9

Topology

x In this chapter we discuss low-energy theories with non-trivial forms of long-range order. We learn how to detect the presence of topologically non-trivial structures, and to understand their physical consequences. Topological terms (θ-terms, Wess–Zumino terms, and Chern–Simons terms) are introduced as contributions to the action, affecting the behavior of low-energy field theories through the topology of the underlying field configurations. Applications discussed in this chapter include persistent currents, quantum spin chains, and the quantum Hall effects.

In the preceding chapters we encountered a wide range of long-range orders, or, to put it more technically, different types of mean–fields. Reflecting the feature of (average) translational invariance, the large majority of these mean-fields turned out to be spatially homogeneous. However, there have also been a number of exceptions: under certain conditions, mean-field configurations with long-range[1] spatial textures are likely to form. One mechanism driving the formation of inhomogeneities is the perpetual competition of energy and entropy: being in conflict with the (average) translational invariance of extended systems, a spatially non-uniform mean-field is energetically costly. On the other hand, this very "disadvantage" implies a state of lowered degree of order, or higher entropy. (Remember, for example, instanton formation in the quantum double well: although energetically unfavorable, once it has been created it can occur at any "time," which brings about a huge entropic factor.) It then depends on the spatio-temporal extension of the system whether or not an entropic proliferation of mean-field textures is favorable.

A second mechanism behind the formation of inhomogeneities can be the topological structure of the order parameter field; does the mean-field accommodate solutions that simply cannot be continuously deformed back into a uniform state? The XY-model discussed at the end of the preceding chapter conveniently illustrates this principle: a vortex cannot be eliminated by any smooth deformation of the field. One might argue that this irreducibility is associated with the behavior of the core region of the vortex, where local order breaks down and the mean-field theoretical description simply does not apply (i.e. the vortex cannot be removed by manipulations of the phase field alone). However, an alternative, and more generally valid, explanation of the phenomenon is that a vortex represents a field configuration characterized by a non-vanishing winding number, i.e. an integer different from

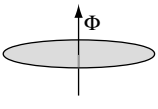
[1] For example, sometimes a system may find it energetically favorable to develop a micro-texture optimally adjusted to the structure of the underlying Hamiltonian (a prominent example being charge density wave formation in one-dimensional systems). Yet even these structures exhibit a discrete translational invariance.

zero. But a non-vanishing integer cannot be continuously "deformed" to zero. (The vortex example also illustrates that the two principles of topology and entropic proliferation tend to cooperate. Once the energy necessary to form a mean-field of non-vanishing winding number has been invested, the system can benefit from the freedom to place the core region anywhere in the system.)

The remarks above touch upon but two of many interesting aspects of systems with topologically non-trivial order parameters. In the next section we employ a trivial example (namely a free particle on a ring) to set the stage for the discussion of further phenomena hinging on topological concepts. Specifically, we introduce the concept of a topological term, i.e. an operator which affects the low-energy behavior of a theory solely on the basis of the topology of its fields. Turning to the more systematic development of the theory, we then introduce homotopy as the key mathematical tool whereby fields can be topologically classified. This discussion provides the conceptual platform on which the rest of the chapter is based. It is followed by a discussion of different classes of topological terms (θ-terms, Wess–Zumino terms, Chern–Simons terms), along with a number of applications. It turns out that, whenever such terms are present in a theory, they tend to massively affect its long-range behavior. At the same time, topological terms are notoriously easy to overlook in "standard" schemes of distilling low-energy theories from their microscopic origins. For this reason, some emphasis is placed on purely operational aspects; i.e. tricks that prevent one from missing the presence of a topological term!

Before getting started, it is worthwhile emphasizing that we are about to plunge into a wide subject area that simply cannot be satisfactorily covered in a single chapter.[2] Consequently, our discussion is example–oriented and often regrettably superficial (with regard to both physical depth and, especially, mathematical structures). In fact, the aim of the present text is to demystify the subject of topology in field theory, to arouse the interest of readers and to motivate them to proceed to more profound and substantial discussions in the literature!

9.1 Example: particle on a ring



Consider the problem of a free quantum particle of charge e confined to one dimension and subject to periodic boundary conditions – a particle on a ring (see figure). To make the problem somewhat more interesting, let us assume that the ring is threaded by a magnetic flux Φ. Measuring the coordinate of the particle in terms of an angular variable $\phi \in [0, 2\pi]$, the free Hamiltonian of the system thus takes the form ($\hbar = e = c = 1$),

$$\hat{H} = \frac{1}{2}(-i\partial_\phi - A)^2, \tag{9.1}$$

[2] For an advanced text specifically targeted on topological considerations in physics, and the underlying mathematical structures, we refer to M. Nakahara, *Geometry, Topology and Physics* (IOP Publishing, 2003).

where $A = \Phi/\Phi_0$ denotes the vector potential corresponding to the magnetic field (exercise), and $\Phi_0 = hc/e = 2\pi$ represents the magnetic flux quantum. (Here, for notational simplicity, we have set the radius of the ring and the particle mass to unity.) Periodicity implies that we are working on a Hilbert space of wavefunctions ψ subject to the condition $\psi(0) = \psi(2\pi)$.

Of course, the problem defined by Eq. (9.1) is embarrassingly simple. One may readily verify that the eigenfunctions and spectrum of the Hamiltonian are given by

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi}} \exp(in\phi), \quad \epsilon_n = \frac{1}{2}\left(n - \frac{\Phi}{\Phi_0}\right)^2, \quad n \in \mathbb{Z}. \tag{9.2}$$

On the other hand, this very simplicity is somewhat deceptive;[3] we shall see in a moment that many of the concepts of topological quantum field theory find a preliminary realization in the problem above.

To explore these connections, let us reformulate the system in the language of the imaginary time path integral (cf. Problem 3.5):

$$\mathcal{Z} = \int\limits_{\phi(\beta)-\phi(0)\in 2\pi\mathbb{Z}} D\phi\, e^{-\int d\tau\, L(\phi,\dot\phi)}, \tag{9.3}$$

where the boundary condition $\phi(\beta) - \phi(0) \in 2\pi\mathbb{Z}$ expresses the fact that the phase is defined only up to integer multiples of 2π, and the Lagrangian is given by

$$L(\phi, \dot\phi) = \frac{1}{2}\dot\phi^2 - iA\dot\phi. \tag{9.4}$$

EXERCISE Verify by Legendre transformation that the Hamiltonian corresponding to this Lagrangian is given by Eq. (9.1). Obtain the spectrum Eq. (9.2) from the path integral, i.e. represent the partition function in the form $\mathcal{Z} = \sum_n \exp(-\beta\epsilon_n)$. (Hint: You may find the Hubbard–Stratonovich decoupling of the quadratic term useful.)

Suppose that we were unaware of the exact solution of the problem. Our canonical approach to controlling the integral would be to subject the theory to a saddle-point analysis. The saddle-point (alias Euler–Lagrange) equations of the action $S[\phi] = \int_0^\beta d\tau\, L(\phi, \dot\phi)$,

$$\frac{\delta S[\phi]}{\delta\phi(\tau)} = 0 \qquad \ddot\phi = 0,$$

have two interesting properties. (i) The vector potential does not enter the equations. On the other hand, we saw above that it *does* have a physical effect (the spectrum explicitly depends on A). We need to understand how these two seemingly contradictory observations can be reconciled with each other. (ii) There exists a whole family of solutions, $\phi_W(\tau) \equiv W 2\pi\tau/\beta$. The action of these configurations, $S[\phi_W]|_{A=0} = \frac{1}{2\beta}(2\pi W)^2$, varies discontinuously with W, i.e. by analogy with other cases where we found saddle-point

[3] Note that the literature is full of erroneous statements on even this simple system. The most frequent of these is that the periodic boundary conditions of the problem force the flux to be quantized in integer multiples of the flux quantum $\Phi = n\Phi_0$. This is, of course, incorrect. Even for non-integer Φ, the wavefunctions ϕ_n are perfectly periodic. The statement is probably triggered by flux-quantization in superconducting systems (a phenomenon that relies on energetic, and not topological, considerations).

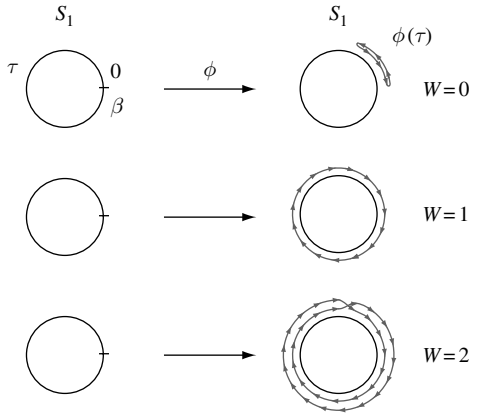


Figure 9.1 Showing mappings $\phi : S_1 \to S_1$ of different winding numbers.

configurations defined by an integer index (cf. instanton solutions in the quantum double well), we may expect the different solutions to be separated by huge energy barriers. However, the present problem is special insofar as, besides energy, there is a much more "profound" principle separating configurations of different W.

To understand this point, let us note that, mathematically, the field ϕ is a mapping

$$\begin{aligned} \phi : S_1 &\to S_1, \\ \tau &\mapsto \phi(\tau), \end{aligned}$$

from the unit circle S_1 (imaginary time with periodic boundary conditions[4]) into another circle (ϕ is a phase!). Mappings of this type can be assigned to a **winding number**, W, i.e. the number of times $\phi(\tau)$ winds around the unit circle as τ progresses from 0 to β: $\phi(\beta) - \phi(0) = 2\pi W$ (see Fig. 9.1).

Indeed, it is not possible to change W by a continuous deformation of ϕ. Since continuity is a paradigm implicit in field integration, the integration over all functions $\phi(\tau)$ can be organized as an integration over functions $\phi(\tau)$ of different winding numbers, or different **topological sectors**:

$$\mathcal{Z} = \sum_W \int\limits_{\phi(\beta)-\phi(0)=2\pi W} D\phi\, e^{-\int d\tau\, L(\phi,\dot\phi)} = \sum_W e^{2\pi i W A} \int\limits_{\phi(\beta)-\phi(0)=2\pi W} D\phi\, e^{-\frac{1}{2}\int d\tau\, \dot\phi^2}. \tag{9.5}$$

Here, we have noted that the A-dependent term in the action,

$$S_{\rm top}[\phi] \equiv iA \int_0^\beta d\tau\, \dot\phi = iA(\phi(\beta) - \phi(0)) = i2\pi W A,$$

involves only the index of the topological sector of ϕ. The representation (9.5) makes the topological aspects of the problem particularly transparent. Specifically, one may note that:

[4] Strictly speaking, imaginary time should be identified with a "circle" of circumference β. However, for our present purposes, all that matters is that the periodic boundary conditions render the interval $[0, \beta]$ isomorphic to a circle.

- ▷ The functional integral assumes the form of a sum of integrals over disjoint topological sectors.
- ▷ The contribution to the action, $S_{\rm top}$, is our first example of a topological term. (More precisely, it belongs to the class of " **-terms**.")
- ▷ Since $S_{\rm top}$ is sensitive only to the topological sector of a field contribution, it cannot affect the equations of motion. This is because these probe how the action responds to an infinitesimal variation of the field configuration, an operation that cannot change the winding number. (You may wish to ponder this point since it will be important.)
- ▷ However, the topological term does affect the outcome of the functional integration: it plays the role of a W-dependent "phase," weighting the contribution of different sectors.
- ▷ The fact that $S_{\rm top}$ knows only about the topological class of a field configuration implies that it is impervious to any changes in the metric of the base manifold of the theory (in our case, imaginary time). For example, we might decide to measure time in different units, i.e. $\tau \to \alpha\tau = \tau'$, where α is some scaling factor. This transformation leaves the topological term invariant.
- ▷ In particular, it remains form invariant under a change from imaginary to real time, $\tau \to -i\tau = t$. In both representations, $S_{\rm top}[\phi] = 2\pi i AW$ is purely imaginary. This, in fact, is a hallmark of topological terms; in both Euclidean and Minkowski space-time, their contribution to the action is always imaginary.

