldots, \mathcal{F}_t) \\ &\quad = \tilde{P}_r(\mathcal{D}\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t) - (r-1)\tilde{P}_r(\theta, \mathcal{D}\mathcal{F}_t, \ldots, \mathcal{F}_t) \\ &\quad = \tilde{P}_r(\mathcal{D}\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t) - (r-1)t\tilde{P}_r(\theta, [\mathcal{F}_t, \theta], \mathcal{F}_t, \ldots, \mathcal{F}_t) \end{aligned} \tag{11.19}$$

where we have added a 0 of the form (11.12) to change d to $\mathcal{D}$. If we take $\Omega_1 = A = \theta, \Omega_2 = \cdots = \Omega_m = \mathcal{F}_t$ in (11.12), we have

$$2\tilde{P}_r(\theta^2, \mathcal{F}_t, \ldots, \mathcal{F}_t) + (r-1)\tilde{P}_r(\theta, [\mathcal{F}_t, \theta], \mathcal{F}_t, \ldots, \mathcal{F}_t) = 0.$$

From (11.18), (11.19) and the previous identity, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} P_r(\mathcal{F}_t) = r\mathrm{d}[\tilde{P}_r(\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t)].$$

We finally find that

$$P_r(\mathcal{F}') - P_r(\mathcal{F}) = \mathrm{d}\left[r\int_0^1 \tilde{P}_r(\mathcal{A}' - \mathcal{A}, \mathcal{F}_t, \ldots, \mathcal{F}_t)\,\mathrm{d}t\right]. \tag{11.20}$$

This shows that $P_r(\mathcal{F}')$ differs from $P_r(\mathcal{F})$ by an exact form. $\square$

We define the **transgression** $TP_r(\mathcal{A}', \mathcal{A})$ of P_r by

$$TP_r(\mathcal{A}', \mathcal{A}) \equiv r \int_0^1 \mathrm{d}t\, \tilde{P}_r\,(\mathcal{A}' - \mathcal{A}, \mathcal{F}_t, \ldots, \mathcal{F}_t) \tag{11.21}$$

where $\tilde{P}_r$ is the polarization of P_r. Transgressions will play an important role when we discuss Chern–Simons forms in section 11.5. Let $\dim M = m$. Since $P_m(\mathcal{F}')$ differs from $P_m(\mathcal{F})$ by an exact form, their integrals over a manifold M without a boundary should be the same:

$$\int_M P_m(\mathcal{F}') - \int_M P_m(\mathcal{F}) = \int_M \mathrm{d}TP_m\,(\mathcal{A}', \mathcal{A}) = \int_{\partial M} P_m(\mathcal{A}', \mathcal{A}) = 0. \tag{11.22}$$

As has been proved, an invariant polynomial is closed and, in general, non-trivial. Accordingly, it defines a cohomology class of M. Theorem 11.1(b) ensures that this cohomology class is independent of the gauge potential chosen. The cohomology class thus defined is called the **characteristic class**. The characteristic class defined by an invariant polynomial P is denoted by $\chi_E(P)$ where E is a fibre bundle on which connections and curvatures are defined. [*Remark:* Since a principal bundle and its associated bundles share the same gauge potentials and field strengths, the Chern–Weil theorem applies equally to both bundles. Accordingly, E can be either a principal bundle or a vector bundle.]

Theorem 11.2. Let P be an invariant polynomial in $I^*(G)$ and E be a fibre bundle over M with structure group G.

(a) The map

$$\chi_E : I^*(G) \to H^*(M) \tag{11.23}$$

defined by $P \to \chi_E(P)$ is a homomorphism (**Weil homomorphism**).

(b) Let $f : N \to M$ be a differentiable map. For the pullback bundle f^*E of E, we have the so-called **naturality**

$$\chi_{f^*E} = f^*\chi_E. \tag{11.24}$$

Proof. (a) Take $P_r \in I^r(G)$ and $P_s \in I^s(G)$. If we write $\mathcal{F} = \mathcal{F}^\alpha T_\alpha$, we have

$$\begin{aligned}(P_rP_s)(\mathcal{F}) &= \mathcal{F}^{\alpha_1} \wedge \ldots \wedge \mathcal{F}^{\alpha_r} \wedge \mathcal{F}^{\beta_1} \wedge \ldots \wedge \mathcal{F}^{\beta_s} \\ &\quad \times \frac{1}{(r+s)!}\tilde{P}_r(T_{\alpha_1}, \ldots, T_{\alpha_r})\tilde{P}_n(T_{\beta_1}, \ldots, T_{\beta_s}) \\ &= P_r(\mathcal{F}) \wedge P_s(\mathcal{F}).\end{aligned}$$

Then (a) follows since $P_r(\mathcal{F}), P_s(\mathcal{F}) \in H^*(M)$.

(b) Let $\mathcal{A}$ be a gauge potential of E and $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. It is easy to verify that the pullback $f^*\mathcal{A}$ is a connection in f^*E. In fact, let $\mathcal{A}_i$ and $\mathcal{A}_j$ be local connections in overlapping charts U_i and U_j of M. If t_{ij} is a transition function on $U_i \cap U_j$, the transition function on f^*E is given by $f^*t_{ij} = t_{ij} \circ f$. The pullback $f^*\mathcal{A}_i$ and $f^*\mathcal{A}_j$ are related as

$$\begin{aligned} f^*\mathcal{A}_j &= f^*(t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\,\mathrm{d}t_{ij}) \\ &= (f^*t_{ij}^{-1})(f^*\mathcal{A}_i)(f^*t_{ij}) + (f^*t_{ij}^{-1})(\mathrm{d}f^*t_{ij}). \end{aligned}$$

This shows that $f^*\mathcal{A}$ is, indeed, a local connection on f^*E. The corresponding field strength on f^*E is

$$\mathrm{d}(f^*\mathcal{A}_i) + f^*\mathcal{A}_i \wedge f^*\mathcal{A}_i = f^*[\mathrm{d}\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i] = f^*\mathcal{F}_i.$$

Hence, $f^*P(\mathcal{F}_i) = P(f^*\mathcal{F}_i)$, that is $f^*\chi_E(P) = \chi_{f^*E}(P)$. □

Corollary 11.1. Characteristic classes of a trivial bundle are trivial.

Proof. Let $E \xrightarrow{\pi} M$ be a trivial bundle. Since E is trivial, there exists a map $f : M \to \{p\}$ such that $E = f^*E_0$ where $E_0 \longrightarrow \{p\}$ is a bundle over a point p. All the de Rham cohomology groups of a point are trivial and so are the characteristic classes. Theorem 11.2(b) ensures that the characteristic classes χ_E $(=f^*\chi_{E_0})$ of E are also trivial. □

11.2 Chern classes

11.2.1 Definitions

Let $E \xrightarrow{\pi} M$ be a complex vector bundle whose fibre is $\mathbb{C}^k$. The structure group G is a subgroup of GL$(k, \mathbb{C})$, and the gauge potential $\mathcal{A}$ and the field strength $\mathcal{F}$ take their values in $\mathfrak{g}$. Define the **total Chern class** by

$$c(\mathcal{F}) \equiv \det\left(I + \frac{\mathrm{i}\mathcal{F}}{2\pi}\right). \tag{11.25}$$

Since $\mathcal{F}$ is a two-form, $c(\mathcal{F})$ is a direct sum of forms of even degrees,

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \cdots \tag{11.26}$$

where $c_j(\mathcal{F}) \in \Omega^{2j}(M)$ is called the jth **Chern class**. In an m-dimensional manifold M, the Chern class $c_j(\mathcal{F})$ with $2j > m$ vanishes trivially. Irrespective of $\dim M$, the series terminates at $c_k(\mathcal{F}) = \det(\mathrm{i}\mathcal{F}/2\pi)$ and $c_j(\mathcal{F}) = 0$ for $j > k$. Since $c_j(\mathcal{F})$ is closed, it defines an element $[c_j(\mathcal{F})]$ of $H^{2j}(M)$.

Example 11.1. Let F be a complex vector bundle with fibre $\mathbb{C}^2$ over M, where $G = \mathrm{SU}(2)$ and $\dim M = 4$. If we write the field $\mathcal{F} = \mathcal{F}^\alpha(\sigma_\alpha/2\mathrm{i})$, $\mathcal{F}^\alpha = \frac{1}{2}\mathcal{F}^\alpha{}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, we have

$$c(\mathcal{F}) = \det\left(I + \frac{\mathrm{i}}{2\pi}\mathcal{F}^\alpha(\sigma_\alpha/2\mathrm{i})\right)$$

$$= \det \begin{pmatrix} 1 + (\mathrm{i}/2\pi)(\mathcal{F}^3/2\mathrm{i}) & (\mathrm{i}/2\pi)(\mathcal{F}^1 - \mathrm{i}\mathcal{F}^2)/2\mathrm{i} \\ (\mathrm{i}/2\pi)(\mathcal{F}^1 + \mathrm{i}\mathcal{F}^2)/2\mathrm{i} & 1 - (\mathrm{i}/2\pi)(\mathcal{F}^3/2\mathrm{i}) \end{pmatrix}$$

$$= 1 + \frac{1}{4}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \left(\mathcal{F}^3 \wedge \mathcal{F}^3 + \mathcal{F}^1 \wedge \mathcal{F}^1 + \mathcal{F}^2 \wedge \mathcal{F}^2\right). \tag{11.27}$$

Individual Chern classes are

$$\begin{aligned} c_0(\mathcal{F}) &= 1 \\ c_1(\mathcal{F}) &= 0 \\ c_2(\mathcal{F}) &= \left(\frac{\mathrm{i}}{2\pi}\right)^2 \sum \frac{\mathcal{F}^\alpha \wedge \mathcal{F}^\alpha}{4} = \det\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right). \end{aligned} \tag{11.28}$$

Higher Chern classes vanish identically.

