(c) Approximating the instanton/anti-instanton pair $q(\tau) = q_{\text{cl}}(\tau+\bar{\tau}) - q_{\text{cl}}(\tau-\bar{\tau})$ by a "top-hat" function, one finds that $q(\omega) = \int_{-\bar{\tau}/2}^{\bar{\tau}/2} d\tau\; q_0 e^{i\omega\tau} = q_0\bar{\tau}\sin(\omega\bar{\tau}/2)/(\omega\bar{\tau}/2)$. Treating the dissipative term as a perturbation, the action then takes the form

$$S_{\text{eff}} - 2S_{\text{part}} = \frac{\eta}{2}\int_0^{\omega_0} \frac{d\omega}{2\pi}|\omega|(q_0\bar{\tau})^2\frac{\sin^2(\omega\bar{\tau}/2)}{(\omega\bar{\tau}/2)^2} \simeq \frac{q_0^2}{\pi}\eta\ln(\omega_0\bar{\tau}),$$

where ω_0 serves as a high-frequency cut-off.

(d) Interpreted as a probability distribution for the instanton separation, one finds

$$\langle\bar{\tau}\rangle = \int d\bar{\tau}\;\bar{\tau}\;\exp\left[-\frac{q_0^2}{\pi}\eta\ln(\omega_0\bar{\tau})\right] \sim \int^{\infty} d\bar{\tau}\;\bar{\tau}^{1-q_0^2\eta/\pi}.$$

The divergence of the integral shows that, for $\eta > 2\pi/q_0^2$, instanton–anti-instanton pairs are confined and particle tunneling is deactivated. Later, in Chapter 8 we revisit the dissipative phase transition from the standpoint of the renormalization group.

Winding numbers

In the main text, we considered the application of the Feynman path integral to model systems where trajectories could be parameterized in terms of their harmonic (Fourier) expansion. However, very often, one is interested in applications of the path integral to spaces that are not simply connected. In this case, one must include classes of trajectories which cannot be simply continued. Rather, trajectories are classified by their "winding number" on the space. To illustrate the point, let us consider the application of the path integral to a particle on a ring.

(a) Starting with the Hamiltonian $\hat{H} = -(1/2I)(\partial^2/\partial\theta^2)$, where θ denotes an angle variable, show from first principles that the quantum partition function $\mathcal{Z} = \text{tr}e^{-\beta\hat{H}}$ is given by

$$\mathcal{Z} = \sum_{n=-\infty}^{\infty} \exp\left[-\beta\frac{n^2}{2I}\right]. \tag{3.61}$$

(b) Formulated as a Feynman path integral, show that the quantum partition function can be cast in the form

$$\mathcal{Z} = \int_0^{2\pi} d\theta \sum_{m=-\infty}^{\infty} \int_{\substack{(0)=\\ (\beta)=\;(0)+2\pi m}} D\theta(\tau)\,\exp\left[-\frac{I}{2}\int_0^{\beta} d\tau\;\dot{\theta}^2\right].$$

(c) Varying the Euclidean action with respect to θ, show that the path integral is minimized by the classical trajectories $\bar{\theta}(\tau) = \theta + 2\pi m\tau/\beta$. Parameterizing a general path as $\theta(\tau) = \bar{\theta}(\tau) + \eta(\tau)$, where $\eta(\tau)$ is a path with no net winding, show that

$$\mathcal{Z} = \mathcal{Z}_0 \sum_{m=-\infty}^{\infty} \exp\left[-\frac{I}{2}\frac{(2\pi m)^2}{\beta}\right], \tag{3.62}$$

where $\mathcal{Z}_0$ represents the quantum partition function for a free particle with open boundary conditions. Making use of the free particle propagator, show that $\mathcal{Z}_0 = \sqrt{I/2\pi\beta}$.

(d) Finally, making use of Poisson's summation formula, $\sum_m h(m) = \sum_n \int_{-\infty}^{\infty} d\phi\, h(\phi) e^{2\pi i n\phi}$, show that Eq. (3.62) coincides with Eq. (3.61).

Answer:

(a) Solving the Schrödinger equation, the wavefunctions obeying periodic boundary conditions take the form $\psi_n = e^{in\theta}/\sqrt{2\pi}, n$ integer, and the eigenvalues are given by $E_n = n^2/2I$. Cast in the eigenbasis representation, the partition function assumes the form Eq. (3.61).

(b) Interpreted as a Feynman path integral, the quantum partition function takes the form of a propagator with

$$\mathcal{Z} = \int_0^{2\pi} d\theta\, \langle\theta|e^{-\beta\hat{H}}|\theta\rangle = \int_0^{2\pi} d\theta \int_{\theta(\beta)\,=\,\theta(0)\,=\,\theta} D\theta(\tau)\, \exp\left[-\int_0^\beta d\tau\, \frac{I}{2}\dot{\theta}^2\right].$$

The trace implies that paths $\theta(\tau)$ must start and finish at the same point. However, to accommodate the invariance of the field configuration θ under translation by 2π we must impose the boundary conditions shown in the question.

(c) Varying the action with respect to θ we obtain the classical equation $I\ddot{\theta} = 0$. Solving this equation subject to the boundary conditions, we obtain the solution given in the question. Evaluating the Euclidean action, we find that

$$\int_0^\beta (\partial_\tau\theta)^2 d\tau = \int_0^\beta d\tau \left[\frac{2\pi m}{\beta} + \partial_\tau\eta\right]^2 = \beta\left(\frac{2\pi m}{\beta}\right)^2 + \int_0^\beta d\tau (\partial_\tau\eta)^2.$$

Thus, we obtain the partition function (3.62), where

$$\mathcal{Z}_0 = \int D\eta(\tau)\, \exp\left[-\frac{I}{2}\int_0^\beta d\tau(\partial_\tau\eta)^2\right] = \sqrt{\frac{I}{2\pi\beta}},$$

denotes the free particle partition function. This can be obtained from direct evaluation of the free particle propagator.

(d) Applying the Poisson summation formula with $h(x) = \exp[-\frac{(2\pi)^2 I}{2\beta}x^2]$, one finds that

$$\sum_{m=-\infty}^{\infty} e^{-\frac{(2\pi)^2 I m^2}{2\beta}} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi\, e^{-\frac{(2\pi)^2 I}{2\beta}\phi^2 + 2\pi i n\phi} = \sqrt{\frac{\beta}{2\pi I}} \sum_{n=-\infty}^{\infty} e^{-\frac{\beta}{2I}n^2}.$$

Multiplication by $\mathcal{Z}_0$ obtains the result.

Particle in a periodic potential

In Section 3.3 it was shown that the quantum probability amplitude for quantum mechanical tunneling can be expressed as a sum over instanton field configurations of the Euclidean action. By generalizing this approach, the aim of the present problem is to explore quantum mechanical tunneling in a periodic potential. Such an analysis allows us to draw a connection to the problem of the Bloch spectrum.