While formulated for the (almost trivial) example of the particle on the ring, all of these features generalize to much more involved settings. However, to discuss these generalizations in a sensible manner, we need to provide somewhat more mathematical background. This will be the subject of the next section.

INFO Owing to its simplicity, the system above frequently appears as an effective model in condensed matter physics. Examples we have encountered previously include the Josephson junction (where the presence of a condensate induces an additional cosine potential) and the physics of normal metal granules subject to strong charging (with the additional complication of dissipative damping of the ϕ-fluctuations). Here we briefly touch upon the physics of **persistent currents** as an example where the topological aspects of the problem play a particularly important role.

Consider then a ring-shaped conductor subject to a magnetic flux. According to a prediction by Byers and Yang[5] the magnetic field induces an equilibrium current

$$I(\Phi) = -\frac{\partial F(\Phi)}{\partial \Phi},$$

periodic in Φ with period Φ_0.

EXERCISE Remembering that a vector potential enters the free energy as $\sim \int d\mathbf{A} \cdot \mathbf{j}$, derive the Byers and Yang formula. Show that, at zero temperature, the persistent current flowing in a perfectly clean one-dimensional metal of non-interacting fermions assumes the form of a Φ_0-periodic sawtooth function, $I(\Phi) = \frac{2\pi v_{\rm F}}{L}[\Phi/\Phi_0]$, where $[x] = x - n$ and n is the largest integer smaller than x. (Hint: For zero temperature, the free energy of a non-interacting system of particles is equal to the sum of all single-particle energies up to the Fermi energy. Notice that the current is

[5] B. Byers and C. N. Yang, Theoretical considerations concerning quantized magnetic flux in superconducting cylinders, *Phys. Rev. Lett.* **7** (1961), 46–9.

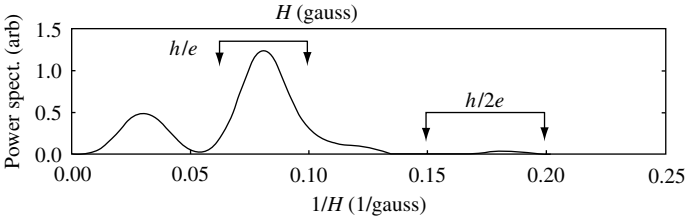


Figure 9.2 Power spectrum of the persistent current carried by a single gold loop (diameter $\mathcal{O}(1\,\mu\text{m})$, $L/\ell \sim 10$). The data shows a peak at h/e and a small satellite modulation at $h/2e$. The magnitude of the current is some two orders of magnitude larger than that predicted by theory. (Reprinted with permission from V. Chandrasekhar, R. A. Wells, M. J. Brady, *et al.* Magnetic response of a single, isolated gold loop, *Phys. Rev. Lett.* **67** (1991), 3578–81. Copyright (1991) by the American Physical Society.)

carried by the last occupied state, i.e. the currents $-\partial_\phi \epsilon_n$ carried by all levels beneath the Fermi energy cancel.)

For a long time it has been believed that this statement was largely academic since surely even a moderate concentration of impurities[6] would be sufficient to render the current undetectable. However, a simple consideration shows that this need not be the case: a gauge transformation $\psi(\phi) \to e^{iA\phi}\psi(\phi)$ removes A from the Hamiltonian while changing the boundary conditions to $\psi(0) = e^{2\pi i A}\psi(2\pi)$. In the gauge-transformed picture, the presence of the magnetic field thus amounts to a twist in the boundary conditions of the wavefunctions, and the persistent current is a measure of the sensitivity of the spectrum to this twist. Now, there is no reason to believe that a wavefunction in a disordered system should be less sensitive to its boundary conditions than that of a clean system.[7] Indeed, it is shown in Problem 9.7.1 that even rings of circumference $L \gg \ell$ may carry a sizeable persistent current.

In a series of beautiful experiments conducted in the early 1990s (see, e.g., Fig. 9.2), persistent currents in both "ballistic" ($L/\ell \sim 1$) and "dirty" ($L/\ell \sim 100$) environments were indeed observed experimentally. Frustratingly, the measured current appears to be some two orders of magnitude larger than the theories of non-interacting particles would predict. This disturbing discrepancy led to the formulation of a plethora of theories of persistent currents in interacting/disordered systems. However, to date, the discrepancy between theory and experiment remains unresolved.

[6] More precisely, it was believed that a sizeable current might only be observed if the circumference of the ring were smaller than the scattering mean free path. The artificial fabrication of rings of that quality requires semiconductor technology which was not available at the time. With regard to *molecular* rings, equilibrium currents had been predicted in the 1930s. However, the fields needed to drive currents in molecules ($\mathcal{O}(10^5\ \text{T})$) cannot be generated by the laboratory magnets currently available.

[7] What does diminish this sensitivity is mechanisms destroying the coherence of wavefunctions (thermal noise, etc.) and strong localization, i.e. a wavefunction confined to a finite portion of the ring will not sense changes in the boundary conditions. In fact, sensitivity with respect to changes in the boundary conditions has been used (largely by theorists) as a popular test for localization.

9.2 Homotopy

9.2.1 Generalities

To appreciate the generalization of the structures above to more complicated contexts, we need to introduce a few more mathematical concepts. In Chapter 1 (see page 6), we defined fields as mappings

$$\begin{aligned} \phi : M &\to T, \\ z &\mapsto \phi(z), \end{aligned}$$

from a "base manifold" M – usually $\mathbb{R}^d$ or a subset thereof – into some "target space" T. By now we have gained enough experience to determine what are the "most frequent" realizations of target spaces: indeed we have mostly been interested in the long-range behavior of Goldstone modes, i.e. long-range modes induced by a mechanism of symmetry breaking. In such cases (cf. page 257), $T = G/H$ will be a coset space obtained by dividing the symmetry group of the problem, G, by some subgroup $H \subset G$ stabilizing the mean-field around which the Goldstone modes fluctuate. Usually (but certainly not invariably) G will be one of the compact classical groups $\mathrm{U}(N)$, $\mathrm{O}(N)$, and $\mathrm{Sp}(N)$, and H some subgroup thereof.

INFO Why is the word "**topology**" so often deployed in this chapter? In the example above, the most relevant characteristic of a field configuration was its winding number, i.e. a quantity that does not change under any continuous deformation of a field, no matter how "large" is this deformation. More generally, in this chapter we are concerned with features of field theories that essentially rely on the concept of continuity, but do not involve the notion of "distance" or, more formally, of a metric.

The most general mathematical structure for which the notion of continuity can be defined is a **topological space**:[8] a mapping $\phi : X \to Y$ between two topological spaces is called **continuous** if, for any open set $U \subset Y$, the set $\phi^{-1}(U) \subset X$ is open in X. This definition is not tied to the existence of a metric.[9] This remark is not entirely academic: there are prominent physical spaces – the phase space of classical mechanics, for example – which do not possess a canonical metric but for which the notion of a continuous mapping certainly exists.

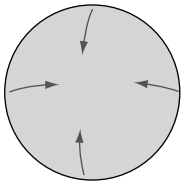
Although our discussion below does not rely on the in-depth mathematics of differential topology, the frequent use of the attribute "topological" emphasizes that what matters in this chapter is "continuity minus metric."

In fact, practical considerations also enable us to be a bit more specific as to the structure of the base manifold. Suppose our microscopic parent theory is defined on some simply connected manifold $M \subset \mathbb{R}^d$. As we shall be typically concerned with some kind of thermodynamic limit, M can be thought of as an "infinitely large" object. On the level of the low-energy theory of the system, this requires that "sensible" field configurations must

[8] As a reminder, let X be a set and $\mathcal{J} = \{Y_i \quad X | i \in I\}$ a collection of its subsets. The pair $(X, \mathcal{J})$ is called a topological space if and only if, (a) $\{\}, X \in \mathcal{J}$, (b) for $J \quad I$, $\quad_{i \quad J} Y_i \in \mathcal{J}$, and, (c) for any finite subset $J \quad I$, $\quad_{i \quad J} Y_i \in \mathcal{J}$. The elements of $\mathcal{J}$ are called **open subsets** of X.

[9] By contrast, in elementary mathematics courses, the continuity of a mapping ϕ is frequently defined via a metric, i.e. one relates the distance between two image points $\phi(x)$ and $\phi(y)$ to that between the arguments x and y.

approach a constant value on the boundary of M: i.e. $\phi|_{\partial M} = \text{const.}$ (lest the action become infinite).



As far as our effective field theory is concerned, we may thus identify all boundary points with a single point at "infinity." Geometrically, this implies that M can be compactified to a large sphere (see the figure). However, as the radius of this sphere carries no significance (indeed, technically, it can be removed by a global rescaling of coordinates), we can, just the same, identify $M \simeq S^d$ with the d-dimensional unit sphere. The practical upshot of all of these considerations is that, in practice, we shall mostly be interested in fields

$$\begin{aligned} \phi : S^d &\rightarrow G/H, \\ z &\mapsto \phi(z), \end{aligned}$$

mapping a unit sphere into some coset space.

We now turn to the discussion of topological aspects of such mappings. By analogy to our discussion of the previous section (where we considered mappings $\phi : S^1 \rightarrow S^1 \simeq \mathrm{O}(2)$), let us consider two fields ϕ_1 and ϕ_2 as topologically equivalent if they can be continuously deformed into each other. Technically, this condition amounts to the existence of a continuous mapping (a **homotopy** in the language of mathematics)

$$\begin{aligned} \phi : S^d \times [0,1] &\rightarrow G/H, \\ (z,t) &\mapsto \hat{\phi}(z,t), \end{aligned}$$

such that $\hat{\phi}(\,.\,,0) = \phi_1$ and $\hat{\phi}_2(\,.\,,1) = \phi_2$. (Notice that $\hat{\phi}$ represents a mapping from $(d{+}1)$-dimensional space into G/H, a fact that will become important when we turn to the discussion of Wess–Zumino terms below.) We denote the **equivalence class** of all fields topologically equivalent to a given representative ϕ by $[\phi]$. (In the example discussed in the previous section, those equivalence classes would contain all fields of a given winding number.) The set of all topological equivalence classes $\{[\phi]\}$ of mappings $\phi : S^d \rightarrow T$ is called the ***d*-th homotopy group**, $\pi_d(T)$.

INFO Some readers might wonder in what sense $\pi_d(T)$ carries a group structure (rather than just being a set). To understand this point, it is convenient to deform our base manifold from a sphere to a d-dimensional unit cube $I^d = [0,1]^d$. As far as topology is concerned, this is a permissible operation, if and only if and only if we identify the boundary ∂I^d of the cube with a single point on the sphere. For example, it is convenient to choose this point to be the representative of the "infinitely large" boundary of our original base manifold. This choice requires that $\phi|_{\partial I^d} = \phi^* = \text{const.}$ be the constant field configuration approached at the "physical infinity".[10]

[10] The assumption of a unique asymptotic configuration ϕ^* is less of a restriction than it may seem; any (constant) boundary field ϕ can be converted to ϕ^* by a global transformation acting on the field (an operation that does not leave the equivalence class).

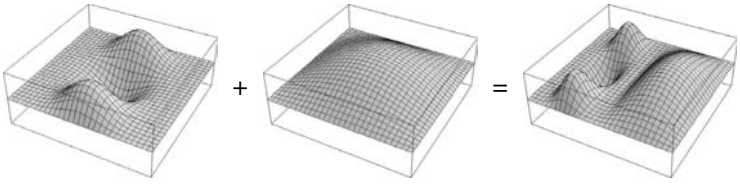


Figure 9.3 Concatenation of two two-dimensional fields into a single one.