For general fibre bundles, it is rather cumbersome to compute the Chern classes by expanding the determinant and it is desirable to find a formula which yields them more easily. This is done by diagonalizing the curvature form. The matrix form $\mathcal{F}$ is diagonalized by an appropriate matrix $g \in \mathrm{GL}(k, \mathbb{C})$ as $g^{-1}(\mathrm{i}\mathcal{F}/2\pi)g = \mathrm{diag}(x_1, \ldots, x_k)$, where x_i is a two-form. This diagonal matrix will be denoted by A. For example, if $G = \mathrm{SU}(k)$, the generators are chosen to be anti-Hermitian and a Hermitian matrix $\mathrm{i}\mathcal{F}/2\pi$ can be diagonalized by $g \in \mathrm{SU}(k)$. We have

$$\begin{aligned} \det(I + A) &= \det[\mathrm{diag}(1 + x_1, 1 + x_2, \ldots, 1 + x_k)] \\ &= \prod_{j=1}^{k} (1 + x_j) \\ &= 1 + (x_1 + \cdots + x_k) + (x_1 x_2 + \cdots + x_{k-1} x_k) \\ &\quad + \cdots + (x_1 x_2 + \cdots + x_k) \\ &= 1 + \mathrm{tr}\, A + \tfrac{1}{2}\{(\mathrm{tr}\, A)^2 - \mathrm{tr}\, A^2\} + \cdots + \det A. \end{aligned} \tag{11.29}$$

Observe that each term of (11.29) is an elementary symmetric function of $\{x_j\}$,

$$\begin{aligned} S_0(x_j) &\equiv 1 \\ S_1(x_j) &\equiv \sum_{j=1}^{k} x_j \\ S_2(x_j) &\equiv \sum_{i<j} x_i x_j \\ &\vdots \\ S_k(x_j) &\equiv x_1 x_2 \ldots x_k. \end{aligned} \tag{11.30}$$

Since $\det(I + A)$ is an invariant polynomial, we have $P(\mathcal{F}) = P(g\mathcal{F}g^{-1}) = P(2\pi A/\mathrm{i})$, see (11.7). Accordingly, we have, for general $\mathcal{F}$,

$$
\begin{aligned}
c_0(\mathcal{F}) &= 1 \\
c_1(\mathcal{F}) &= \operatorname{tr} A = \operatorname{tr}\left(g\frac{\mathrm{i}\mathcal{F}}{2\pi}g^{-1}\right) = \frac{\mathrm{i}}{2\pi}\operatorname{tr}\mathcal{F} \\
c_2(\mathcal{F}) &= \tfrac{1}{2}[(\operatorname{tr}\mathcal{A})^2 - \operatorname{tr}\mathcal{A}^2] = \tfrac{1}{2}(\mathrm{i}/2\pi)^2[\operatorname{tr}\mathcal{F}\wedge\operatorname{tr}\mathcal{F} - \operatorname{tr}(\mathcal{F}\wedge\mathcal{F})] \\
&\ \vdots \\
c_k(\mathcal{F}) &= \det A = (\mathrm{i}/2\pi)^k \det\mathcal{F}.
\end{aligned}
\tag{11.31}
$$

Example 11.1 is easily verified from (11.31). [Note that the Pauli matrices (in general, any element of the Lie algebra $\mathfrak{su}(n)$ of $\mathrm{SU}(n)$) are traceless, $\operatorname{tr}\sigma_\alpha = 0$.]

11.2.2 Properties of Chern classes

We will deal with several vector bundles in the following. We often denote the Chern class of a vector bundle E by $c(E)$. If the specification of the curvature is required, we write $c(\mathcal{F}_E)$.

Theorem 11.3. Let $E \xrightarrow{\pi} M$ be a vector bundle with $G = \mathrm{GL}(k, \mathbb{C})$ and $F = \mathbb{C}^k$.

(a) (Naturality) Let $f : N \to M$ be a smooth map. Then

$$c(f^*E) = f^*c(E). \tag{11.32}$$

(b) Let $F \xrightarrow{\pi'} M$ be another vector bundle with $F = \mathbb{C}^l$ and $G = \mathrm{GL}(l, \mathbb{C})$. The total Chern class of a Whitney sum bundle $E \oplus F$ is

$$c(E \oplus F) = c(E) \wedge c(F). \tag{11.33}$$

Proof.

(a) The naturality follows directly from theorem 11.2(a). Since the curvature of f^*E is $\mathcal{F}_{f^*E} = f^*\mathcal{F}_E$, the total Chern class of f^*E is

$$
\begin{aligned}
c(f^*E) &= \det\left(I + \frac{\mathrm{i}}{2\pi}\mathcal{F}_{f^*E}\right) = \det\left(I + \frac{\mathrm{i}}{2\pi}f^*\mathcal{F}_E\right) \\
&= f^*\det\left(I + \frac{\mathrm{i}}{2\pi}\mathcal{F}_E\right) = f^*c(E).
\end{aligned}
$$

(b) Let us consider the Chern polynomial of a matrix

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}.$$

[Note that the curvature of a Whitney sum bundle is block diagonal: $\mathcal{F}_{E\oplus F} = \operatorname{diag}(\mathcal{F}_E, \mathcal{F}_F)$.] We find that

$$\det\left(I + \frac{\mathrm{i}A}{2\pi}\right) = \det\begin{pmatrix} I + \frac{\mathrm{i}B}{2\pi} & 0 \\ 0 & I + \frac{\mathrm{i}C}{2\pi} \end{pmatrix}$$
$$= \det\left(I + \frac{\mathrm{i}B}{2\pi}\right)\det\left(I + \frac{\mathrm{i}C}{2\pi}\right) = c(B)c(C).$$

This relation remains true when B and C are replaced by $\mathcal{F}_E$ and $\mathcal{F}_F$, namely

$$c(\mathcal{F}_{E\oplus F}) = c(\mathcal{F}_E) \wedge c(\mathcal{F}_F)$$

which proves (11.33). □

Exercise 11.1. (a) Let E be a trivial bundle. Use corollary 11.1 to show that

$$c(E) = 1. \tag{11.34}$$

(b) Let E be a vector bundle such that $E = E_1 \oplus E_2$ where E_1 is a vector bundle of dimension k_1 and E_2 is a trivial vector bundle of dimension k_2. Show that

$$c_i(E) = 0 \qquad k_1 + 1 \leq i \leq k_1 + k_2. \tag{11.35}$$

11.2.3 Splitting principle

Let E be a Whitney sum of n complex line bundles,

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_n. \tag{11.36}$$

From (11.33), we have

$$c(E) = c(L_1)c(L_2)\ldots c(L_n) \tag{11.37}$$

where the product is the exterior product of differential forms. Since $c_r(L) = 0$ for $r \geq 2$, we write

$$c(L_i) = 1 + c_1(L_i) \equiv 1 + x_i. \tag{11.38}$$

Then (11.37) becomes

$$c(E) = \prod_{i=1}^{n}(1 + x_i). \tag{11.39}$$

Comparing this with (11.29), we find that the Chern class of an n-dimensional vector bundle E is identical with that of the Whitney sum of n complex line bundles. Although E is not a Whitney sum of complex line bundles in general, as far as the Chern classes are concerned, we may pretend that this is the case. This is called the **splitting principle** and we accept this fact without proof. The general proof is found in Shanahan (1978) and Hirzebruch (1966), for example.

Intuitively speaking, if the curvature $\mathcal{F}$ is diagonalized, the complex vector space on which g acts splits into k independent pieces: $\mathbb{C}^k \to \mathbb{C} \oplus \cdots \oplus \mathbb{C}$. An eigenvalue x_i is a curvature in each complex line bundle. Since diagonalizable matrices are dense in $M(n, \mathbb{C})$, any matrix may be approximated by a diagonal one as closely as we wish. Hence, the splitting principle applies to any matrix. As an exercise, the reader may prove (11.33) using the splitting principle.

11.2.4 Universal bundles and classifying spaces

By now the reader must have some acquaintance with characteristic classes. Before we close this section, we examine these from a slightly different point of view emphasizing their role in the classification of fibre bundles. Let $E \xrightarrow{\pi} M$ be a vector bundle with fibre $\mathbb{C}^k$. It is known that we can always find a bundle $\bar{E} \xrightarrow{\pi'} M$ such that

$$E \oplus \bar{E} \cong M \times \mathbb{C}^n \tag{11.40}$$

for some $n \geq k$. The fibre F_p of E at $p \in M$ is a k-plane lying in $\mathbb{C}^n$. Let $G_{k,n}(\mathbb{C})$ be the Grassmann manifold defined in example 8.4. The manifold $G_{k,n}(\mathbb{C})$ is the set of k-planes in $\mathbb{C}^n$. Similarly to the canonical line bundle, we define the canonical k-plane bundle $L_{k,n}(\mathbb{C})$ over $G_{k,n}(\mathbb{C})$ with the fibre $\mathbb{C}^k$. Consider a map $f : M \to G_{k,n}(\mathbb{C})$ which maps a point p to the k-plane F_p in $\mathbb{C}^n$.

Theorem 11.4. Let M be a manifold with $\dim M = m$ and let $E \xrightarrow{\pi} M$ be a complex vector bundle with the fibre $\mathbb{C}^k$. Then there exists a natural number N such that for $n > N$,

(a) there exists a map $f : M \to G_{k,n}(\mathbb{C})$ such that

$$E \cong f^* L_{k,n}(\mathbb{C}) \tag{11.41}$$

(b) $f^* L_{k,n}(\mathbb{C}) \cong g^* L_{k,n}(\mathbb{C})$ if and only if $f, g : M \to G_{k,n}(\mathbb{C})$ are homotopic.

The proof is found in Chern (1979). For example, if $E \xrightarrow{\pi} M$ is a complex line bundle, then there exists a bundle $\bar{E} \xrightarrow{\pi'} M$ such that $E \oplus \bar{E} \cong M \times \mathbb{C}^n$ and a map $f : M \to G_{1,n}(\mathbb{C}) \cong \mathbb{C}P^{n-1}$ such that $E = f^* L$, L being the canonical line bundle over $\mathbb{C}P^{n-1}$. Moreover, if $f \sim g$, then $f^* L$ is equivalent to $g^* L$. Theorem 11.4 shows that the classification of vector bundles reduces to that of the homotopy classes of the maps $M \to G_{k,n}(\mathbb{C})$.

It is convenient to define the **classifying space** $G_k(\mathbb{C})$. Regarding a k-plane in $\mathbb{C}^n$ as that in $\mathbb{C}^{n+1}$, we have natural inclusions.

$$G_{k,k}(\mathbb{C}) \hookrightarrow G_{k,k+1}(\mathbb{C}) \hookrightarrow \cdots \hookrightarrow G_k(\mathbb{C}) \tag{11.42}$$

where

$$G_k(\mathbb{C}) \equiv \bigcup_{n=k}^{\infty} G_{k,n}(\mathbb{C}). \tag{11.43}$$

Correspondingly, we have the **universal bundle** $L_k \to G_k(\mathbb{C})$ whose fibre is $\mathbb{C}^k$. For any complex vector bundle $E \xrightarrow{\pi} M$ with fibre $\mathbb{C}^k$, there exists a map $f : M \to G_k(\mathbb{C})$ such that $E = f^* L_k(\mathbb{C})$.

Let $E \xrightarrow{\pi} M$ be a vector bundle. A characteristic class χ is defined as a map $\chi : E \to \chi(E) \in H^*(M)$ such that

$$\chi(f^* E) = f^* \chi(E) \qquad \text{(naturality)} \tag{11.44a}$$

$$\chi(E) = \chi(E') \qquad \text{if } E \text{ is equivalent to } E'. \tag{11.44b}$$

The map f^* on the LHS of (11.44a) is a pullback of the bundle while f^* on the RHS is that of the cohomology class. Since the homotopy class $[f]$ of $f : M \to G_k(\mathbb{C})$ uniquely defines the pullback

$$f^* : H^*(G_k) \to H^*(M) \tag{11.45}$$

an element $\chi(E) = f^* \chi(G_k)$ proves to be useful in classifying complex vector bundles over M with $\dim E = k$. For each choice of $\chi(G_k)$, there exists a characteristic class in E.

The Chern class $c(E)$ is also defined axiomatically by

$$\text{(i)} \qquad c(f^* E) = f^* c(E) \qquad \text{(naturality)} \tag{11.46a}$$

$$\text{(ii)} \qquad c(E) = c_0(E) \oplus c_1(E) \oplus \cdots \oplus c_k(E)$$

$$c_i(E) \in H^{2i}(M);\; c_i(E) = 0 \qquad i > k \tag{11.46b}$$

$$\text{(iii)} \qquad c(E \oplus F) = c(E) c(E) \qquad \text{(Whitney sum)} \tag{11.46c}$$

$$\text{(iv)} \qquad c(L) = 1 + x \qquad \text{(normalization)} \tag{11.46d}$$

L being the canonical line bundle over $\mathbb{C}P^n$. It can be shown that these axioms uniquely define the Chern class as (11.25).

11.3 Chern characters

11.3.1 Definitions

Among the characteristic classes, the Chern characters are of special importance due to their appearance in the Atiyah–Singer index theorem. The **total Chern character** is defined by

$$\mathrm{ch}(\mathcal{F}) \equiv \mathrm{tr} \exp\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right) = \sum_{j=1} \frac{1}{j!} \mathrm{tr} \left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^j . \tag{11.47}$$

The jth **Chern character** $\mathrm{ch}_j(\mathcal{F})$ is

$$\mathrm{ch}_j(\mathcal{F}) \equiv \frac{1}{j!}\,\mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^j. \tag{11.48}$$

If $2j > m = \dim M$, $\mathrm{ch}_j(\mathcal{F})$ vanishes, hence $\mathrm{ch}(\mathcal{F})$ is a polynomial of finite order.

Let us diagonalize $\mathcal{F}$ as

$$\frac{\mathrm{i}\mathcal{F}}{2\pi} \to g^{-1}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)g = A \equiv \mathrm{diag}(x_1, \ldots, x_k) \qquad g \in \mathrm{GL}(k, \mathbb{C}).$$

The total Chern character is expressed as

$$\mathrm{tr}[\exp(A)] = \sum_{j=1}^{k} \exp(x_j). \tag{11.49}$$

In terms of the elementary symmetric functions $S_r(x_j)$, the total Chern character becomes

$$\begin{aligned}\sum_{j=1}^{k}\exp(x_j) &= \sum_{j=1}^{k}\left(1 + x_j + \frac{1}{2!}x_j^2 + \frac{1}{3!}x_j^3 + \cdots\right)\\ &= k + S_1(x_j) + \frac{1}{2!}[S_1(x_j)^2 - 2S_2(x_j)] + \cdots.\end{aligned} \tag{11.50}$$

Accordingly, each Chern character is expressed in terms of the Chern classes as

$$\mathrm{ch}_0(\mathcal{F}) = k \tag{11.51a}$$

$$\mathrm{ch}_1(\mathcal{F}) = c_1(\mathcal{F}) \tag{11.51b}$$

$$\mathrm{ch}_2(\mathcal{F}) = \tfrac{1}{2}[c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})] \tag{11.51c}$$

$$\vdots$$

where k is the fibre dimension of the bundle.

Example 11.2. Let P be a U(1) bundle over S^2. If $\mathcal{A}_\mathrm{N}$ and $\mathcal{A}_\mathrm{S}$ are the local connections on U_N and U_S defined in section 10.5, the field strength is given by $\mathcal{F}_i = \mathrm{d}\mathcal{A}_i$ ($i = \mathrm{N}, \mathrm{S}$). We have

$$\mathrm{ch}(\mathcal{F}) = 1 + \frac{\mathrm{i}\mathcal{F}}{2\pi} \tag{11.52}$$

where we have noted that $\mathcal{F}^n = 0$ ($n \geq 2$) on S^2. This bundle describes the magnetic monopole. The magnetic charge $2g$ given by (10.94) is an integer expressed in terms of the Chern character as

$$N = \frac{\mathrm{i}}{2\pi}\int_{S^2}\mathcal{F} = \int_{S^2}\mathrm{ch}_1(\mathcal{F}). \tag{11.53}$$

Let P be an SU(2) bundle over S^4. The total Chern class of P is given by (11.27). The total Chern character is

$$\mathrm{ch}(\mathcal{F}) = 2 + \mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right) + \frac{1}{2}\mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^2. \tag{11.54}$$

$\mathrm{Ch}(\mathcal{F})$ terminates at $\mathrm{ch}_2(\mathcal{F})$ since $\mathcal{F}^n = 0$ for $n \geq 3$. Moreover, $\mathrm{tr}\,\mathcal{F} = 0$ for $G = \mathrm{SU}(2)$, $n \geq 2$. As we found in section 10.5, the instanton number is given by

$$\frac{1}{2}\int_{S^4} \mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^2 = \int_{S^4} \mathrm{ch}_2(\mathcal{F}). \tag{11.55}$$

In both cases, ch_j measures how the bundle is twisted when local pieces are patched together.

Example 11.3. Let P be a U(1) bundle over a $2m$-dimensional manifold M. The mth Chern character is

$$\begin{aligned}
\frac{1}{m!}\mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^m &= \frac{1}{m!}\left(\frac{\mathrm{i}}{2\pi}\right)^m \left[\frac{1}{2}\mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right]^m \\
&= \frac{1}{m!}\left(\frac{\mathrm{i}}{4\pi}\right)^m \mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_m\nu_m}\,\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\nu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_m}\wedge\mathrm{d}x^{\nu_m} \\
&= \left(\frac{\mathrm{i}}{4\pi}\right)^m \epsilon^{\mu_1\nu_1\ldots\mu_m\nu_m}\mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_m\nu_m}\,\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^{2m}
\end{aligned}$$

which describes the U(1) anomaly in $2m$-dimensional space, see chapter 13.

Example 11.4. Let L be a complex line bundle. It then follows that

$$\mathrm{ch}(L) = \mathrm{tr}\exp\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right) = \mathrm{e}^x = 1 + x \qquad x \equiv \frac{\mathrm{i}\mathcal{F}}{2\pi}. \tag{11.56}$$

For example, let $L \xrightarrow{\pi} \mathbb{C}P^1$ be the canonical line bundle over $\mathbb{C}P^1 = S^2$. The Fubini–Study metric yields the curvature

$$\mathcal{F} = -\partial\bar{\partial}\ln(1 + |z|^2) = -\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + z\bar{z})^2} \tag{11.57}$$

see example 8.8. In real coordinates $z = x + \mathrm{i}y = r\exp(\mathrm{i}\theta)$, we have

$$\mathcal{F} = 2\mathrm{i}\frac{\mathrm{d}x \wedge \mathrm{d}y}{(1 + x^2 + y^2)^2} = 2\mathrm{i}\frac{r\,\mathrm{d}r \wedge \mathrm{d}\theta}{(1 + r^2)^2}. \tag{11.58}$$

From $\mathrm{ch}(\mathcal{F}) = 1 + \mathrm{tr}(\mathrm{i}\mathcal{F}/2\pi)$, we have

$$\mathrm{ch}_1(\mathcal{F}) = -\frac{1}{\pi}\frac{r\,\mathrm{d}r \wedge \mathrm{d}\theta}{(1 + r^2)^2}. \tag{11.59}$$

$\mathrm{Ch}_1(L)$, the integral of $\mathrm{ch}_1(\mathcal{F})$ over S^2 is an integer,

$$\mathrm{Ch}_1(L) = -\frac{1}{\pi}\int \frac{r\,\mathrm{d}r\mathrm{d}\theta}{(1 + r^2)^2} = -\int_1^\infty t^{-2}\,\mathrm{d}t = -1. \tag{11.60}$$

11.3.2 Properties of the Chern characters

Theorem 11.5. (a) (Naturality) Let $E \xrightarrow{\pi} M$ be a vector bundle with $F = \mathbb{C}^k$. Let $f : N \to M$ be a smooth map. Then

$$\mathrm{ch}(f^*E) = f^*\mathrm{ch}(E). \tag{11.61}$$

(b) Let E and F be vector bundles over a manifold M. The Chern characters of $E \otimes F$ and $E \oplus F$ are given by

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F) \tag{11.62a}$$

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) \oplus \mathrm{ch}(F). \tag{11.62b}$$

Proof. (a) follows from theorem 11.2(a).

(b) These results are immediate from the definition of the ch-polynomial. Let

$$\mathrm{ch}(A) = \sum \frac{1}{j!} \operatorname{tr}\left(\frac{\mathrm{i}A}{2\pi}\right)^j$$

be a polynomial of a matrix A. Suppose A is a tensor product of B and C, $A = B \otimes C = B \otimes I + I \otimes C$ (note that $\mathcal{F}_{E\otimes F} = \mathcal{F}_E \otimes I + I \otimes \mathcal{F}_F$). Then we find that

$$\begin{aligned}
\mathrm{ch}(B \otimes C) &= \sum_j \frac{1}{j!}\left(\frac{\mathrm{i}}{2\pi}\right)^j \operatorname{tr}(B \otimes I + I \otimes C)^j \\
&= \sum_j \frac{1}{j!}\left(\frac{\mathrm{i}}{2\pi}\right)^j \sum_{m=1}^{j} \binom{j}{m} \operatorname{tr}(B^m) \operatorname{tr}(C^{j-m}) \\
&= \sum_m \frac{1}{m!} \operatorname{tr}\left(\frac{\mathrm{i}B}{2\pi}\right)^m \sum_n \frac{1}{n!} \operatorname{tr}\left(\frac{\mathrm{i}C}{2\pi}\right)^n = \mathrm{ch}(B)\mathrm{ch}(C).
\end{aligned}$$

Equation (11.62a) is proved if B is replaced by $\mathcal{F}_E$ and C by $\mathcal{F}_F$.

If A is block diagonal,

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} = B \oplus C$$

we have

$$\begin{aligned}
\mathrm{ch}(B \oplus C) &= \sum \frac{1}{j!}\left(\frac{\mathrm{i}}{2\pi}\right)^j \operatorname{tr}(B \oplus C)^j \\
&= \sum \frac{1}{j!}\left(\frac{1}{2\pi}\right)^j [\operatorname{tr}(B^j) + \operatorname{tr}(C^j)] = \mathrm{ch}(B) + \mathrm{ch}(C).
\end{aligned}$$

This relation remains true when A, B and C are replaced by $\mathcal{F}_{E\oplus F}$, $\mathcal{F}_E$ and $\mathcal{F}_F$ respectively. $\square$

Let us see how the splitting principle works in this case. Let L_j $(1 \le j \le k)$ be complex line bundles. From (11.62b) we have, for $E = L_1 \oplus L_2 \oplus \cdots \oplus L_k$,

$$\mathrm{ch}(E) = \mathrm{ch}(L_1) \oplus \mathrm{ch}(L_2) \oplus \cdots \oplus \mathrm{ch}(L_k). \tag{11.63}$$

Since $\mathrm{ch}(L_i) = \exp(x_i)$, we find

$$\mathrm{ch}(E) = \prod_{j=1}^{k} \exp(x_j) \tag{11.64}$$

which is simply (11.50). Hence, the Chern character of a general vector bundle E is given by that of a Whitney sum of k complex line bundles. The characteristic classes themselves cannot differentiate between two vector bundles of the same base space and the same fibre dimension. What is important is their *integral* over the base space.