The purpose of the cube construction is that it presents us with ways to glue two fields ϕ_1 and ϕ_2 together to form a new field $\phi_3 \equiv \phi_1 * \phi_2$ (see Fig. 9.3). For example, we might define

$$\phi_3(x_1, x_2, \ldots, x_d) = \begin{cases} \phi_1(2x_1, x_2, \ldots, x_d), & x_1 \in [0, 1/2], \\ \phi_2(2x_1 - 1, x_2, \ldots, x_d), & x_1 \in [1/2, 1]. \end{cases}$$

One might justly object that, due to its explicit coordinate dependence, this is not an acceptable mathematical definition. However, as far as our discussion of homotopy is concerned, this objection is immaterial. Indeed, we may define a group operation on $\pi_d(T)$ by

$$[\phi_1] * [\phi_2] \equiv [\phi_3]. \tag{9.6}$$

This definition is canonical in the sense that any other coordinate convention in our concatenation operation above would not leave the equivalence class of ϕ_3 (a point on which you might like to dwell). Similarly one may verify that "$*$" obeys all the criteria for a group mapping. In short, the very possibility to "concatenate" fields induces a well-defined group operation on the set of topological equivalence classes.

9.2.2 Examples of homotopies

The general group theoretical analysis of homotopies is a mathematical subject that reaches far beyond the scope of the present text. Here we restrict ourselves to the discussion of a few examples of practical interest. In simple cases, the homotopy group can be identified by common sense reasoning. For example, in the previous section we saw that mappings $S^1 \mapsto S^1$ can be classified in terms of winding numbers: $\pi_1(S^1) = \mathbb{Z}$. Similarly, it is clear that any mapping $S^1 \to S^2$ – a closed curve on the 2-sphere – is continuously contractible to a point: $\pi_1(S^2) = \emptyset$. The same applies, by definition, to any curve in a simply connected space. Prominent examples of such spaces are the higher-dimensional spheres $S^{d>1}$ and $\mathrm{SU}(N)$: $\pi_1(S^{d>1}) = \pi_1(\mathrm{SU}(N)) = \emptyset$. By contrast, the first homotopy groups of non-simply connected spaces are categorically non-trivial. For example, curves on the d-dimensional torus T^d are classified by (exercise: convince yourself of the veracity of this statement)

$$\pi_1(T^d) = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{d}.$$

Turning to higher dimensions, it becomes more and more difficult to identify homotopy classes simply by invoking one's imagination. One of the last intuitively accessible examples is $\pi_2(S^2) = \mathbb{Z}$: maps of the 2-sphere into itself can be classified according to how often they

Table 9.1 *Homotopy groups of a number of frequently encountered mappings. A blank entry means that no general statements can be made.*

	S^1	S^2	$S^{d>2}$	T^d	SU(2)	SU(N)
$k=1$	$\mathbb{Z}$			$\underbrace{\mathbb{Z} \quad \cdots \quad \mathbb{Z}}_{d}$		
$k=2$		$\mathbb{Z}$				
$k>2$		[a]	$k<d$: $k=d$: $\mathbb{Z}$ $k>d$:		$\pi_3(\mathrm{SU}(2))=\mathbb{Z}$	

[a] But $\pi_3(S^2)=\mathbb{Z}$.

wrap around the sphere. This statement generalizes to $\pi_d(S^d)=\mathbb{Z}$, while $\pi_k(S^{d>k})=\emptyset$. Interestingly, the situation can be non-trivial for mappings $S^k \mapsto S^{d<k}$. For example, Hopf has shown that $\pi_3(S^2)=\mathbb{Z}$. For a summary of these, and a few more results, see Table 9.1.

Far from being complete, the list of examples in Table 9.1 fails to cover a number of homotopies of outstanding relevance. For example, in condensed matter physics (unlike particle physics where space and time are intertwined by relativistic covariance) we are typically confronted with a situation where time is separately compactified to a circle – imaginary time with periodic boundary conditions. On top of that, the finite-action arguments outlined above motivate a compactification of $\mathbb{R}^d \to S^d$ of real space to a d-dimensional sphere. This implies that, in quantum statistical field theory, one often encounters the base manifold $M \simeq S^1 \times S^d$ (instead of S^{d+1} as found in particle physics). Clearly the construction of homotopic groups $\pi(S^1 \times S^d, T)$ corresponding to mappings $S^1 \times S^d \to T$ is more complex than that involved in the definition of the group structures above. However, rather than dwelling on the near-endless field of homotopy theory, we now return to field theory and explore general implications of the homotopic classification scheme.[2]

9.3 -Terms

Returning to field theory considerations, let us address the question of what can be learned from the concepts introduced in the previous section. Each field $\phi : M \to T$ can be uniquely assigned to a certain homotopy class. Consequently, the functional integration defining a field theory can be organized as

$$\mathcal{Z} = \sum_{W \in G} \int D\phi_W \, e^{-S[\phi]},$$

where G is the homotopy group and $\int D\phi_W$ denotes integration over the homotopy class defined by a given element $W \in G$ – a "**topological sector**" of the theory. It may happen that the action of our field theory,

$$S[\phi] = S_0[\phi] + S_{\mathrm{top}}[\phi],$$

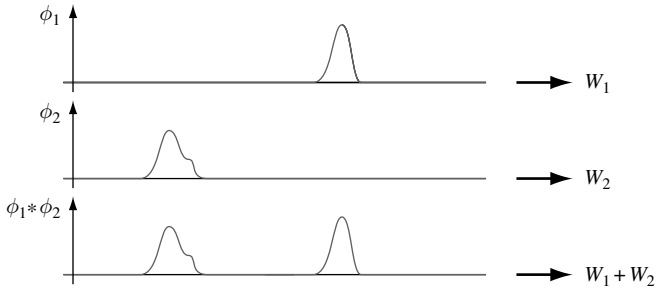


Figure 9.4 The combination of two fields with winding numbers W_1 and W_2 leads to a new field with winding number $W_1 + W_2$.

contains a **topological action** component: i.e. a contribution $S_{\rm top}[\phi] \equiv F(W)$ that depends only on the topological class of the field ϕ (which, for integer-valued W, is sometimes called the **topological charge** of the configuration). In such cases, one may pull the function $F(W)$ in front of the functional integral to write

$$\mathcal{Z} = \sum_{W \in G} e^{-F(W)} \int D\phi_W \, e^{-S_0[\phi]}. \tag{9.7}$$

In the following, we would like to understand in what way the topological action depends on the index W. This question can also be addressed by the type of reasoning ubiquitous in this chapter. Suppose we are given two fields ϕ_1 and ϕ_2 which are constant everywhere save for two well-localized regions of variation somewhere in space-time. Let us assume that the two regions where the fields vary are "infinitely" far away from each other (see Fig. 9.4). Now, suppose we had glued these fields together (e.g. by the prescription formulated in the previous section) to form a new field $\phi_1 * \phi_2$. The infinitely large separation of the two constituent fields implies that they are completely "uncorrelated," i.e. the action of the composite field $S[\phi_1 * \phi_2] = S[\phi_1] + S[\phi_2]$ is obtained simply by adding the actions of the constituents. In particular, $F(W_1 + W_2) = S_{\rm top}[\phi_1 * \phi_2] = S_{\rm top}[\phi_1] + S_{\rm top}[\phi_2] = F(W_1) + F(W_2)$, where in the first equality we have used the fact that the composite field has topological index $W_1 + W_2$.

The identity $F(W_1 + W_2) = F(W_1) + F(W_2)$ tells us that the topological action is linear in the topological index. For example, consider the simple (and at the same time most frequently encountered) case where $\pi(M, T) \simeq \mathbb{Z}$, i.e. $W_i \in \mathbb{Z}$ are just numbers. The linearity then uniquely determines the topological action,

$$F[W] = i\theta W,$$

up to a constant (which we choose to be real lest the action become ill-defined at large values of the topological charge).

These considerations tell us that the factor $\exp(-F[W]) = \exp(-i\theta W)$ weighing the different sectors assumes the form of a *phase*. Relatedly, the constant θ is usually referred to as a **topological angle**. (Since W is integer, θ is defined only mod 2π – an angular

variable.) For historical reasons[11] the topological action $S_{\rm top}$ is commonly referred to as a -**term**. Notwithstanding its simplicity, a drawback of this representation is that it explicitly relies on the decomposition of the field integral into a sum over distinct topological sectors. It would be much more desirable to work with a representation $S_{\rm top}[\phi]$ directly in terms of the field (rather than through its winding number). This would enable us to formulate the field integral more directly as $\int D\phi \exp(-S_0[\phi] - S_{\rm top}[\phi])$, without the necessity to explicitly sum over winding numbers.

Indeed, it is almost always possible to represent the topological action as an integral over a topological Lagrangian density,

$$S_{\rm top}[\phi] = \int d^d x \, \mathcal{L}_{\rm top}[\phi, \partial_\mu \phi].$$

Unfortunately, no canonical recipe for the construction of this representation exists. However, this is not as serious a problem as one might suspect. For one thing, the Lagrangian densities of the "usual suspects" of topologically non-trivial field theories are known. Conversely, it is usually straightforward to deduce whether or not a given term in the action is a θ-term in disguise. To illustrate these points, we next discuss the concept of a topological Lagrangian density on a few relevant examples.

In fact, we have already encountered the simplest representative of a θ-term in our analysis of the particle on a ring:

$$S_{\rm top}[\phi] \equiv i\theta \int_0^\beta \frac{d\tau}{2\pi} \dot{\phi} = iW\theta,$$

where $\theta = 2\pi A$ was proportional to the magnetic flux through the ring. We reiterate the key features of this term: it does not affect the equations of motion (a small distortion, or variation, of the field does not change its topological index, i.e. it leaves $S_{\rm top}$ invariant), and it is invariant under arbitrary coordinate reparameterizations $\tau \mapsto s = s(\tau)$. As a less simple example illustrating these features we now discuss a two-dimensional field theory.

9.3.1 A case study: $\pi_2(S^2)$

Consider a field theory in a two-dimensional compactified space with the two-sphere as a target manifold. Important examples falling into this category include the theory of quantum spin chains (see Section 9.3.3), the two-dimensional classical Heisenberg model, and the field theory of the integer quantum Hall effect (see Section 9.3.4). Technically, the fields of this theory are mappings

$$\begin{aligned} \mathbf{n} : S^2 &\to S^2, \\ \mathbf{x} &\mapsto \mathbf{n}(\mathbf{x}), \quad |\mathbf{n}| = 1, \end{aligned}$$

[11] One of the first major applications where terms of this type played a dominant role was 'tHooft's analysis of SU(2) gauge field instantons in $(3+1)$-dimensional compactified space-time. (G. 'tHooft, Magnetic monopoles in unified gauge theories, *Nucl. Phys.* **B79** (1974), 276–84.) There he systematically labeled the topological angle by θ; hence the name θ-term.

and the relevant homotopy group is $\pi_2(S^2) \simeq \mathbb{Z}$. Let us suppose that the topological action of this field theory is given by

$$S_{\rm top}[\mathbf{n}] = \frac{i\theta}{4\pi} \int dx_1\, dx_2\, \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}). \tag{9.8}$$

How does one verify an assertion of this type? A first condition for $S_{\rm top}$ to be of topological nature is an insensitivity to small variations of the field $\mathbf{n}$. Suppose then we vary $\mathbf{n}$ by a small amount, i.e. $\mathbf{n}(\mathbf{x}) \to \mathbf{n}(\mathbf{x}) + \epsilon^a(\mathbf{x}) R^a \mathbf{n}$, where the functions $\epsilon^a, a = 1, 2, 3$, are infinitesimal and R^a are the generators of rotations around the three coordinate axes. By integrating a few times by parts, it is straightforward to show that the variation of $S_{\rm top}$ assumes the form

$$\delta S_{\rm top}[\mathbf{n}] = \frac{3i\theta}{4\pi} \int dx_1\, dx_2\, \epsilon^a R^a \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}).$$

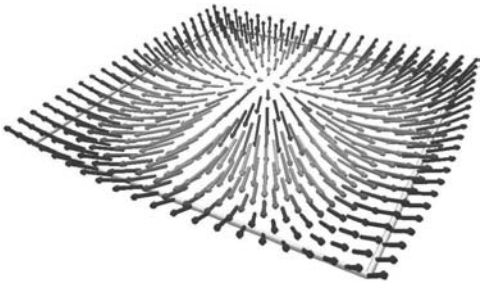
However, $R^a \mathbf{n}$ is perpendicular to $\mathbf{n}$ while (exercise) $(\partial_1 \mathbf{n} \times \partial_2 \mathbf{n})$ lies parallel, i.e. $\delta S_{\rm top} = 0$. (As a corollary we note that $S_{\rm top}$ will not affect the equations of motion.)