11.3.3 Todd classes

Another useful characteristic class associated with a complex vector bundle is the **Todd class** defined by

$$\mathrm{Td}(\mathcal{F}) = \prod_j \frac{x_j}{1 - \mathrm{e}^{-x_j}} \tag{11.65}$$

where the splitting principle is understood. If expanded in powers of x_j, $\mathrm{Td}(\mathcal{F})$ becomes

$$\begin{aligned}
\mathrm{Td}\,(\mathcal{F}) &= \prod_j \left(1 + \frac{1}{2}x_j + \sum_{k\ge 1}(-1)^{k-1}\frac{B_k}{(2k)!}x_j^{2k}\right) \\
&= 1 + \tfrac{1}{2}\sum_j x_j + \tfrac{1}{12}\sum_j x_j^2 + \tfrac{1}{4}\sum_{j<k} x_j x_k + \cdots \\
&= 1 + \tfrac{1}{2}c_1(\mathcal{F}) + \tfrac{1}{12}[c_1(\mathcal{F})^2 + c_2(\mathcal{F})] + \cdots
\end{aligned} \tag{11.66}$$

where the B_k are the **Bernoulli numbers**

$$B_1 = \tfrac{1}{6} \qquad B_2 = \tfrac{1}{30} \qquad B_3 = \tfrac{1}{42} \qquad B_4 = \tfrac{1}{30} \qquad B_5 = \tfrac{5}{66} \qquad \ldots.$$

The first few terms of (11.66) are:

$$\mathrm{Td}_0(\mathcal{F}) = 1 \tag{11.67a}$$

$$\mathrm{Td}_1(\mathcal{F}) = \tfrac{1}{2}c_1 \tag{11.67b}$$

$$\mathrm{Td}_2(\mathcal{F}) = \tfrac{1}{12}(c_1^2 + c_2) \tag{11.67c}$$

$$\mathrm{Td}_3(\mathcal{F}) = \tfrac{1}{24}c_1 c_2 \tag{11.67d}$$

$$\mathrm{Td}_4(\mathcal{F}) = \tfrac{1}{720}(-c_1^4 + 4c_1^2 c_2 + 3c_2^2 + c_1 c_3 - c_4) \tag{11.67e}$$

$$\mathrm{Td}_5(\mathcal{F}) = \tfrac{1}{1440}(-c_1^3 c_2 + 3c_1 c_2^2 + c_1^2 c_3 - c_1 c_4) \tag{11.67f}$$

where c_i stands for $c_i(\mathcal{F})$.

Exercise 11.2. Let E and F be complex vector bundles over M. Show that

$$\mathrm{Td}(E \oplus F) = \mathrm{Td}(E) \wedge \mathrm{Td}(F). \tag{11.68}$$

11.4 Pontrjagin and Euler classes

In the present section we will be concerned with the characteristic classes associated with a real vector bundle.

11.4.1 Pontrjagin classes

Let E be a real vector bundle over an m-dimensional manifold M with $\dim_{\mathbb{R}} E = k$. If E is endowed with the fibre metric, we may introduce orthonormal frames at each fibre. The structure group may be reduced to $\mathrm{O}(k)$ from $\mathrm{GL}(k, \mathbb{R})$. Since the generators of $\mathfrak{o}(k)$ are skew symmetric, the field strength $\mathcal{F}$ of E is also skew symmetric. A skew-symmetric matrix A is not diagonalizable by an element of a subgroup of $\mathrm{GL}(k, \mathbb{R})$. It is, however, reducible to block diagonal form as

$$\begin{aligned} A &\to \begin{pmatrix} 0 & \lambda_1 & & & 0 \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ 0 & & & & \ddots \end{pmatrix} \\ &\to \begin{pmatrix} \mathrm{i}\lambda_1 & & & & \\ & -\mathrm{i}\lambda_1 & & 0 & \\ & & \mathrm{i}\lambda_2 & & \\ & & & -\mathrm{i}\lambda_2 & \\ & 0 & & & \ddots \end{pmatrix} \end{aligned} \tag{11.69}$$

where the second diagonalization is achieved only by an element of $\mathrm{GL}(k, \mathbb{C})$. If k is odd, the last diagonal element is set to zero. For example, the generator of $\mathfrak{o}(3) = \mathfrak{so}(3)$ generating rotations around the z-axis is

$$T_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The **total Pontrjagin class** is defined by

$$p(\mathcal{F}) \equiv \det\left(I + \frac{\mathcal{F}}{2\pi}\right). \tag{11.70}$$

From the skew symmetry $\mathcal{F}^t = -\mathcal{F}$, it follows that

$$\det\left(I + \frac{\mathcal{F}}{2\pi}\right) = \det\left(I + \frac{\mathcal{F}^t}{2\pi}\right) = \det\left(I - \frac{\mathcal{F}}{2\pi}\right).$$

Therefore, $p(\mathcal{F})$ is an *even* function in $\mathcal{F}$. The expansion of $p(\mathcal{F})$ is

$$p(\mathcal{F}) = 1 + p_1(\mathcal{F}) + p_2(\mathcal{F}) + \cdots \tag{11.71}$$

where $p_j(\mathcal{F})$ is a polynomial of order $2j$ and is an element of $H^{4j}(M; \mathbb{R})$. We note that $p_j(\mathcal{F}) = 0$ for either $2j > k = \dim E$ or $4j > \dim M$.[1]

Let us diagonalize $\mathcal{F}/2\pi$ as

$$\frac{\mathcal{F}}{2\pi} \to A \equiv \begin{pmatrix} -\mathrm{i}x_1 & & & & \\ & \mathrm{i}x_1 & & 0 & \\ & & -\mathrm{i}x_2 & & \\ & 0 & & \mathrm{i}x_2 & \\ & & & & \ddots \end{pmatrix} \tag{11.72}$$

where $x_k \equiv -\lambda_k/2\pi$, λ_k being the eigenvalues of $\mathcal{F}$. The sign has been chosen to simplify the Euler class defined here. The generating function of $p(\mathcal{F})$ is given by

$$p(\mathcal{F}) = \det(I + A) = \prod_{i=1}^{[k/2]} (1 + x_i^2) \tag{11.73}$$

where

$$[k/2] \Longrightarrow \begin{cases} k/2 & \text{if } k \text{ is even} \\ (k-1)/2 & \text{if } k \text{ is odd.} \end{cases}$$

In (11.73) only *even* powers appear, reflecting the skew symmetry. Each **Pontrjagin class** is computed from (11.73) as

$$p_j(\mathcal{F}) = \sum_{i_1 < i_2 < \ldots < i_j}^{[k/2]} x_{i_1}^2 x_{i_2}^2 \ldots x_{i_j}^2. \tag{11.74}$$

To write $p_j(\mathcal{F})$ in terms of the curvature two-form $\mathcal{F}/2\pi$, we first note that

$$\operatorname{tr}\left(\frac{\mathcal{F}}{2\pi}\right)^{2j} = \operatorname{tr} A^{2j} = 2(-1)^j \sum_{i=1}^{[k/2]} x_i^{2j}.$$

[1] Although $p_m(\mathcal{F}) = 0$, $p_m(B)$ need not vanish for a matrix B. p_m will be used to define the Euler class later.

It then follows that

$$p_1(\mathcal{F}) = \sum_i x_i^2 = -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \operatorname{tr}\mathcal{F}^2 \tag{11.75a}$$

$$\begin{aligned} p_2(\mathcal{F}) &= \sum_{i<j} x_i^2 x_j^2 = \frac{1}{2}\left[\left(\sum_i x_i^2\right)^2 - \sum_i x_i^4\right] \\ &= \frac{1}{8}\left(\frac{1}{2\pi}\right)^4 [(\operatorname{tr}\mathcal{F}^2)^2 - 2\operatorname{tr}\mathcal{F}^4] \end{aligned} \tag{11.75b}$$

$$\begin{aligned} p_3(\mathcal{F}) &= \sum_{i<j<k} x_i^2 x_j^2 x_k^2 \\ &= \frac{1}{48}\left(\frac{1}{2\pi}\right)^6 [-(\operatorname{tr}\mathcal{F}^2)^3 + 6\operatorname{tr}\mathcal{F}^2\operatorname{tr}\mathcal{F}^4 - 8\operatorname{tr}\mathcal{F}^6] \end{aligned} \tag{11.75c}$$

$$\begin{aligned} p_4(\mathcal{F}) &= \sum_{i<j<k<l} x_i^2 x_j^2 x_k^2 x_l^2 \\ &= \frac{1}{384}\left(\frac{1}{2\pi}\right)^8 [(\operatorname{tr}\mathcal{F}^2)^4 - 12(\operatorname{tr}\mathcal{F}^2)^2\operatorname{tr}\mathcal{F}^4 + 32\operatorname{tr}\mathcal{F}^2\operatorname{tr}\mathcal{F}^6 \\ &\quad + 12(\operatorname{tr}\mathcal{F}^4)^2 - 48\operatorname{tr}\mathcal{F}^8] \end{aligned} \tag{11.75d}$$

$$\vdots$$

$$p_{[k/2]}(\mathcal{F}) = x_1^2 x_2^2 \dots x_{[k/2]}^2 = \left(\frac{1}{2\pi}\right)^k \det\mathcal{F}. \tag{11.75e}$$

The reader should verify that

$$p(E \oplus F) = p(E) \wedge p(F). \tag{11.76}$$

It is easy to guess that the Pontrjagin classes are written in terms of Chern classes. Since Chern classes are defined only for complex vector bundles, we must complexify the fibre of E so that complex numbers make sense. The resulting vector bundle is denoted by $E^{\mathbb{C}}$. Let A be a skew-symmetric real matrix. We find that

$$\begin{aligned} \det(I + \mathrm{i}A) &= \det\begin{pmatrix} 1+x_1 & & & 0 & \\ & 1-x_1 & & & \\ & & 1+x_2 & & \\ & 0 & & 1-x_2 & \\ & & & & \ddots \end{pmatrix} \\ &= \prod_{i=1}^{[k/2]}(1 - x_i^2) = 1 - p_1(A) + p_2(A) - \cdots \end{aligned}$$

from which it follows that

$$p_j(E) = (-1)^j c_{2j}(E^{\mathbb{C}}). \tag{11.77}$$

Example 11.5. Let M be a four-dimensional Riemannian manifold. When the orthonormal frame $\{\hat{e}_\alpha\}$ is employed, the structure group of the tangent bundle TM may be reduced to O(4). Let $\mathcal{R} = \frac{1}{2}\mathcal{R}_{\alpha\beta}\theta^\alpha \wedge \theta^\beta$ be the curvature two-form ($\mathcal{R}$ should not be confused with the scalar curvature). For the tangent bundle, it is common to write $p(M)$ instead of $p(\mathcal{R})$. We have

$$\det\left(I + \frac{\mathcal{R}}{2\pi}\right) = 1 - \frac{1}{8\pi^2}\operatorname{tr}\mathcal{R}^2 + \frac{1}{128\pi^4}[(\operatorname{tr}\mathcal{R}^2)^2 - 2\operatorname{tr}\mathcal{R}^4]. \tag{11.78}$$

Each **Pontrjagin class** is given by

$$p_0(M) = 1 \tag{11.79a}$$

$$p_1(M) = -\frac{1}{8\pi^2}\operatorname{tr}\mathcal{R}^2 = -\frac{1}{8\pi^2}\mathcal{R}_{\alpha\beta}\mathcal{R}_{\beta\alpha} \tag{11.79b}$$

$$p_2(M) = \frac{1}{128\pi^4}[(\operatorname{tr}\mathcal{R}^2)^2 - 2\operatorname{tr}\mathcal{R}^4] = \left(\frac{1}{2\pi}\right)^4 \det\mathcal{R}. \tag{11.79c}$$

Although $p_2(M)$ vanishes as a differential form, we need it in the next subsection to compute the Euler class.

11.4.2 Euler classes

Let M be a $2l$-dimensional orientable Riemannian manifold and let TM be the tangent bundle of M. We denote the curvature by $\mathcal{R}$. It is always possible to reduce the structure group of TM down to SO($2l$) by employing an orthonormal frame. The **Euler class** e of M is defined by the square root of the $4l$-form p_l,

$$e(A)e(A) = p_l(A). \tag{11.80}$$

Both sides should be understood as functions of a $2l \times 2l$ matrix A and not of the curvature $\mathcal{R}$, since $p_1(\mathcal{R})$ vanishes identically. However, $e(M) \equiv e(\mathcal{R})$ thus defined is a $2l$-form and, indeed, gives a volume element of M. If M is an odd-dimensional manifold we define $e(M) = 0$, see later.

Example 11.6. Let $M = S^2$ and consider the tangent bundle TS^2. From example 7.14, we find the curvature two-form,

$$\mathcal{R}_{\theta\phi} = -\mathcal{R}_{\phi\theta} = \sin^2\theta\frac{\mathrm{d}\theta \wedge \mathrm{d}\phi}{\sin\theta} = \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi$$

where we have noted that $g_{\theta\theta} = \sin^2\theta$. Although $p_1(S^2) = 0$ as a differential form, we compute it to find the Euler form. We have

$$\begin{aligned} p_1(S^2) = &-\frac{1}{8\pi^2}\operatorname{tr}\mathcal{R}^2 = -\frac{1}{8\pi^2}[\mathcal{R}_{\theta\phi}\mathcal{R}_{\phi\theta} + \mathcal{R}_{\phi\theta}\mathcal{R}_{\theta\phi}] \\ = &\left(\frac{1}{2\pi}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi\right)^2 \end{aligned}$$

from which we read off

$$e(S^2) = \frac{1}{2\pi} \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{11.81}$$

It is interesting to note that

$$\int_{S^2} e(S^2) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta = 2 \tag{11.82}$$

which is the Euler characteristic of S^2, see section 2.4. This is not just a coincidence. Let us take another convincing example, a torus T^2. Since T^2 admits a flat connection, the curvature vanishes identically. It then follows that $e(T^2) \equiv 0$ and $\chi(T^2) = 0$. These are special cases of the **Gauss–Bonnet theorem**,

$$\int_M e(M) = \chi(M) \tag{11.83}$$

for a compact orientable manifold M. If M is odd dimensional both e and χ vanish, see (6.39).

In general, the determinant of a $2l \times 2l$ skew-symmetric matrix A is a square of a polynomial called the **Pfaffian** $\mathrm{Pf}(A)$, [2]

$$\det A = \mathrm{Pf}(A)^2. \tag{11.84}$$

We show that the Pfaffian is given by

$$\mathrm{Pf}(A) = \frac{(-1)^l}{2^l l!} \sum_P \mathrm{sgn}(P) A_{P(1)P(2)} A_{P(3)P(4)} \dots A_{P(2l-1)P(2l)} \tag{11.85}$$

where the phase has been chosen for later convenience. We first note that a skew-symmetric matrix A can be block diagonalized by an element of $\mathrm{O}(2l)$ as

$$S^{\mathrm{t}} A S = \Lambda = \begin{pmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & 0 & \\ & & 0 & \lambda_2 & & & & \\ & & -\lambda_2 & 0 & & & & \\ & & & & \ddots & & & \\ & 0 & & & & & 0 & \lambda_l \\ & & & & & & -\lambda_l & 0 \end{pmatrix}. \tag{11.86}$$

It is easy to see that

$$\det A = \det \Lambda = \prod_{i=1}^{l} \lambda_i^2.$$

[2] See proposition 1.3. The definition here differs in phase from that in section 1.5. It turns out to be convenient to choose the present phase convention in the definition of the Euler class.

To compute $\mathrm{Pf}(\Lambda)$, we note that the non-vanishing terms in (11.85) are of the form $A_{12}A_{34}\ldots A_{2l-1,2l}$. Moreover, there are 2^l ways of changing the suffices as $A_{ij} \to A_{ji}$, such as

$$A_{12}A_{34}\ldots A_{2l-1,2l} \to A_{21}A_{34}\ldots A_{2l-1,2l}$$

and $l!$ permutations of the pairs of indices, for example,

$$A_{12}A_{34}\ldots A_{2l-1,2l} \to A_{34}A_{12}\ldots A_{2l-1,2l}.$$

Hence, we have

$$\mathrm{Pf}(\Lambda) = (-1)^l A_{12}A_{34}\ldots A_{2l-1,2l} = (-1)^l \prod_{i=1}^{l} \lambda_i.$$

Thus, we conclude that a block diagonal matrix Λ satisfies

$$\det \Lambda = \mathrm{Pf}(\Lambda)^2.$$

To show that (11.84) is true for any skew-symmetric matrices (not necessarily block diagonal) we use the following lemma,[3]

$$\mathrm{Pf}(X^{\mathrm{t}}AX) = \mathrm{Pf}(A)\det X. \tag{11.87}$$

If $S^{\mathrm{t}}AS = \Lambda$ for $S \in \mathrm{O}(2l)$, we have $A = S\Lambda S^{\mathrm{t}}$, hence

$$\mathrm{Pf}(S\Lambda S^{\mathrm{t}}) = \mathrm{Pf}(\Lambda)\det S = (-1)^l \prod_{i=1}^{l} \lambda_i \det S.$$

We finally find $\det A = \mathrm{Pf}(A)^2$ for a skew-symmetric matrix A.

Note that $\mathrm{Pf}(A)$ is $\mathrm{SO}(2l)$ invariant but changes sign under an improper rotation S ($\det S = -1$) of $\mathrm{O}(2l)$.

Exercise 11.3. Show that the determinant of an odd-dimensional skew-symmetric matrix vanishes. This is why we put $e(M) = 0$ for an odd-dimensional manifold.

The **Euler class** is defined in terms of the curvature $\mathcal{R}$ as

$$\begin{aligned} e(M) &= \mathrm{Pf}(\mathcal{R}/2\pi) \\ &= \frac{(-1)^l}{(4\pi)^l l!} \sum_P \mathrm{sgn}(P) \mathcal{R}_{P(1)P(2)} \ldots \mathcal{R}_{P(2l-1)P(2l)}. \end{aligned} \tag{11.88}$$

[3] Since $\det(X^{\mathrm{t}}AX) = (\det X)^2 \det A$, we have $\mathrm{Pf}(X^{\mathrm{t}}AX) = \pm\mathrm{Pf}(A)\det X$. Here the plus sign should be chosen since $\mathrm{Pf}(I^{\mathrm{t}}AI) = \mathrm{Pf}(A)$.

The generating function is obtained by taking $x_j = -\lambda_i/2\pi$,

$$e(x) = x_1 x_2 \dots x_l = \prod_{i=1}^{l} x_i. \tag{11.89}$$

The phase $(-1)^l$ has been chosen to simplify the RHS.

Example 11.7. Let M be a four-dimensional orientable manifold. The structure group of TM is SO(4), see example 11.5. The Euler class is obtained from (11.88) as

$$e(M) = \frac{1}{2(4\pi)^2} \epsilon^{ijkl} \mathcal{R}_{ij} \wedge \mathcal{R}_{kl}. \tag{11.90}$$

This is in agreement with the result of example 11.5. The relevant Pontrjagin class is

$$p_2(M) = \frac{1}{128\pi^4}[(\operatorname{tr}\mathcal{R}^2)^2 - 2\operatorname{tr}\mathcal{R}^4] = x_1^2 x_2^2.$$

Since $e(M) = x_1 x_2$, we have $p_2(M) = e(M) \wedge e(M)$. This is written as a matrix identity,

$$\frac{1}{128\pi^4}[(\operatorname{tr} A^2)^2 - 2\operatorname{tr} A^4] = \left(\frac{1}{2(4\pi)^4} \epsilon^{ijkl} A_{ij} A_{kl}\right)^2.$$

11.4.3 Hirzebruch L-polynomial and $\hat{A}$-genus

The **Hirzebruch L-polynomial** is defined by

$$\begin{aligned} L(x) &= \prod_{j=1}^{k} \frac{x_j}{\tanh x_j} \\ &= \prod_{j=1}^{k} \left(1 + \sum_{n\geq 1} (-1)^{n-1} \frac{2^{2n}}{(2n)!} B_n x_j^{2n}\right) \end{aligned} \tag{11.91}$$

where the B_n are Bernoulli numbers, see (11.66). The function $L(x)$ is even in x_j and can be written in terms of the Pontrjagin classes,

$$L(\mathcal{F}) = 1 + \tfrac{1}{3} p_1 + \tfrac{1}{45}(-p_1^2 + 7p_2) + \tfrac{1}{945}(2p_1^3 - 13p_1 p_2 + 62 p_3) + \cdots \tag{11.92}$$

where p_j stands for $p_j(\mathcal{F})$. From the splitting principle, we find that

$$L(E \oplus F) = L(E) \wedge L(F). \tag{11.93}$$

The $\hat{A}$ **(A-roof) genus** $\hat{A}(\mathcal{F})$ is defined by

$$\begin{aligned} \hat{A}(\mathcal{F}) &= \prod_{j=1}^{k} \frac{x_j/2}{\sinh(x_j/2)} \\ &= \prod_{j=1}^{k} \left(1 + \sum_{n\geq 1} (-1)^n \frac{(2^{2n} - 2)}{(2n)!} B_n x_j^{2n}\right). \end{aligned} \tag{11.94}$$

This is an even function of x_j and can be expanded in p_j. $\hat{A}$ is also called the **Dirac genus** by physicists. It satisfies

$$\hat{A}(E \oplus F) = \hat{A}(E) \wedge \hat{A}(F). \tag{11.95}$$

$\hat{A}$ is written in terms of the Pontrjagin classes as

$$\begin{aligned}\hat{A}(\mathcal{F}) = 1 &- \tfrac{1}{24}p_1 + \tfrac{1}{5760}(7p_1^2 - 4p_2) \\ &+ \tfrac{1}{967\,680}(-31p_1^3 + 44p_1p_2 - 16p_3) + \cdots .\end{aligned} \tag{11.96}$$