The invariance of $S_{\rm top}$ implies that we can evaluate its value on any convenient test field configuration $\mathbf{n}_0$; each field that can be reached by continuous deformation of $\mathbf{n}_0$ will have the same topological action. Consider, for example, the family of field configurations

$$\begin{aligned} \mathbf{n}^{(W)} :\ \mathbb{R}^2 &\to S^2, \\ (x_1, x_2) &\mapsto \left(\phi = W \tan^{-1}\left(\frac{x_2}{x_1}\right), \theta = 2 \tan^{-1}\left(\frac{a^2}{x_1^2 + x_2^2}\right)\right), \end{aligned}$$

where we used polar coordinates to parameterize $\mathbf{n}^{(W)}(\theta, \phi)$. For historical reasons,[12] these field configurations are commonly referred to as **skyrmions**. (Notice that the suffix "-on" indicates that we are dealing with highly stable [particle-like] excitations.)

A coarse visualization of the simplest skyrmion configuration, $\mathbf{n}^{(1)}$, is shown in the figure: a texture of unit vectors varying on a scale set by the parameter a. Skyrmions of higher winding numbers, $\mathbf{n}^{(W>1)}$, are difficult to visualize. However, it is straightforward to verify (by substitution of the unit vector $\mathbf{n}^{(W)}(\phi, \theta)$ into the integral (9.8)) that their topological charge is given by W. Equivalently, the topological action reads $S_{\rm top}[\phi^{(W)}] = i\theta W$.



EXERCISE Verify these statements. Show that the topological charge is insensitive to coordinate changes on both the target and the base manifold. Try to invent other simple field configurations of non-vanishing topological charge.

INFO A word on **semantics**: depending on the context, topologically non-trivial field configurations are described as **solitons, instantons, skyrmions**, etc. While there seem to be no

[12] See T. H. R. Skyrme, A nonlinear field theory, *Proc. Roy. Soc. London A* **260** (1961), 127-38 where these excitations appeared in the context of an effective model of nuclear matter.

discernible systematics in this scheme, a rule of thumb is that topological excitations in dynamical quantum theories (i.e. theories where the base manifold represents space-time) are called "instantons." By contrast, "solitons" are topological solutions of classical equations of motions. However, this rule is also sometimes broken.

The ubiquitous presence of the suffix "-on" reflects a widespread tendency in physics to associate excitations that are protected from ordinary decay mechanisms (presently, by the presence of topological indices) with different kinds of "particles".

9.3.2 Functional integration and topological textures: generalities

How can our present understanding of topologically non-trival field configurations be extended to a working scheme of field integration? As we saw above, the different topological sectors of the theory essentially lead their "own lives." It is then an obvious idea to try to carry out the "canonical program" of field integration (analysis of mean-field configurations $\rightsquigarrow$ integration over fluctuations) in each sector separately. Our first step would thus be to seek solutions of the equation

$$\left.\frac{\delta S[\phi]}{\delta\phi}\right|_{\phi\in\phi_W} = \left.\frac{\delta S_0[\phi]}{\delta\phi}\right|_{\phi\in\phi_W} = 0. \tag{9.9}$$

(The later equality expresses the fact that the topological action does not change under field variation.) In our prototypical example of Section 9.1, the solutions of these equations were readily identified as $\phi_W(\tau) = 2\pi W\tau/\beta$. However, in general, finding solutions of Eq. (9.9) for $W \neq 0$ is a task more complicated than the analysis of the $W = 0$ mean-field. (This is because field configurations with $W \neq 0$ generally exhibit some non-trivial spatial variations, i.e. we cannot rely on the standard homogeneity assumptions.)

INFO However, there exists an elegant trick whereby the **identification of topologically non-trivial mean-field configurations** can sometimes be drastically simplified. Consider the expression

$$\begin{aligned} 0 \qquad & \frac{1}{2}\int d^2x\,(\partial_\mu \mathbf{n} + \epsilon_\mu\ \mathbf{n}\times\partial\ \mathbf{n})\cdot(\partial_\mu \mathbf{n} + \epsilon_\mu\ {}'\mathbf{n}\times\partial\ {}'\mathbf{n}) \\ = \quad & \int d^2x\,(\partial_\mu \mathbf{n}\cdot\partial_\mu \mathbf{n} + \epsilon_\mu\ \mathbf{n}\cdot(\partial_\mu \mathbf{n}\times\partial\ \mathbf{n}))\,, \end{aligned}$$

where the inequality simply expresses the fact that we are integrating a positive definite quantity. Now, we know that the second term in the latter integral yields just -8π times the topological charge, i.e.

$$W \qquad \frac{1}{8\pi}\int d^2x\,\partial_\mu \mathbf{n}\cdot\partial_\mu \mathbf{n} = S_0[\phi].$$

We thus conclude that W represents a lower bound for the action of field configurations of topological charge W. This limit is reached for extremal configurations

$$\partial_\mu \mathbf{n} + \epsilon_\mu\ \mathbf{n}\times\partial\ \mathbf{n} = 0, \tag{9.10}$$

on which the integral above vanishes. Since any continuous variation of these fields leads to a non-vanishing integral, they must be stationary field configurations. We have thus managed

to reduce the identification of stationary phase configurations to the solution of the *first*-order differential equation (9.10).

In passing we note that Eq. (9.10) is most elegantly solved by introducing complex coordinates $z \equiv x_1 + ix_2$ and representing $\mathbf{n}$ in terms of a stereographic projection:

$$n_1 + in_2 = \frac{2w}{1+|w|^2}, \quad n_3 = \frac{1-|w|^2}{1+|w|^2},$$

where $w \in \mathbb{C}$. Straightforward substitution then shows that Eq. (9.10) assumes the form $\partial_{\bar{z}} w(z) = 0$ (exercise). This means that any meromorphic function

$$w = \prod_{i=1}^{W} \frac{z-a_i}{z-b_i},$$

solves the equation. To understand the identification of the order of the product, W, with the topological charge, notice that the inverse $z(w)$ will be a W-valued function, i.e. W indeed measures the number of times the sphere (w-space) is covered by the compactified plane (z-space). (Exercise: Verify that, up to a different choice of the coordinate axes, the skyrmion $\phi^{(1)}$ above is one of these extremal configurations.)

As an attractive by-product, the scheme above automatically yields the action $S_0[\bar{\phi}] = W$ of the extremal configurations $\bar{\phi}$. A slightly modified variant of the same trick helps to find other topological mean-field configurations; e.g. the famous SU(2) instantons central to the analysis of θ-vacua in QCD (for a pedagogical discussion of this subject see, e.g. Ryder.[13]).

At this stage, the further course of action seems to be clear. We ought to compute the action of the mean-field solutions of a given topological index, and then integrate over fluctuations. In practice, however, things turn out to be not quite as straightforward. To understand what is going on, let us revisit a problem we discussed back in Chapter 3, namely the motion of a quantum particle in a steep periodic potential. The topological nature of that system follows from the fact that, (i) by virtue of the periodicity of the potential, $V(x) = V(x+a)$, we can identify the configuration space with a ring of periodicity a and, (ii) with exponentially large probability, at times $t \to \pm\infty$ the quantum particle will rest in any one of the minima

[13] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, 1996).

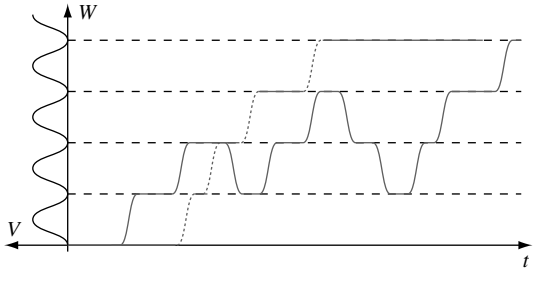


Figure 9.5 Snapshot of two typical field configurations of the U(1) model. Dashed: minimal field configuration with $W = 4$ instantons. Solid: field with 7 instantons and $7 - W = 3$ anti-instantons.

of the potential (see Fig. 9.5). This latter fact implies that the two points $t = \pm\infty$ forming the boundary of the base manifold can be identified (in analogy to our discussion of higher-dimensional compactification above). Within the path integral approach to the problem, we are thus integrating over mappings $S^1 \to S^1$; the spatial distance (in units of a) our particle traverses in the course of time translates to the winding number of the mappings.

This number is a topological invariant, i.e. it does not change under any continuous deformation of the path. However, what may well happen is that, on its way from the minimum at $x = 0$ to another one at $x = Wa$, the particle includes one or several detours (see Fig. 9.5). As is evident from the figure, any of those non-direct paths *can* be continuously deformed to a straight tunneling path $0 \to a \to 2a \to \cdots \to Wa$. To connect to the notions of topology, suppose that the individual tunneling events forming a path of total winding number W are widely separated in time. By analogy with our discussion in Section 9.2, we can then imagine the full path as the result of the superposition of $V \geq W$ single-instanton ($W = 1$) field configurations and $V-W$ anti-instantons ($W = -1$). The homotopic equivalence of a path with a non-vanishing number of anti-instantons to the direct path (no anti-instantons) amounts to the fact that an instanton and an anti-instanton annihilate, $1 + (-1) \to 0$, when their temporal coordinates approach each other; only the difference between the numbers of instantons and anti-instantons is a topological invariant.

For steep potentials, instantons and anti-instantons are widely separated in time, implying that correlations/annihilations are vanishingly improbable. For obvious reasons, such configurations are referred to as **dilute instanton gases**. However, for sufficiently shallow potentials, instantons begin to proliferate in number. The increase in the instanton "density" gives rise to correlation effects or "interactions" between the instantons. (Technically, the interaction of two nearby tunneling/anti-tunneling events no longer is just the sum of the two partial actions, but contains correlation terms.) Of course, these **instanton liquids** are much more difficult to describe than dilute instanton gases.

Clearly, these phenomena are not limited to the one-dimensional example above. For general $\pi(M, T) \simeq \mathbb{Z}$, instantons exist as "particles" ($W > 0$) and "anti-particles" ($W < 0$). Only the difference in the number of particles and anti-particles is a topological quantum number. The general plan of an instanton analysis will, therefore, typically take the following form:

- ▷ Solve mean-field equations – find the instantons/anti-instantons.
- ▷ Analyze the dilute instanton phase.
- ▷ Identify correlations between instantons and explore when the diluteness assumption breaks down.
- ▷ If possible, try to understand the physics of the correlated instanton system.

In Chapter 3, we exemplified this program on the U(1) example (albeit not emphasizing the underlying topology). A pedagogical discussion of the generalization to the S^2-instanton gas (skyrmions/anti-skyrmions) can be found in the text by Polyakov.[14] However, at this stage, we will not pursue further the complexities of the dense instanton systems. Rather

[14] A. M. Polyakov, *Gauge Fields and Strings* (Harwood, 1987).

we shall turn to the – long overdue – discussion of examples where S^2-instanton formation plays an important physical role.

9.3.3 Spin chains

In Section 2.2 we applied the Holstein–Primakoff transformation to explore the dispersion relation $\epsilon(\mathbf{p})$ of excitations in ferro- and antiferromagnetic spin systems. Within this approach we found that, for antiferromagnetic systems, the dynamics of long-wavelength spin-wave excitations is characterized by a linear dispersion relation

$$\epsilon(\mathbf{p}) \sim v_{\rm s}|\mathbf{p}|, \tag{9.11}$$

where $v_{\rm s}$ is the spin wave velocity. However, let us recall that this result was based on a crude semi-classical expansion valid only to leading order in $1/S$, where S is the magnitude of the spins. Yet, what happens when a $1/S$ expansion is seemingly unjustified? Do interaction processes between the elementary spin-waves significantly renormalize the observable excitation spectrum? At any rate, we are dealing with a "strong coupling problem" for which no obvious approximation scheme is in sight.

In situations like this it is usually a good idea to identify an exactly solvable reference system where the physics is known. In the present context, the $S = 1/2$ **antiferromagnetic spin chain** plays this role. Indeed, we have seen (in Problem 2.4[15]) that, at small wavevectors, the $S = 1/2$ antiferromagnet is equivalently described as a system of one-dimensional chiral fermions. We know that the excitations of the fermions (charge density waves) have a linear dispersion, i.e. they obey a one-dimensional variant of Eq. (9.11). This seems quite reassuring: both, $S \gg 1$ and the exactly solvable point $S = 1/2$ are characterized by a linear dispersion. It is then, perhaps, not too bold to speculate that, in the analytically inaccessible intermediate regime $S \simeq 1$, interactions will also not corrupt the linear dispersion of antiferromagnetic spin chains.

Surprisingly, though, this expectation does not conform with experimental observation. Neutron scattering experiments on one-dimensional spin 1/2 antiferromagnets have indeed shown that, in the vicinity of the Néel ordering wavevector $q = \pi/a$, the dispersion is linear. However, spin $S = 1$ chains show altogether different behavior! It turns out that these systems do not support low-energy magnetic excitations at all (see Fig. 9.6). More generally, the cumulative experimental finding is that antiferromagnetic chains of half-integer spin do support a relativistic low-energy excitation, while chains of integer spins are gapped.

[15] To be precise, in Problem 2.4 we applied a Jordan–Wigner transformation to represent the $S = 1/2$ chain in terms of a half-filled system of one-dimensional fermions. The fermion system contained an interaction term whose strength was determined by the anisotropy Δ of the magnetic correlations. In the XY-limit, $\Delta = 0$ (vanishing coupling of the z-components), the fermion system becomes free. (That is, both the fermion system and its equivalent partner, the spin system, support long-range excitations.) For general Δ, bosonization techniques can be applied to map the problem onto a two-dimensional sine–Gordon model. This model falls into the universality class of the two-dimensional (!) classical XY-model (cf. discussion on page 469). Translated back to the context of the spin chain, the RG flow behavior of the latter implies that, for values of the anisotropy all the way up to the Heisenberg limit $\Delta = 1$, the system remains in a gapless phase (the interaction stays irrelevant). For further discussion of the spin 1/2 chain we refer to A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, 1998).

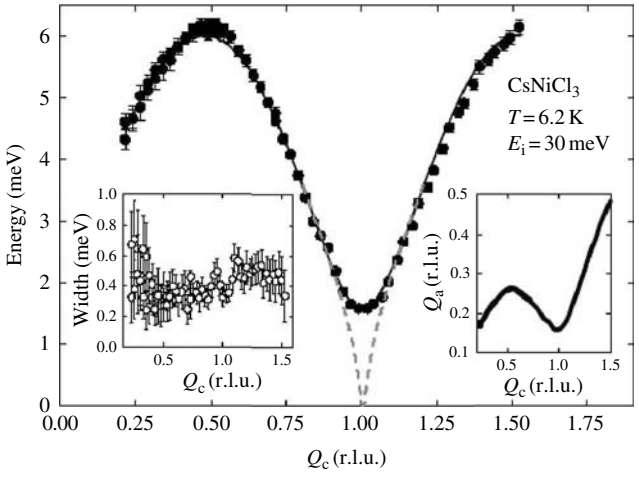


Figure 9.6 Neutron scattering data from the $S = 1$ compound $CsNiCl_3$. The main panel shows the excitation energy for wavevectors close to the antiferromagnetic nesting vector $Q_c = \pi$. In contrast to half integer spin systems, the spectrum is gapped. (Reprinted with permission from M. Kenzelmann, R. A. Cowley, W. J. L Buyers, *et al.*, Properties of Haldane excitations and multiparticle states in the antiferromagnetic spin-1 chain compound $CsNiCl_3$, *Phys. Rev. B* **66** (2002), 24407. Copyright (2002) by The American Physical Society.)

As a rule, physical phenomena depending on the parity of an integer quantum number (presently, $2S$ being even or odd) tend to be of topological origin. To understand why topology appears in the present context recall that, classically, the configuration space of a spin is a sphere (of radius S). The spin configuration of a spin chain is thus described by some mapping from $(1+1)$-dimensional space-time into the sphere. Compactification of space-time then leads to the mappings $S^2 \to S^2$ discussed above.

To substantiate this picture, let us start out from the quantum partition function of an isolated spin derived in Section 3.3:

$$\mathcal{Z}^{(1)} = \int D\mathbf{n}\, e^{iS\int_0^\beta d\tau\, \mathcal{L}_{\rm WZ}(\mathbf{n},\partial_\tau\mathbf{n})}, \quad \mathcal{L}_{\rm WZ}(\mathbf{n},\partial_\tau\mathbf{n}) = (1-\cos(\theta))\dot{\phi},$$

where the integration extends over all paths $\mathbf{n} : \tau \mapsto \mathbf{n}(\tau)$ and (ϕ, θ) are two angles parameterizing the unit vector $\mathbf{n}$.

INFO In the light of the discussion above, the action of this integral looks rather suspicious: it is purely imaginary and remains invariant under reparameterizations of time $\tau \mapsto \tau'(\tau)$ – both hallmarks of topological terms. Indeed, $\mathcal{L}_{\rm WZ}[\mathbf{n}, \partial_\tau\mathbf{n}]$ is the Lagrangian of a "**Wess-Zumino action**".[16] For the topological character of these actions, and their connection to the θ-terms discussed presently, see the next section.

[16] Notice that, from the point of view of a purist, the notation $\mathcal{L}_{\rm WZ}(\mathbf{n}, \partial_\tau\mathbf{n})$ is problematic. As discussed earlier, the Wess–Zumino action does not admit a globally coordinate invariant representation in terms of $\mathbf{n}$ and its derivatives. To formulate one, we have to deploy an explicit coordinate representation. For the underlying reason, see the next section.

The generalization of the $(0+1)$-dimensional path integral to a $(1+1)$-dimensional field integral of the spin chain is a good exercise in guessing effective actions. As a warm-up exercise, we begin our discussion with the **ferromagnetic spin chain**. The interaction between the neighboring spins is mediated by an operator

$$-J\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+1} \longrightarrow -JS^2 \mathbf{n}_i \cdot \mathbf{n}_{i+1} \longrightarrow \frac{JS}{2}(\mathbf{n}_i - \mathbf{n}_{i+1})^2, \tag{9.12}$$

where J is the positive exchange constant, i labels the sites on the chain, and the first arrow maps to the representation $\hat{\mathbf{S}} \to S\mathbf{n}$ of the spin operators in the field integral language. In the second term we have noted that, up to irrelevant constants ($\mathbf{n}^2 = 1$), the interaction can be represented as a "discrete derivative." Adding to this interaction term the Wess–Zumino terms of the individual spins, we are led to the partition function $\mathcal{Z} = \int D\mathbf{n} \exp(-S[\mathbf{n}])$, with the effective action

$$S[\mathbf{n}] = \int d\tau \sum_i \left[\frac{JS^2}{2}(\mathbf{n}_i - \mathbf{n}_{i+1})^2 + iS\mathcal{L}_{\mathrm{WZ}}(\mathbf{n}_i, \partial_\tau \mathbf{n}_i) \right]. \tag{9.13}$$

Anticipating that, for a ferromagnetic system, the configuration $\{\mathbf{n}_i\}$ will typically be smooth, we may then take the continuum limit to arrive at the action

$$S_{\mathrm{ferro}}[\mathbf{n}] = a^{-1} \int d\tau\, dx \left[\frac{JS^2 a^2}{2}(\partial \mathbf{n})^2 + iS\mathcal{L}_{\mathrm{WZ}}(\mathbf{n}, \partial_\tau \mathbf{n}) \right], \tag{9.14}$$

where a denotes the lattice spacing. The action (9.14) does not contain a θ-term.

EXERCISE Derive the equations of motion of this action. Show that the mean-field dispersion of the ferromagnetic chain, $\omega \sim q^2$, is quadratic. (Hint: Recapitulate what we know about the variation of the Wess–Zumino term from Section 3.3.)

We now turn to the more interesting case of the **antiferromagnetic spin chain**. The exchange coupling is now negative, implying that the neighboring spins prefer antiparallel alignment. We thus start out from a configurational *ansatz*

$$\mathbf{n}_i = (-)^i \mathbf{n}'_i,$$

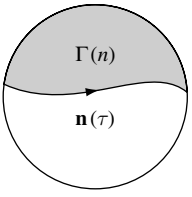
where $\mathbf{n}'$ is the antiferromagnetic order parameter field. To derive the antiferromagnetic analog of the continuum action (9.14), we ought to substitute this *ansatz* into Eq. (9.13) and perform a gradient expansion. However, rather than going through the technical details of this expansion (see e.g., Jackson[17]), we will here fix the structure of the action by qualitative reasoning. For one thing, we know that the system supports a wave-like mode. The minimal action consistent with the global rotational invariance of the model ($\mathbf{n} \to R\mathbf{n}$, where $R \in \mathrm{O}(3)$) and the presence of a wave-like mode is given by

$$S_0[\mathbf{n}] = \frac{S}{4} \int d\tau\, dx \left(\frac{1}{v_{\mathrm{s}}}(\partial_x \mathbf{n})^2 + v_{\mathrm{s}}(\partial_\tau \mathbf{n})^2 \right),$$

[17] J. D. Jackson, *Classical Electrodynamics* (Wiley, 1975).

where we have relabeled $\mathbf{n}' \to \mathbf{n}$ for notational simplicity and $v_s = 2aJS$ is the spin-wave velocity. Of course, the detailed structure of the coupling constants is beyond the scope of our present plausibility argument. However, what we *can* say is that the overall coupling constant must be proportional to a positive power of S. This is because, for large S, "interactions" between the elementary spin waves ought to become weak.

EXERCISE To show this, expand $\mathbf{n}$ around any preferential axis (e.g. by setting $\mathbf{n}_{1,2} \equiv r_{1,2}$, $\mathbf{n}_3 \equiv (1 - r_1^2 - r_2^2)^{1/2}$, $r_1^2 + r_2^2 \in [0,1]$). Expand the action in powers of r_i and show that, for large S, the contribution of anharmonic terms (spin-wave "interactions") becomes small.



Rescaling variables, $\tau \to v_s^{-1/2}\tau \equiv x_0$, $x \to v_s^{1/2}x \equiv x_1$,

$$S_0[\mathbf{n}] \to \frac{1}{\lambda}\int d^2x\, \partial_{x_\mu}\mathbf{n}\cdot\partial_{x_\mu}\mathbf{n}, \quad \lambda = 4/S,$$

assumes the form of the action of the O(3) **nonlinear σ-model**.[18] However, as argued above, we expect the full action of the problem to contain not only a dynamical piece S_0 but also a θ-term. How can this be made plausible from the prototypical action (9.13)? We expect that, somehow, the θ-term must appear as a descendant of the Wess–Zumino action. In the ferromagnetic case, the Wess–Zumino actions of the spins basically added to give a contribution that was not a θ-term. However, in the present case, we are dealing with a staggered spin configuration that leaves room for more interesting things to happen. Let us first recall the interpretation of $iS\int_0^\tau d\mathcal{L}_{\rm WZ}(\mathbf{n}, \partial_\tau\mathbf{n})$ as the oriented area $\Gamma[\mathbf{n}]$ on the sphere swept out by the curve $\mathbf{n}(\tau)$ (see figure).