Example 11.8. Let M be a compact connected and orientable four-dimensional manifold. Let us consider the symmetric bilinear form $\sigma : H^2(M;\mathbb{R}) \times H^2(M;\mathbb{R}) \to \mathbb{R}$ defined by

$$\sigma([\alpha],[\beta]) = \int_M \alpha \wedge \beta. \tag{11.97}$$

σ is a $b^2 \times b^2$ symmetric matrix where $b^2 = \dim H^2(M;\mathbb{R})$ is the Betti number. Clearly σ is non-degenerate since $\sigma([\alpha],[\beta]) = 0$ for any $[\alpha] \in H^2(M;\mathbb{R})$ implies $[\beta] = 0$. Let p (q) be the number of positive (negative) eigenvalues of σ. The **Hirzebruch signature** of M is

$$\tau(M) \equiv p - q. \tag{11.98}$$

According to the **Hirzebruch signature theorem** (see section 12.5), this number is also given in terms of the L-polynomial as

$$\tau(M) = \int_M L_1(M) = \tfrac{1}{3}\int_M p_1(M). \tag{11.99}$$

11.5 Chern–Simons forms

11.5.1 Definition

Let $P_j(\mathcal{F})$ be an arbitrary $2j$-form characteristic class. Since $P_j(\mathcal{F})$ is closed, it can be written locally as an exact form by Poincaré's lemma. Let us write

$$P_j(\mathcal{F}) = \mathrm{d}Q_{2j-1}(\mathcal{A},\mathcal{F}) \tag{11.100}$$

where $Q_{2j-1}(\mathcal{A},\mathcal{F}) \in \mathfrak{g} \otimes \Omega^{2j-1}(M)$. [*Warning:* This cannot be true globally. If $P_j = \mathrm{d}Q_{2j-1}$ globally on a manifold M without boundary, we would have

$$\int_M P_{m/2} = \int_M \mathrm{d}Q_{m-1} = \int_{\partial M} Q_{m-1} = 0$$

where $m = \dim M$.] The $2j-1$ from $Q_{2j-1}(\mathcal{A},\mathcal{F})$ is called the **Chern–Simons form** of $P_j(\mathcal{F})$. From the proof of theorem 11.2(b), we find that Q is given by the transgression of P_j,

$$Q_{2j-1}(\mathcal{A},\mathcal{F}) = TP_j(\mathcal{A},0) = j\int_0^1 \tilde{P}_j(\mathcal{A},\mathcal{F}_t,\ldots,\mathcal{F}_t)\,\mathrm{d}t \tag{11.101}$$

where $\tilde{P}_j$ is the polarization of P_j, $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}^2$ and we set $\mathcal{A}' = \mathcal{F}' = 0$. Since Q_{2j-1} depends on $\mathcal{F}$ and $\mathcal{A}$, we explicitly quote the $\mathcal{A}$-dependence. Of course, $\mathcal{A}'$ can be put equal to zero only on a local chart over which the bundle is trivial.

Suppose M is an even-dimensional manifold ($\dim M = m = 2l$) such that $\partial M \neq \emptyset$. Then it follows from Stokes' theorem that

$$\int_M P_l(\mathcal{F}) = \int_M \mathrm{d}Q_{m-1}(\mathcal{A}, \mathcal{F}) = \int_{\partial M} Q_{m-1}(\mathcal{A}, \mathcal{F}). \tag{11.102}$$

The LHS takes its value in integers, and so does the RHS. Thus Q_{m-1} is a characteristic class in its own right and it describes the topology of the boundary ∂M.

11.5.2 The Chern–Simons form of the Chern character

As an example, let us work out the Chern–Simons form of a Chern character $\mathrm{ch}_j(\mathcal{F})$. The connection $\mathcal{A}_t$ which interpolates between 0 and $\mathcal{A}$ is

$$\mathcal{A}_t = t\mathcal{A} \tag{11.103}$$

the corresponding curvature being

$$\mathcal{F}_t = t\,\mathrm{d}\mathcal{A} + t^2\mathcal{A}^2 = t\mathcal{F} + (t^2 - t)\mathcal{A}^2. \tag{11.104}$$

We find from (11.21) that

$$Q_{2j-1}(\mathcal{A}, \mathcal{F}) = \frac{1}{(j-1)!}\left(\frac{\mathrm{i}}{2\pi}\right)^j \int_0^1 \mathrm{d}t\, \mathrm{str}(\mathcal{A}, \mathcal{F}_t^{j-1}). \tag{11.105}$$

For example,

$$Q_1(\mathcal{A}, \mathcal{F}) = \frac{\mathrm{i}}{2\pi}\int_0^1 \mathrm{d}t\ \mathrm{tr}\,\mathcal{A} = \frac{\mathrm{i}}{2\pi}\,\mathrm{tr}\,\mathcal{A} \tag{11.106a}$$

$$\begin{aligned} Q_3(\mathcal{A}, \mathcal{F}) &= \left(\frac{\mathrm{i}}{2\pi}\right)^2 \int_0^1 \mathrm{d}t\ \mathrm{str}(\mathcal{A}, t\mathrm{d}\mathcal{A} + t^2\mathcal{A}^2) \\ &= \frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \mathrm{tr}\left(\mathcal{A}\mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right). \end{aligned} \tag{11.106b}$$

$$\begin{aligned} Q_5(\mathcal{A}, \mathcal{F}) &= \frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^3 \int_0^1 \mathrm{d}t\ \mathrm{str}[\mathcal{A}, (t\mathrm{d}\mathcal{A} + t^2\mathcal{A}^2)^2] \\ &= \frac{1}{6}\left(\frac{\mathrm{i}}{2\pi}\right)^3 \mathrm{tr}\left[\mathcal{A}(\mathrm{d}\mathcal{A})^2 + \frac{3}{2}\mathcal{A}^3\mathrm{d}\mathcal{A} + \frac{3}{5}\mathcal{A}^5\right]. \end{aligned} \tag{11.106c}$$

Exercise 11.4. Let $\mathcal{F}$ be the field strength of the SU(2) gauge theory. Write down the component expression of the identity $\mathrm{ch}_2(\mathcal{F}) = \mathrm{d}Q_3(\mathcal{A}, \mathcal{F})$ to verify that (cf lemma 10.3)

$$\mathrm{tr}[\epsilon^{\kappa\lambda\mu\nu}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\mu\nu}] = \partial_\kappa[2\epsilon^{\kappa\lambda\mu\nu}\,\mathrm{tr}(\mathcal{A}_\lambda\partial_\mu\mathcal{A}_\nu + \tfrac{2}{3}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu)]. \tag{11.107}$$

11.5.3 Cartan's homotopy operator and applications

For later purposes, we define Cartan's homotopy formula following Zumino (1985) and Alvarez-Gaumé and Ginsparg (1985). Let

$$\mathcal{A}_t = \mathcal{A}_0 + t(\mathcal{A}_1 - \mathcal{A}_0) \qquad \mathcal{F}_t = \mathrm{d}\mathcal{A}_t + \mathcal{A}_t^2 \tag{11.108}$$

as before. Define an operator l_t by

$$l_t\mathcal{A}_t = 0 \qquad l_t\mathcal{F}_t = \delta t(\mathcal{A}_1 - \mathcal{A}_0). \tag{11.109}$$

We require that l_t be an anti-derivative,

$$l_t(\eta_p\omega_q) = (l_t\eta_p)\omega_q + (-1)^p\eta_p(l_t\omega_q) \tag{11.110}$$

for $\eta_p \in \Omega^p(M)$ and $\omega_q \in \Omega^q(M)$. We verify that

$$(\mathrm{d}l_t + l_t\mathrm{d})\mathcal{A}_t = l_t(\mathcal{F}_t - \mathcal{A}_t^2) = \delta t(\mathcal{A}_1 - \mathcal{A}_0) = \delta t\frac{\partial\mathcal{A}_t}{\partial t}$$

and

$$\begin{aligned}(\mathrm{d}l_t + l_t\mathrm{d})\mathcal{F}_t &= \mathrm{d}[\delta t(\mathcal{A}_1 - \mathcal{A}_0)] + l_t[\mathcal{D}_t\mathcal{F}_t - \mathcal{A}_t\mathcal{F}_t + \mathcal{F}_t\mathcal{A}_t] \\ &= \delta t[\mathrm{d}(\mathcal{A}_1 - \mathcal{A}_0) + \mathcal{A}_t(\mathcal{A}_1 - \mathcal{A}_0) + (\mathcal{A}_1 - \mathcal{A}_0)\mathcal{A}_t] \\ &= \delta t\mathcal{D}_t(\mathcal{A}_1 - \mathcal{A}_0) = \delta t\frac{\partial\mathcal{F}_t}{\partial t}\end{aligned}$$

where we have used the Bianchi identity $\mathcal{D}_t\mathcal{F}_t = 0$. This shows that for any polynomial $S(\mathcal{A}, \mathcal{F})$ of $\mathcal{A}$ and $\mathcal{F}$, we obtain

$$(\mathrm{d}l_t + l_t\mathrm{d})S(\mathcal{A}_t, \mathcal{F}_t) = \delta t\frac{\partial}{\partial t}S(\mathcal{A}_t, \mathcal{F}_t). \tag{11.111}$$

On the RHS, S should be a polynomial of $\mathcal{A}$ and $\mathcal{F}$ *only* and not of $\mathrm{d}\mathcal{A}$ or $\mathrm{d}\mathcal{F}$: if S does contain them, $\mathrm{d}\mathcal{A}$ should be replaced by $\mathcal{F} - \mathcal{A}^2$ and $\mathrm{d}\mathcal{F}$ by $\mathcal{D}\mathcal{F} - [\mathcal{A}, \mathcal{F}] = -[\mathcal{A}, \mathcal{F}]$. Integrating (11.111) over [0, 1], we obtain **Cartan's homotopy formula**

$$S(\mathcal{A}_1, \mathcal{F}_1) - S(\mathcal{A}_0, \mathcal{F}_0) = (\mathrm{d}k_{01} + k_{01}\mathrm{d})S(\mathcal{A}_t, \mathcal{F}_t) \tag{11.112}$$

where the **homotopy operator** k_{01} is defined by

$$k_{01}S(\mathcal{A}_t, \mathcal{F}_t) \equiv \int_0^1 \delta t\, l_t S(\mathcal{A}_t, \mathcal{F}_t). \tag{11.113}$$

To operate k_{01} on $S(\mathcal{A}, \mathcal{F})$, we first replace $\mathcal{A}$ and $\mathcal{F}$ by $\mathcal{A}_t$ and $\mathcal{F}_t$, respectively, then operate l_t on $S(\mathcal{A}_t, \mathcal{F}_t)$ and integrate over t.