By an elementary geometrical consideration, $\Gamma[-\mathbf{n}] = 4\pi - \Gamma[\mathbf{n}] = -\Gamma[\mathbf{n}] \bmod 4\pi$. This implies that the continuum version of the Wess–Zumino action

$$S_{\rm top}[\mathbf{n}] = iS\sum_i (-)^i \int_0^\beta d\tau\, \mathcal{L}_{\rm WZ}(\mathbf{n}_i, \partial_\tau\mathbf{n}_i) = iS\sum_i (-)^i \Gamma[\mathbf{n}_i],$$

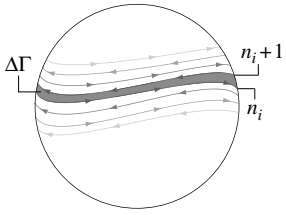
evaluated on a staggered configuration $(-)^i\mathbf{n}_i$ must contain a spatial derivative besides the temporal derivative inherent in $\mathcal{L}_{\rm WZ}$ (since for $\mathbf{n}_i = $ const. the factor $(-)^i$ would lead to a global cancellation). Indeed, the Wess–Zumino actions of two neighboring configurations $\mathbf{n}_{i+1}$ and $\mathbf{n}_i$ evaluate, respectively, to the areas bounded by the curves $\Gamma[\mathbf{n}_{i+1}]$ and $\Gamma[\mathbf{n}_i]$. For $|\mathbf{n}_{i+1} - \mathbf{n}_i|$ small, the area difference can be approximated by

$$\Gamma[\mathbf{n}_{i+1}] - \Gamma[\mathbf{n}_i] \simeq \int_0^\beta d\tau\, \mathbf{n}_i\cdot((\mathbf{n}_{i+1} - \mathbf{n}_i)\times\partial_\tau\mathbf{n}).$$

Summing over i and taking the continuum limit, we obtain (see figure)

$$S_{\rm top}[\mathbf{n}] = i\theta\int d\tau\, dx\, \mathbf{n}\cdot(\partial_x\mathbf{n}\times\partial_\tau\mathbf{n}), \quad \theta = \frac{S}{2}, \tag{9.15}$$

[18] Although the action above is commonly referred to as the action of the O(3)-model, a better terminology would be the (O(3)/O(2))-model. Indeed, the degrees of freedom of the theory span the 2-sphere $S^2 \simeq$ O(3)/O(2). Although the nonlinear σ-model on the sphere has to be distinguished from the σ-model on O(3) (cf. the discussion of Section 8.5 and Problem 8.8.3) we will follow the widespread convention to describe the model above as the O(3)-model.



which we identify as our earlier representation of the θ-term. (Notice that the integral $\int d\tau\, dx$ really represents an integral over the sphere. This is because we agreed that, for $|\tau| \to \infty$ and/or $|x| \to \infty$, all trajectories approach a common reference point on S^2.)

Summarizing, we have obtained $S_0[\mathbf{n}] + S_{\text{top}}[\mathbf{n}]$ as the effective action of the antiferromagnet. In Problem 8.8.3, we have seen that at large length scales a two-dimensional system described by the action $S_0[\mathbf{n}]$ flows into a disordered phase. But how will the presence of the topological term modify this behavior? To get some idea of what might happen, let us reformulate the partition function as a sum over disjoint topological sectors,

$$\mathcal{Z} = \sum_{W \in \mathbb{Z}} \int D\mathbf{n}_W \, e^{2\pi i S W} e^{-S_0[\mathbf{n}_W]}, \tag{9.16}$$

where $\mathbf{n}_W$ denotes field configurations of winding number W. Equation (9.16) provides a preliminary explanation for the observation of the half-integer/integer spin staggering phenomena mentioned above. For integer spin, $\exp(2\pi i S W) = 1$ and the topological term is not operational. By contrast, for half-integer spin, $\exp(2\pi i S W) = (-)^W$, i.e. consecutive topological sectors are weighted by alternating signs. Notice that the topological term is susceptible to the parity of $2S$ and nothing else. To understand heuristically the consequences of this feature we emphasize that, at the mean-field level (this stabilized by large S), the partition function is governed by a mode with linear dispersion. Quantum fluctuations around this configuration will alter its dispersion, potentially by the creation of an excitation gap. Now, in the integer case, these fluctuations additively ($\exp(-S_0) \in \mathbb{R}^+$) contribute to the partition function. For small S they may (and in fact do) totally mask the mean-field sector. By contrast, for half-integer spin, fluctuations contribute with alternating sign, thereby partially canceling each other; the mean-field sector has a better chance to survive. These observations form the basis of **Haldane's conjecture**,[19] according to which spin chains of integer S will flow into a disordered phase with no long-range excitations whilst, in chains of half integer spin, they remain in a gapless phase.

Reassuringly, the predictions born out of these rather abstract constructions are in full agreement with neutron scattering measurements of the dispersion of various quasi-one-dimensional magnets. But why should one, nonetheless, use the attribute "conjecture" (as opposed to, say, "theory")? The scenario above is based on a number of shaky suppositions. Most seriously, it is tacitly assumed that the σ-model will seamlessly scale into the strong coupling region without changing its form. (That the model remains form invariant under renormalization is a perturbative prediction which can be trusted only for $\lambda \ll 1$.) Indeed, a subsequent analysis by Affleck and Haldane[20] has shown that, *en route* to the strong coupling regime, much more drastic things happen. Specifically, the appropriate critical

[19] F. D.M. Haldane, Nonlinear field theory of large-spin Heisenberg antiferromagnets: semiclassically quantized solitons of the one-dimensional easy-axis Néel state, *Phys. Rev. Lett.* **50** (1983), 1153–6.

[20] I. Affleck and F. D. M. Haldane, Critical theory of quantum spin chains, *Phys. Rev. B* **36** (1987), 5291–300.

theory describing the behavior of half-integer chains at large length scales turns out to be not the O(3) nonlinear σ-model but rather a field theory with SU(2)-valued degrees of freedom. We discuss this theory in Section 9.4.4 after the concept of Wess–Zumino terms has been introduced.

Introducing another group of phenomena where mechanisms of topology are crucial, we now turn to a discussion of the quantum Hall effects (QHE). The QHE belongs here because (a) it is surely a compulsory part of any modern treatise on condensed matter phenomena, and (b) it is, in many respects, a topological phenomenon. On the other hand, our discussion will lead us somewhat astray inasmuch as it involves, necessarily, a review of experimental observation, the elementary quantum mechanics of electrons in the presence of strong magnetic fields, etc. Readers wishing to maintain a more streamlined discussion of topology in condensed matter field theory are therefore invited to skip the next section, and turn directly to the section on Wess–Zumino terms below.

9.3.4 Integer quantum Hall effect

In fact, it is quite misleading to talk about *the* quantum Hall effect: at the least, one should speak of the quantum Hall effect"s" – a spectrum of quite different phenomena, almost unparalleled in diversity and conceptual depth. To get a preliminary impression of the phenomena, observe the raw experimental data displayed in Fig. 9.7. The figure shows the Hall resistance $\rho_{xy} \equiv R_{\rm H}$ and the longitudinal resistance ρ_{xx} (the ragged curve in the bottom of the plot) of a two-dimensional electron gas as functions of a strong perpendicular magnetic field. Instead of a dull linearly increasing curve (the classical Hall resistance ρ_{xy}) or an approximate constant (the Drude resistance ρ_{xx}) one finds a profile that could hardly be more structured.

On close inspection of the data, one may notice a number of characteristic sub-structures: (1) Shubnikov–de Haas oscillations at small magnetic fields, followed by (2) the characteristic **quantum Hall plateaus** $\rho_{xy} = \nu^{-1} h/e^2$ at rational "filling fractions" (see below) $\nu \in \mathbb{Q}$ to which the effect owes its name. These are accompanied (3) by a dramatic **drop in the longitudinal resistance** ρ_{xx}. The functional form of the increase (4) from one plateau to the next is described by certain well-defined power laws as a function of temperature, indicative of a second-order phase transition – the zero-temperature **quantum Hall transition**. Barely visible, (5) a second generation of Shubnikov–de Haas oscillations is observed at $\nu = 1/2$. Finally, there appears much structure in the pattern of rationals ν for which plateaus are found. For example, for some low-lying rationals (such as $\nu = 1/4$) *no* plateau is formed. The set of rationals for which the effect occurs is known as the **quantum Hall hierarchy**.

These are but a few of the most striking observations gathered under the label quantum Hall effect. Undoubtedly, to account even superficially for all of these phenomena would present a task that is well beyond the scope of the present text. Rather, we will have to restrict ourselves to a brief (and, alas, painfully superficial) discussion of a number of conceptual basics. In the present section we shall focus on the discussion of the conductance plateaus observed at integer filling factors – the so-called **integer quantum Hall effect**

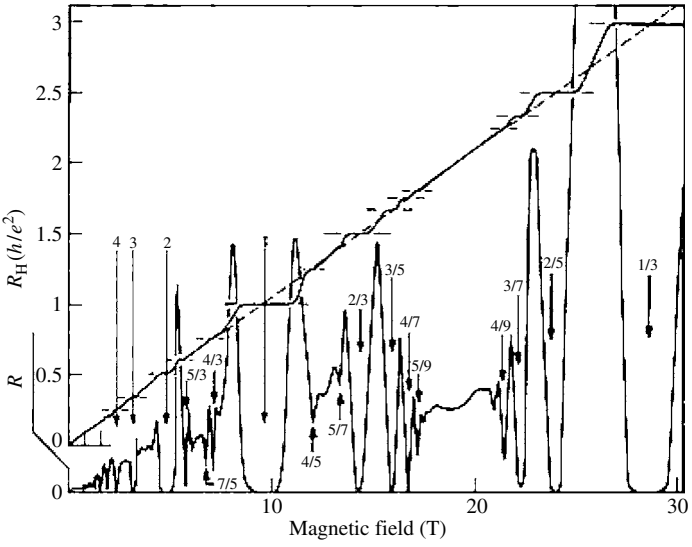


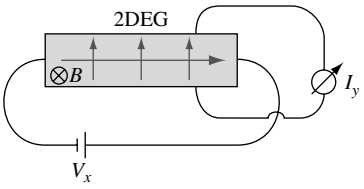
Figure 9.7 Hall conductance and longitudinal conductance of a two-dimensional electron gas as functions of a perpendicular magnetic field. For a discussion, see the main text. (Reprinted with permission from H. L. Störmer, D. C. Tsui, and A. C Gossard, The fractional quantum Hall effect, *Rev. Mod. Phys.* **71** (1999), S298–305. Copyright (1999) by the American Physical Society.)

(IQHE). The generalization to plateaus at rational filling fractions – which, curiously, hinges on altogether different physical concepts – is discussed in Section 9.5.1 below.

When specified in units of the conductance quantum e^2/h, the plateau conductance $\sigma_{xy} = \rho_{xy}^{-1}$ of the IQHE is an integer. By its very nature an integer cannot vary continuously (upon changing some physical control parameter). Indeed, more often than not, observables quantized in integer (or rational) units are linked to some topological origin. In the following, we will formulate two different "explanations" of the quantization phenomenon. The quotes indicate that, far from being rigorous, both lines of argument involve some degree of "bootstrap character": *assuming* that the effect is of topological nature, we feel free to subject the "real world" arrangement of a QHE experiment to all kinds of abuses (deformation of the sample boundaries, etc.) to then discover that, yes, the Hall conductance emerges as a topological invariant. The backbone of a more rigorous approach to the problem is discussed in Section 9.3.7 below.

9.3.5 Background information on the IQHE

Let us begin our discussion with a nutshell summary of the phenomenology of the IQHE. This is followed by a brief reminder of the phenomenology of Landau level quantization – formulated in a language that highlights the symmetries of the problem.