Example 11.9. Let us compute the Chern–Simons form of the Chern character using the homotopy formula. Let $S(\mathcal{A}, \mathcal{F}) = \mathrm{ch}_{j+1}(\mathcal{F})$ and $\mathcal{A}_1 = \mathcal{A}$, $\mathcal{A}_0 = 0$. Since $\mathrm{d}\,\mathrm{ch}_{j+1}(\mathcal{F}) = 0$, we have

$$\mathrm{ch}_{j+1}(\mathcal{F}) = (\mathrm{d}k_{01} + k_{01}\mathrm{d})\mathrm{ch}_{j+1}(\mathcal{F}_t) = \mathrm{d}[k_{01}\mathrm{ch}_{j+1}(\mathcal{F}_t)].$$

Thus, $k_{01}\mathrm{ch}_{j+1}(\mathcal{F})$ is identified with the Chern–Simons form $Q_{2j+1}(\mathcal{A}, \mathcal{F})$. We find that

$$\begin{aligned} k_{01}\mathrm{ch}_{j+1}(\mathcal{F}_t) &= \frac{1}{(j+1)!} k_{01}\,\mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^{j+1} \\ &= \frac{1}{(j+1)!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1}\int_0^1 \delta t\, l_t\, \mathrm{tr}(\mathcal{F}_t^{j+1}) \\ &= \frac{1}{j!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1}\int_0^1 \delta t\, \mathrm{str}(\mathcal{A}, \mathcal{F}_t^{j}) \end{aligned} \tag{11.114}$$

in agreement with (11.105).

Although a characteristic class is gauge invariant, the Chern–Simons form need not be so. As an application of Cartan's homotopy formula, we compute the change in $Q_{2j+1}(\mathcal{A}, \mathcal{F})$ under $\mathcal{A} \to \mathcal{A}^g = g^{-1}(\mathcal{A} + \mathrm{d})g$, $\mathcal{F} \to \mathcal{F}^g = g^{-1}\mathcal{F}g$. Consider the interpolating families $\mathcal{A}_t^g$ and $\mathcal{F}_t^g$ defined by

$$\mathcal{A}_t^g \equiv t g^{-1}\mathcal{A}g + g^{-1}\mathrm{d}g \tag{11.115a}$$

$$\mathcal{F}_t^g \equiv \mathrm{d}\mathcal{A}_t^g + (\mathcal{A}_t^g)^2 = g^{-1}\mathcal{F}_t g \tag{11.115b}$$

where $\mathcal{F}_t \equiv t\mathcal{F} + (t^2 - t)\mathcal{A}^2$. Note that $\mathcal{A}_0^g = g^{-1}\mathrm{d}g$, $\mathcal{A}_1^g = \mathcal{A}^g$, $\mathcal{F}_0^g = 0$ and $\mathcal{F}_1^g = \mathcal{F}^g$. Equation (11.112) yields

$$Q_{2j+1}(\mathcal{A}^g, \mathcal{F}^g) - Q_{2j+1}(g^{-1}\mathrm{d}g, 0) = (\mathrm{d}k_{01} + k_{01}\mathrm{d})Q_{2j+1}(\mathcal{A}_t^g, \mathcal{F}_t^g). \tag{11.116}$$

For example, let Q_{2j+1} be the Chern–Simons form of the Chern character $\mathrm{ch}_{j+1}(\mathcal{F})$. Since $\mathrm{d}Q_{2j+1}(\mathcal{A}_t^g, \mathcal{F}_t^g) = \mathrm{ch}_{j+1}(\mathcal{F}_t^g) = \mathrm{ch}_{j+1}(\mathcal{F}_t)$, we have

$$\begin{aligned} k_{01}\,\mathrm{d}Q_{2j+1}(\mathcal{A}_t^g, \mathcal{F}_t^g) &= k_{01}\mathrm{ch}_{j+1}(\mathcal{F}_t^g) \\ &= k_{01}\mathrm{ch}_{j+1}(\mathcal{F}_t) = Q_{2j+1}(\mathcal{A}, \mathcal{F}) \end{aligned} \tag{11.117}$$

where the result of example 11.9 has been used to obtain the final equality. Collecting these results, we write (11.116) as

$$Q_{2j+1}(\mathcal{A}^g, \mathcal{F}^g) - Q_{2j+1}(\mathcal{A}, \mathcal{F}) = Q_{2j+1}(g^{-1}\mathrm{d}g, 0) + \mathrm{d}\alpha_{2j} \tag{11.118}$$

where α_{2j} is a $2j$-form defined by

$$\begin{aligned} \alpha_{2j}(\mathcal{A}, \mathcal{F}, v) &\equiv k_{01}Q_{2j+1}(\mathcal{A}_t^g, \mathcal{F}_t^g) \\ &= k_{01}Q_{2j+1}(\mathcal{A}_t + v, \mathcal{F}_t) \end{aligned} \tag{11.119}$$

where $v \equiv \mathrm{d}g \cdot g^{-1}$. [Note that $Q_{2j+1}(\mathcal{A}, \mathcal{F}) = Q_{2j+1}(g\mathcal{A}g^{-1}, g\mathcal{F}g^{-1})$.] The first term on the RHS of (11.118) is

$$\begin{aligned} Q_{2j+1}(g^{-1}\mathrm{d}g, 0) &= \frac{1}{j!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1} \int_0^1 \delta t\, \mathrm{tr}[g^{-1}\mathrm{d}g\{(t^2-t)(g^{-1}\mathrm{d}g)^2\}^j] \\ &= \frac{1}{j!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1} \mathrm{tr}[(g^{-1}\mathrm{d}g)^{2j+1}] \int_0^1 \delta t\, (t^2-t)^j \\ &= (-1)^j \frac{j!}{(2j+1)!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1} \mathrm{tr}[(g^{-1}\mathrm{d}g)^{2j+1}] \end{aligned} \tag{11.120}$$

where we have noted that $\mathcal{F}_t = (t^2 - t)(g^{-1}\mathrm{d}g)^2$ and

$$\int_0^1 \delta t(t^2-t)^j = (-1)^j B(j+1, j+1) = (-1)^j \frac{(j!)^2}{(2j+1)!}$$

B being the beta function. The $2j+1$ form $Q_{2j+1}(g\mathrm{d}g, 0)$ is closed and, hence, locally exact: $\mathrm{d}Q_{2j+1}(g^{-1}\mathrm{d}g, 0) = \mathrm{ch}_{j+1}(0) = 0$.

As for α_{2j} we have, for example,

$$\begin{aligned} \alpha_2 &= \frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \int_0^1 l_t\, \mathrm{tr}[(\mathcal{A}_t + v)\mathcal{F}_t - \tfrac{1}{3}(\mathcal{A}_t + v)^3] \\ &= \frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \int_0^1 \delta t\, \mathrm{tr}(-t\mathcal{A}^2 - v\mathcal{A}) \\ &= -\frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \mathrm{tr}(v\mathcal{A}) \end{aligned} \tag{11.121}$$

where we have noted that

$$\mathrm{tr}\,\mathcal{A}^2 = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\ \mathrm{tr}(\mathcal{A}_\mu \mathcal{A}_\nu) = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu\ \mathrm{tr}(\mathcal{A}_\nu \mathcal{A}_\mu) = 0.$$

Example 11.10. In three-dimensional spacetime, a gauge theory may have a gauge-invariant mass term given by the Chern–Simons three-form (Jackiw and Templeton 1981, Deser *et al* 1982a, b). Since the Chern–Simons form changes by a locally exact form under a gauge transformation, the action remains invariant. We restrict ourselves to the U(1) gauge theory for simplicity. Consider the Lagrangian (we put $\mathcal{A} = \mathrm{i}A$, $\mathcal{F} = \mathrm{i}F$)

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{4}m\epsilon^{\lambda\mu\nu}F_{\lambda\mu}A_\nu \tag{11.122}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the second term is the Chern–Simons form of the second Chern character F^2 (modulo a constant factor) of the U(1) bundle. The field equation is

$$\partial_\mu F^{\mu\nu} + m * F^\nu = 0 \tag{11.123}$$

where

$$*F^{\mu} = \tfrac{1}{2}\epsilon^{\mu\kappa\lambda}F_{\kappa\lambda} \qquad F^{\mu\nu} = \epsilon^{\mu\nu\lambda} * F_{\lambda}.$$

The Bianchi identity

$$\partial_{\mu} * F^{\mu} = 0 \tag{11.124}$$

follows from (11.123) as a consequence of the skew symmetry of $F^{\mu\nu}$. It is easy to verify that the field equation is invariant under a gauge transformation,

$$A_{\mu} \to A_{\mu} + \partial_{\mu}\theta \tag{11.125}$$

while the Lagrangian changes by a total derivative,

$$\mathcal{L} \to -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{4}m\epsilon^{\lambda\mu\nu}F_{\lambda\mu}(A_{\nu} + \partial_{\nu}\theta) = \mathcal{L} + \tfrac{1}{2}m\partial_{\nu}(*F^{\nu}\theta). \tag{11.126}$$

Equation (11.106b) shows that the last term on the RHS is identified with

$$Q_3(A^{\theta}, F^{\theta}) - Q_3(A, F) \sim (A + \mathrm{d}\theta)\,\mathrm{d}A - A\,\mathrm{d}A \sim \mathrm{d}(\theta\mathrm{d}A).$$

If we assume that F falls off at large spacetime distances, this term does not contribute to the action:

$$\int \mathrm{d}^3x\mathcal{L} \to \int \mathrm{d}^3x\mathcal{L} + \frac{m}{2}\int \mathrm{d}^3x\,\partial_{\nu}(*F^{\nu}\theta) = \int \mathrm{d}^3x\mathcal{L}. \tag{11.127}$$

Let us show that (11.122) describes a *massive* field. We first write (11.123) as

$$\epsilon^{\mu\nu\alpha}\partial_{\mu} * F_{\alpha} = -m * F^{\nu}.$$

Multiplying $\varepsilon_{\kappa\lambda\nu}$ on both sides, we have

$$\partial_{\lambda} * F_{\kappa} - \partial_{\kappa} * F_{\lambda} = -mF_{\kappa\lambda}.$$

Taking the ∂^{λ}-derivative and using (11.124), we find that

$$(\partial^{\lambda}\partial_{\lambda} + m^2) * F_{\kappa} = 0 \tag{11.128}$$

which shows that $*F_{\kappa}$ is a massive vector field of mass m.

11.6 Stiefel–Whitney classes

The last example of the characteristic classes is the Stiefel–Whitney class. In contrast to the rest of the characteristic classes, the Stiefel–Whitney class cannot be expressed in terms of the curvature of the bundle. The Stiefel–Whitney class is important in physics since it tells us whether a manifold admits a spin or not. Let us start with a brief review of a spin bundle.

11.6.1 Spin bundles

Let $TM \xrightarrow{\pi} M$ be a tangent bundle with $\dim M = m$. The bundle TM is assumed to have a fibre metric and the structure group G is taken to be $\mathrm{O}(m)$. If, furthermore, M is orientable, G can be reduced down to $\mathrm{SO}(m)$. Let LM be the frame bundle associated with TM. Let t_{ij} be the transition function of LM which satisfies the consistency condition (9.6)

$$t_{ij}t_{jk}t_{ki} = I \qquad t_{ii} = I.$$

A spin structure on M is defined by the transition function $\tilde{t}_{ij} \in \mathrm{SPIN}(m)$ such that

$$\varphi(\tilde{t}_{ij}) = t_{ij} \qquad \tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = I \qquad \tilde{t}_{ii} = I \tag{11.129}$$

where φ is the double covering $\mathrm{SPIN}(m) \to \mathrm{SO}(m)$. The set of $\tilde{t}_{ij}$ defines a **spin bundle** $PS(M)$ over M and M is said to admit a **spin structure** (of course, M may admit many spin structures depending on the choice of $\tilde{t}_{ij}$).