The QHE is observed in two-dimensional electron gases subject to a strong magnetic field. The prototypical setup of a QH experiment is shown in the figure. A voltage drop V_x applied across the sample induces a current I_x in the x-direction plus, by virtue of the Lorentz force, a Hall current I_y in the y-direction. The relation between I_x and I_y determines the Hall conductance. More precisely, the key quantities of interest are the entries σ_{xx} and σ_{xy} determining the conductance tensor

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix},$$

which is defined as usual by $\mathbf{I} = \sigma\mathbf{V}$ where $\mathbf{I} = (I_x, I_y)^T$, $\mathbf{V} = (V_x, V_y)^T$ (note that, by symmetry, $\sigma_{xx} = \sigma_{yy}$). The inverse $\rho = \sigma^{-1}$ defines the resistance tensor $\mathbf{V} = \rho\mathbf{I}$. For a system of linear extension L, conduct*ance* g and conduct*ivity* σ are connected by the relation $g = \sigma L^{d-2}$. In two dimensions, the two quantities coincide. This implies that no device-geometry-dependent factors interfere when we pass from the basic quantity determined by the microscopic physics of the system (σ) to experimental observables (g); otherwise, no completely universal Hall conductance could possibly be observed. In the context of the QHE, it is truly important to keep the tensorial structure of σ and ρ in mind. For example, somewhat paradoxically, for $\sigma_{xy} \neq 0$, a vanishing of the longitudinal resistance ρ_{xx} implies a vanishing (as opposed to a divergence) of the longitudinal conductance σ_{xx}.

Within the context of the QHE, the natural unit for the strength of the applied magnetic field is the "**filling fraction**". This is defined as the ratio

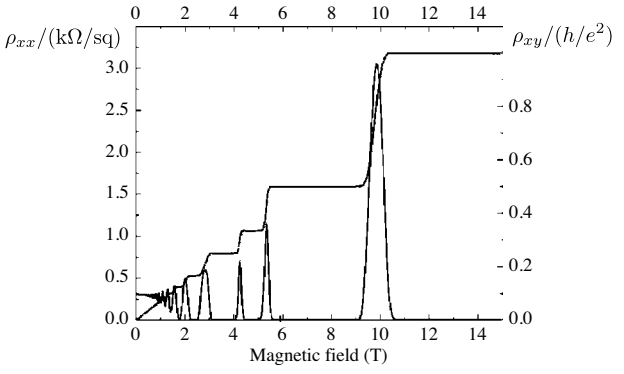
$$\nu \equiv \frac{2\pi N l_0^2}{A}, \tag{9.17}$$

where N denotes the number of electrons in the system, A is the sample area and $l_0 = \sqrt{\Phi_0/2\pi|B|}$ the **magnetic length**, i.e. the external magnetic flux through the area l_0^2 is equal to (2π times) one flux quantum.[21]

Experimentally, one finds that (see the figure overleaf, courtesy of D. Leadley), for field strengths close to an integer filling fraction $\nu \in \mathbb{N}$, the Hall resistance is quantized, $\rho_{xy} = \nu^{-1}h/e^2$ to an accuracy of $\mathcal{O}(10^{-10})$.[22] At the same time, the longitudinal resistivity/conductivity drops by as much as 13 orders of magnitude. In passing we note that the rapid oscillations visible in the figure at small field strengths represent the familiar Shubnikov–de Haas oscillations.

[21] One may recall that the flux quantum is defined through the relation $\Phi_0 = h/e$. In our standard units $\hbar = e = c = 1$, $\Phi_0 = 2\pi$ and $l_0 = |B|^{-1/2}$.

[22] Due to the striking precision of the experimental data, the unit of electrical resistance is nowadays maintained as $h/e^2 = 25\,812.80\,\Omega$ through quantum Hall measurements.



As a (presumed) topological phenomenon, we expect the quantization of the Hall conductance to be robust against sample imperfections and/or the presence of **static disorder**. What is more surprising is that the phenomenon, in fact, critically *relies* on the presence of disorder. To see this, let us for a moment assume that the opposite is true: we have a homogeneous electron gas accommodated by a translationally invariant device. We further assume the absence of external voltage gradients, $\mathbf{E} = 0$, so that no current is flowing in the system, $\mathbf{I} = 0$. Now suppose that we observe the system from a frame moving with velocity v in, say, the 1-direction. An experimentalist working in that frame would observe a current density $\mathbf{j} = -v\rho\mathbf{e}_1$, where ρ is the density of the electron gas. Further, the Lorentz covariance of electrodynamics implies that one would measure a finite electric field $\mathbf{E} = v\mathbf{e}_1 \times \mathbf{B} = -vB\mathbf{e}_2$, where $\mathbf{B} = B\mathbf{e}_3$ is the applied magnetic field. With $j_1 = \sigma_{12}E_2$, we obtain $\sigma_{12} = \rho B^{-1}$ for the Hall conductivity in the moving frame. Being independent of the boost velocity, v, this result holds in all moving frames, including the static frame, $v \to 0$. We conclude that, in any translationally invariant environment, the Hall conductivity is linearly related to the magnetic field.

Reciprocating the argument above, we see that the presence of (translational invariance breaking) disorder must be necessary for the observability of the QHE. In fact, we shall see in a moment that the effect is born out of a conspiracy of disorder induced localization and the phenomenon of **Landau level quantization**. However, before turning to the discussion of the combined effect of these two mechanisms, let us briefly recapitulate the formation of Landau levels in a clean two-dimensional electron gas subject to a magnetic field.

Let us temporarily consider a geometry where the electron gas assumes the form of a perfect disk.[23] To explore the quantum mechanics of this problem, we represent the in-plane vector potential in the so-called symmetric gauge, $A_i = (B/2)\epsilon_{ij}x_j$, $i = 1, 2$ (where the coordinates are measured with respect to the center of the disk), whereupon the free electron Hamiltonian, $\hat{H} = (\hat{\mathbf{p}} - \hat{\mathbf{A}})^2/(2m^*)$, assumes the form

$$\hat{H} = \frac{1}{2m^*}\left[\left(-i\partial_1 - \frac{x_2}{2l_0^2}\right)^2 + \left(-i\partial_2 + \frac{x_1}{2l_0^2}\right)^2\right]. \tag{9.18}$$

(To avoid confusion with the quantum number to appear shortly below, we designate the electron mass by m^*.) We wish to solve the Schrödinger equation $\hat{H}\psi_n = \epsilon_n\psi_n$. This task is greatly simplified by subjecting the eigenvalue problem to the similarity transformation

[23] Remember our fundamental working hypothesis whereby the quantization phenomena forming the QHE will not depend on details of the geometry.

$\hat{H} \to \hat{H}' = S\hat{H}S^{-1}$, $\psi_n \to \psi_n' = S\psi_n$, where $S = \exp\left[\frac{1}{4l_0^2}\left(x_1^2 + x_2^2\right)\right]$. The reason is that the transformed problem $\hat{H}'\psi_n' = \epsilon_n\psi_n'$ is governed by the effective Hamiltonian

$$\hat{H}' = \frac{1}{2m^*}\left[\left(-i\partial_1 + i\frac{1}{2l_0^2}(x_1 + ix_2)\right)^2 + \left(-i\partial_2 + \frac{1}{2l_0^2}(x_1 + ix_2)\right)^2\right],$$

i.e. an operator whose (vector) potential depends only on the linear combination $x_1 + ix_2$ rather than on two linearly independent coordinates x_1 and x_2. To benefit from this simplification, we may switch to complex coordinates, $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, in which the Hamiltonian assumes the form (exercise)

$$\hat{H}' = \frac{1}{2m^*}\left(-4\partial_z\partial_{\bar{z}} + 2\frac{z}{l_0^2}\partial_z + \frac{1}{l_0^2}\right).$$

This Hamiltonian possesses a family of eigenstates $\psi_n' \equiv z^n$ with eigenvalues $\epsilon_n = \frac{1}{m^* l_0^2}(n + 1/2) = \frac{B}{m^*}(n + 1/2)$. Undoing the similarity transform, one can conclude that the original Hamiltonian is diagonal on the states

$$\psi_n = z^n e^{-\frac{1}{4l_0^2}z\bar{z}}, \quad \epsilon_n = \omega_c(n + 1/2),$$

whose eigenvalues ϵ_n are the celebrated **Landau levels**. The Landau levels differ by integer multiples of the **cyclotron frequency** $\omega_c \equiv B/m^*$. On the other hand, we know that, for a system of linear extension L, there are of $\mathcal{O}(k_F L)^2$ states below the Fermi energy $k_F^2/2m^*$. This implies a typical level spacing $\sim 1/(m^* L)^2$ which is by a factor $\sim BL^2$ smaller than the spacing between Landau levels. Anticipating that the clean problem does not support energies other than ϵ_n, one can conclude that the Landau levels must be hugely degenerate: each of them hosting $BL^2 = (L/l_0)^2$ states.

To reveal the origin of the massive degeneracy of the Landau levels, we have to identify a symmetry of the Hamiltonian (9.18) or, equivalently, a set of linearly independent operators commuting with $\hat{H}$. In the present context, these are the "**magnetic translation operators**"

$$k = \partial_z - \frac{\bar{z}}{4l_0^2}, \quad \bar{k} = \partial_{\bar{z}} + \frac{z}{4l_0^2}, \quad \bar{k}\psi_n = 0.$$

(In the absence of a magnetic field $l_0 \to \infty$, these would be ordinary translation operators $\sim -i\partial_i$ transformed to complex coordinates; hence the name magnetic "translation" operators.) It is straightforward (exercise) to verify the following properties:

$$[\hat{H}, k] = [\hat{H}, \bar{k}] = 0, \quad [k, \bar{k}] = \frac{1}{2l_0^2}.$$

We next use these operators as generators for the creation of states degenerate with the reference states ψ_n. To this end, let us define

$$T = \exp\left[\frac{4\pi l_0^2}{L}k\right], \quad U = \exp\left[i\frac{2\pi l_0^2}{L}\bar{k}\right].$$

Using the commutation relations between k and $\bar{k}$, one may verify that these operators obey the relation

$$TU = UT\exp\left[4\pi^2 i\left(\frac{l_0}{L}\right)^2\right],$$

sometimes referred to as the **magnetic algebra**. We now have all elements to construct the entire set of states: the fact that $\bar{k}$ annihilates ψ_n implies that $U\psi_n = \psi_n$. Consider now the family of states

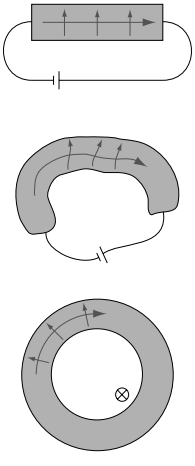
$$\psi_{n,m} \equiv \mathcal{N}_{n,m} T^m \psi_n, \tag{9.19}$$

where $\mathcal{N}_{n,m}$ is a normalization factor. Since $\hat{H}$ commutes with k (and therefore with T), $\psi_{n,m}$ form eigenstates with eigenenergy ϵ_n – they all populate the nth Landau level. However, we do not yet know whether $\psi_{n,m}$ actually form a set of linearly independent states. To prove their independence, one can employ Eq. (9.19) and $U\psi_n = \psi_n$ to show that $U\psi_{n,m} = \exp(-i4\pi^2 m(l_0/L)^2)\psi_{n,m}$. Therefore, as with Bloch states in a periodic potential, $\psi_{n,m}$ are eigenstates of a generalized translation operator. For $m = 1, \ldots, 2\pi(L/l_0)^2$, the corresponding eigenvalues are different, which proves that

$$\{\psi_{n,m} | 0 \leq m < 2\pi(L/l_0)^2\},$$

is a linearly independent set of eigenstates in the nth Landau level. The magnitude of this set $2\pi(L/l_0)^2$ coincides with our estimate of the degeneracy of the Landau levels above, i.e. we have succeeded in constructing a complete eigenbasis of the magnetic Hamiltonian.