It is interesting to note that not all manifolds admit spin structures. Non-admittance of spin structures is measured by the second Stiefel–Whitney class which takes values in the Čech cohomology group $H^2(M; \mathbb{Z}_2)$.

11.6.2 Čech cohomology groups

Let $\mathbb{Z}_2$ be the *multiplicative* group $\{-1, +1\}$. A **Čech *r*-cochain** is a function $f(i_0, i_1, \ldots, i_r) \in \mathbb{Z}_2$, defined on $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_r} \neq \emptyset$, which is totally symmetric under an arbitrary permutation P,

$$f(i_{P(0)}, \ldots, i_{P(r)}) = f(i_0, \ldots, i_r).$$

Let $C^r(M, \mathbb{Z}_2)$ be the multiplicative group of Čech r-cochains. We define the coboundary operator $\delta : C^r(M; \mathbb{Z}_2) \to C^{r+1}(M; \mathbb{Z}_2)$ by

$$(\delta f)(i_0, \ldots, i_{r+1}) = \prod_{j=0}^{r+1} f(i_0, \ldots, \hat{i}_j, \ldots, i_{r+1}) \tag{11.130}$$

where the variable below the $\hat{}$ is omitted. For example,

$$(\delta f_0)(i_0, i_1) = f_0(i_1)f_0(i_0) \qquad f_0 \in C^0(M; \mathbb{Z}_2)$$

$$(\delta f_1)(i_0, i_1, i_2) = f_1(i_1, i_2)f_1(i_0, i_2)f_1(i_0, i_1) \qquad f_1 \in C^1(M; \mathbb{Z}_2).$$

Since we employ the multiplicative notation, the unit element of $C^r(M; \mathbb{Z}_2)$ is denoted by 1. We verify that δ is nilpotent:

$$(\delta^2 f)(i_0, \ldots, i_{r+2}) = \prod_{j,k=1}^{r+1} f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_k, \ldots, i_{r+2}) = 1$$

since -1 always appears an even number of times in the middle expression (for example if $f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_k, \ldots, i_{r+2}) = -1$, we have $f(i_0, \ldots, \hat{i}_k, \ldots, \hat{i}_j, \ldots, i_{r+2}) = -1$ from the symmetry of f). Thus, we have proved, for any Čech r-cochain f, that

$$\delta^2 f = 1. \tag{11.131}$$

The **cocycle group** $Z^r(M; \mathbb{Z}_2)$ and the **coboundary group** $B^r(M; \mathbb{Z}_2)$ are defined by

$$Z^r(M; \mathbb{Z}_2) = \{f \in C^r(M; \mathbb{Z}_2) | \delta f = 1\} \tag{11.132}$$

$$B^r(M; \mathbb{Z}_2) = \{f \in C^r(M; \mathbb{Z}_2) | f = \delta f', \ f' \in C^{r-1}(M; \mathbb{Z}_2). \tag{11.133}$$

Now the rth **Čech cohomology group** $H^r(M; \mathbb{Z}_2)$ is defined by

$$H^r(M; \mathbb{Z}_2) = \ker\delta_r / \mathrm{im}\delta_{r-1} = Z^r(M; \mathbb{Z}_2)/B^r(M; \mathbb{Z}_2). \tag{11.134}$$

11.6.3 Stiefel–Whitney classes

The **Stiefel–Whitney class** w_r is a characteristic class which takes its values in $H^r(M; \mathbb{Z}_2)$. Let $TM \xrightarrow{\pi} M$ be a tangent bundle with a Riemannian metric. The structure group is $\mathrm{O}(m)$, $m = \dim M$. We assume $\{U_i\}$ is a *simple* open covering of M, which means that the intersection of any number of charts is either empty or contractible. Let $\{e_{i\alpha}\}$ $(1 \le \alpha \le m)$ be a local orthonormal frame of TM over U_i. We have $e_{i\alpha} = t_{ij} e_{j\alpha}$ where $t_{ij} : U_i \cap U_j \to \mathrm{O}(m)$ is the transition function. Define the Čech 1-cochain $f(i, j)$ by

$$f(i, j) \equiv \det(t_{ij}) = \pm 1. \tag{11.135}$$

This is, indeed, an element of $C^1(M; \mathbb{Z}_2)$ since $f(i, j) = f(j, i)$. From the cocycle condition $t_{ij} t_{jk} t_{ki} = I$, we verify that

$$\begin{aligned} \delta f(i, j, k) &= \det(t_{ij}) \det(t_{jk}) \det(t_{ki}) \\ &= \det(t_{ij} t_{jk} t_{ki}) = 1. \end{aligned} \tag{11.136}$$

Hence, $f \in Z^1(M, \mathbb{Z}_2)$ and it defines an element $[f]$ of $H^1(M; \mathbb{Z}_2)$. Now we show that this element is independent of the local frame chosen. Let $\{\bar{e}_{i\alpha}\}$ be another frame over U_i such that $\bar{e}_{i\alpha} = h_i e_{i\alpha}$, $h_i \in \mathrm{O}(m)$. From $\bar{e}_{i\alpha} = \bar{t}_{ij} \bar{e}_{j\alpha}$, we find $\bar{t}_{ij} = h_i t_{ij} h_j^{-1}$. If we define the 0-cochain f_0 by $f_0(i) \equiv \det h_i$, we find that

$$\begin{aligned} \tilde{f}(i, j) &= \det(h_i t_{ij} h_j^{-1}) = \det(h_i) \det(h_j) \det(t_{ij}) \\ &= \delta f_0(i, j) f(i, j) \end{aligned}$$

where use has been made of the identity $\det h_j^{-1} = \det h_j$ for $h_j \in \mathrm{O}(m)$. Thus, f changes by an exact amount and still defines the same cohomology class $[f]$.[4]

[4] Note that the multiplicative notation is being used.

This special element $w_1(M) \equiv [f] \in H^1(M; \mathbb{Z}_2)$ is called the **first Stiefel–Whitney class**.

Theorem 11.6. Let $TM \xrightarrow{\pi} M$ be a tangent bundle with fibre metric. M is orientable if and only if $w_1(M)$ is trivial.

Proof. If M is orientable, the structure group may be reduced to $\mathrm{SO}(m)$ and $f(i, j) = \det(t_{ij}) = 1$, and hence $w_1(M) = 1$, the unit element of $\mathbb{Z}_2$. Conversely, if $w_1(M)$ is trivial, f is a coboundary; $f = \delta f_0$. Since $f_0(i) = \pm 1$, we can always choose $h_i \in \mathrm{O}(m)$ such that $\det(h_i) = f_0(i)$ for each i. If we define the new frame $\bar{e}_{i\alpha} = h_i e_{i\alpha}$, we have transition functions $\tilde{t}_{ij}$ such that $\det(\tilde{t}_{ij}) = 1$ for any overlapping pair (i, j) and M is orientable. [Suppose $f(i, j) = \det t_{ij} = -1$ for some pair (i, j). Then we may take $f_0(i) = -1$ and $f_0(j) = +1$, hence $\det \tilde{t}_{ij} = -\det t_{ij} = +1$.] $\square$

Theorem 11.6 shows that the first Stiefel–Whitney class is an obstruction to the orientability. Next we define the second Stiefel–Whitney class. Suppose M is an m-dimensional orientable manifold and TM is its tangent bundle. For the transition function $t_{ij} \in \mathrm{SO}(m)$, we consider a 'lifting' $\tilde{t}_{ij} \in \mathrm{SPIN}(m)$ such that

$$\varphi(\tilde{t}_{ij}) = t_{ij} \qquad \tilde{t}_{ji} = \tilde{t}_{ij}^{-1} \tag{11.137}$$

where $\varphi : \mathrm{SPIN}(m) \to \mathrm{SO}(m)$ is the $2:1$ homomorphism (note that we have an option $t_{ij} \leftrightarrow \tilde{t}_{ij}$ or $-\tilde{t}_{ij}$). This lifting always exists locally. Since

$$\varphi(\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki}) = t_{ij}t_{jk}t_{ki} = I$$

we have $\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} \in \ker\varphi = \{\pm I\}$. For $\tilde{t}_{ij}$ to define a spin bundle over M, they must satisfy the cocycle condition,

$$\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = I. \tag{11.138}$$

Define the Čech 2-cochain $f : U_i \cap U_j \cap U_k \to \mathbb{Z}_2$ by

$$\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = f(i, j, k)I. \tag{11.139}$$

It is easy to see that f is symmetric and closed. Thus, f defines an element $w_2(M) \in H^2(M, \mathbb{Z}_2)$ called the **second Stiefel–Whitney class**. It can be shown that $w_2(M)$ is independent of the local frame chosen.

Exercise 11.5. Suppose we take another lift $-\tilde{t}_{ij}$ of t_{ij}. Show that f changes by an exact amount under this change. Accordingly, $[f]$ is independent of the lift. [*Hint:* Show that $f(i, j, k) \to f(i, j.k)\delta f_1(i, j, k)$ where $f_1(i, j)$ denotes the sign of $\pm\tilde{t}_{ij}$.]

Theorem 11.7. Let TM be the tangent bundle over an orientable manifold M. There exists a spin bundle over M if and only if $w_2(M)$ is trivial.

Proof. Suppose there exists a spin bundle over M. Then we define a set of transition functions $\tilde{t}_{ij}$ such that $\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = I$ over any overlapping charts U_i, U_j and U_k, hence $w_2(M)$ is trivial. Conversely, suppose $w_2(M)$ is trivial, namely

$$f(i, j, k) = \delta f_1(i, j, k) = f_1(j, k) f_1(i, k) f_1(k, i)$$

f_1 being a 1-cochain. We consider the 1-cochain $f_1(i, j)$ defined in exercise 11.5. If we choose new transition functions $\tilde{t}'_{ij} \equiv \tilde{t}_{ij} f_1(i, j)$, we have

$$\tilde{t}'_{ij}\tilde{t}'_{jk}\tilde{t}'_{ki} = [\delta f_1(i, j, k)]^2 = I$$

and, hence, $\{\tilde{t}'_{ij}\}$ defines a spin bundle over M. □

We outline some useful results:

(a)

$$w_1(\mathbb{C}P^m) = 1 \qquad w_2(\mathbb{C}P^m) = \begin{cases} 1 & m \text{ odd} \\ x & m \text{ even} \end{cases} \tag{11.140}$$

x being the generator of $H^2(\mathbb{C}P^m; \mathbb{Z}_2)$.

(b)

$$w_1(S^m) = w_2(S^m) = 1 \tag{11.141}$$

(c)

$$w_1(\Sigma_g) = w_2(\Sigma_g) = 1 \tag{11.142}$$

Σ_g being the Riemann surface of genus g.