9.3.6 IQHE as a topological phenomenon



Shortly after the experimental discovery of the IQHE,[24] Laughlin presented an ingenious argument whereby the quantization of the Hall conductance could be explained under fairly general conditions.[25] Slightly later it became clear[26] that Laughlin's argument in fact implied a number of curious features of electrons subject to a magnetic field *and* static disorder. However, before turning to a more detailed discussion of Laughlin's ideas, and their subsequent refinement by Halperin, let us first outline the basic skeleton of the argument.

Given its complete universality, the quantization phenomenon must be insensitive to (continuous) deformations of the sample geometry. Using this freedom, Laughlin proposed to subject the basic quantum Hall geometry to the sequence of transformations indicated in the figure. From a "Hall bar" geometry we pass to an annular geometry of higher symmetry.

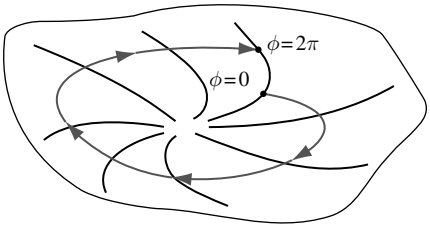
[24] K. von Klitzing, G. Dorda, and M. Pepper, New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance, *Phys. Rev. Lett.* **45** (1990), 494–7.

[25] R. B. Laughlin, Quantized Hall conductivity in two dimensions, *Phys. Rev. B* **23** (1981), 5632–3.

[26] See B. I. Halperin, Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential, *Phys. Rev. B* **25** (1982), 2185–90.

In the last step of the construction, the external bias voltage is replaced by the electromotive force generated by a weakly time-dependent flux through the annulus. Laughlin then suggested monitoring the fate of the system upon an adiabatic (i.e. infinitely slow) variation of the flux threading the annulus. Before discussing the response of the system in more detail, let us try to motivate this idea.

We know that, for specific values of the annular flux, namely integer multiples of a flux quantum $\phi = 2\pi n$, the Hamiltonian of the system is gauge equivalent to the Hamiltonian in the absence of flux. This is because an integer multiple of the flux quantum can be removed by the gauge transformation $\psi_a \to e^{in\theta}\psi_a$ acting on the wavefunctions of the system. (For non-integer fluxes, this transformation alters the boundary condition $\psi(r, 2\pi) = \psi(r, 0)$ and, therefore, the Hilbert space of the problem.)



Now, let us see what happens as we gradually increase the flux from $\phi = 0$ to $\phi = 2\pi$. The situation is visualized in the figure where each line symbolically represents a basis of eigenstates of the Hamiltonian for a given value of the flux. Assuming that a gauge transformation has been applied to move the flux dependence of the problem to a change in the azimuthal boundary conditions (see our discussion in Section 9.1), the eigenstates for different values of the flux are truly distinct, i.e. as one increases the flux from $\phi = 0$, each state moves along a path "perpendicular" to the collective set of eigenstates, as shown schematically in the figure. Eventually, for $\phi = 2\pi$, we arrive back at the original $\phi = 0$ basis. (This follows from the fact that $\hat{H}|_{\phi=2\pi}$ can be mapped onto $\hat{H}|_{\phi=0}$ by a gauge transformation that does *not* alter the boundary conditions.) That, however, does not necessarily imply that individual basis states map onto themselves upon the completion of the path $\phi = 0 \to \phi = 2\pi$. I.e., while the set of eigenstates as a whole gets reproduced, permutations of individual states are consistent with the gauge invariance of the problem.

EXERCISE If this statement does not make much sense to you consider, as an example, a clean one-dimensional ring subject to a magnetic flux. Explore what happens to the eigenstates of $\hat{H} = \frac{1}{2m}(\hat{p} - A)$ as the flux is increased from 0 to 2π. Show that the non-invariance of individual states is compatible with the invariance of the global spectrum.

The non-invariance of individual states upon completion of a round trip back to a gauge-equivalent Hamiltonian is a phenomenon called **spectral flow**. The spectral flow of the eigenstates of our magnetic environment lies at the heart of Laughlin's argument. Specifically, we shall see that, upon the sending of a flux quantum through the ring, n states radially centered at the inner perimeter are pushed above the Fermi energy (n is the number of Landau levels below the Fermi energy). At the same time, n states at the outer perimeter sink below the Fermi energy. To regain thermal equilibrium, the system responds by transferring n electrons from the inner to the outer perimeter. This process takes place during the time t_0 it takes to adiabatically send the flux quantum through the system, i.e. the transverse current $I_2 = n/t_0$ (remember, $e = 1$). The electromotive force driving the process is $V_1 = \dot{\phi} = 2\pi/t_0$. The corresponding Hall conductance is therefore given by

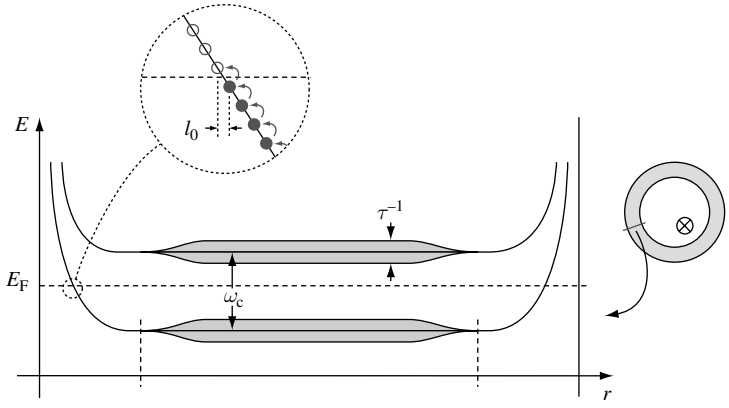


Figure 9.8 Energy levels of a quantum Hall annulus as functions of the radial coordinate. For a discussion, see the main text.

$\sigma_{12} = I_2/V_1 = n/2\pi$. Expressed in physical units, this can be cast in the form $\sigma_{12} = ne^2/h$ – the quantum Hall effect!

To substantiate this picture, let us consider a situation in which the Fermi level E_F is placed somewhere between the first and the second Landau level. For simplicity we shall also make the (artificial but physically immaterial, cf. Halperin's paper[26]) assumption that the disorder is confined to the inner regions of the sample. We require the disorder strength, as set by the inverse elastic scattering time, τ^{-1}, to be smaller than the separation between Landau levels, $\omega_c\tau \gg 1$. This condition is, in fact, necessary to prevent the Hall plateaus from becoming "washed out." As a result we obtain the level diagram shown in Fig. 9.8. The figure schematically shows the energies of the single-particle states as functions of the radial coordinate of the annular region. In the bulk of the annulus, disorder leads to a broadening of the Landau levels to energy bands of width τ^{-1}. At the outer/inner perimeter of the annulus, the confining potential pushes the levels energetically up. Crucially, this implies the presence of as many Fermi energy states – "**edge states**" – as there are occupied Landau levels (in our case, just one). These states, and not so much the bulk states buried deep below the Fermi energy, are likely to be the carriers of longitudinal currents in the system. As to the bulk states, let us presume that they are localized by disorder on a length scale ξ much smaller than the circumference $L_\parallel$ of the system. (Intriguingly, we shall soon see that this assumption leads to a contradiction.)

To explore the phenomenon of spectral flow in this environment, we need to turn to a refined description where individual levels are resolved. This is the subject of the following.

EXERCISE We wish to explore the radial structure of the states occupying the lowest Landau level. This task is most conveniently accomplished in a basis different from the set $\{\psi_{0,m}\}$ discussed above: revisit the construction on page 520 to show that all states $\phi_{0,m} \equiv \mathcal{N}_{0,m}\bar{z}^m \exp[-\frac{1}{4l_0^2} z\bar{z}]$ lie in the lowest Landau level. Switch to polar coordinates (r, θ) and verify that these states can be approximated as

$$\phi_{0,m}(r,\theta) \approx \mathcal{N}_{n,m} e^{-im}\ e^{-\frac{3}{4l_0^2}(r-r_m)^2}, \tag{9.20}$$

where r_m defines the area through which m flux quanta pass; $\pi r_m^2 B = mh/e$. Show that, for any reference angle θ_0, the azimuthal current carried by these states is given by

$$\begin{aligned} I_\parallel \equiv \frac{1}{m^*}\int_0^\infty dr\ \langle \hat{\mathbf{j}}_{(r,\ 0)}\rangle \cdot \mathbf{e} \qquad &= \frac{1}{m^*}\int_0^\infty dr\ |\phi_{0,m}|^2 \left(\frac{m}{r} - \frac{Br}{2}\right) \\ &\simeq \frac{B}{m^*}\int_0^\infty dr\ |\phi_{0,m}|^2 (r - r_m)\,. \end{aligned} \tag{9.21}$$

The wavefunctions $\phi_{0,m}(r,\theta)$ describe the system far away from its boundaries. They are symmetrically centered around r_m and carry angular momentum $L = m$. From Eq. (9.21) and the symmetry around the center coordinate r_m, we further conclude that the azimuthal current carried by these states vanishes, $I_\parallel = 0$. However, in the vicinity of the boundaries, this picture becomes perturbed. Firstly, the confining potential will push the states $\phi_{0,m}$ up in energy (see Fig. 9.8 where the circles represent the states centered at coordinates r_m close to the inner boundary of the annulus). Secondly, the boundary potential will render the states radially asymmetric, implying that the integral (9.21) no longer vanishes: azimuthal currents flow at the boundaries. The surface currents flowing at the inner/outer perimeter are opposite to each other.

We next discuss what happens as some weakly time-dependent flux $\tilde{\phi}$ is sent through the annulus to generate an electromotive force $E_\parallel = d_t\tilde{\phi}/r$. The vector potential generalized to the presence of the field $E_\parallel$ takes the form $\mathbf{A} = \frac{\phi(r)+\tilde{\phi}}{2r}\mathbf{e}_\theta$, where $\phi(r) = Br^2$. Referring for a more detailed discussion to Halperin's paper, we note that the flux $\tilde{\phi}$ adds to the background flux $\phi(r)$. As a consequence, the center coordinates $r_m(\phi + \tilde{\phi})$ "contract." Once a full flux quantum has been added to the system, $\tilde{\phi} = 2\pi$, the center coordinates have contracted by one unit, $r_m(\phi + 2\pi) = r_{m-1}(\phi)$, and the original set of levels (and therefore the spectrum of the system) is restored. However, the individual levels have changed – spectral flow: at the inner edge of the system, one occupied level has been pushed above the Fermi energy, at the outer edge one empty level dived below the Fermi energy (see Fig. 9.8). To repair this energy imbalance, the system will want to transfer one electron from the inner edge to the outer edge.

But how will it do that? Before the advent of the QHE, it had been common wisdom that the states of a two-dimensional electron gas in the presence of disorder (such as the states in the bulk of our annulus) are localized on a certain scale ξ. For $\xi/L_\parallel \gg 1$, these states would be completely oblivious to the presence of our driving flux $\tilde{\phi}$. This would imply that, as far as the action of $\tilde{\phi}$ is concerned, the inner and the outer edge of the system are decoupled. How, then, would the system know that it ought to transfer one electron between the edges. (After all, this transfer has to be mediated through the bulk.) The only way out of this dilemma is to courageously *postulate* (as Halperin did) that, notwithstanding the presence of disorder, there must be at least one **delocalized bulk state** below the Fermi energy. Subsequently, it was indeed found that the localization length diverges upon approaching the center of the Landau band. The delocalized states in the center of the band establish the contact between the edges and may act as conduits of electronic charge.

Laughlin's gauge argument hints at the topological nature of the QHE: independent of system-specific details, the addition of a flux quantum through the annulus transfers an integer charge across the system. This suggests that the effect should be understandable in terms of some kind of topological index lurking behind the mapping of a parameter space (presently, the amount of flux through the ring) into the Hilbert space of the problem. Subsequently, Avron and Seiler,[27] and Thouless, Kohmoto, Nightingale, and den Nijs[28] indeed succeeded in rigorously identifying the quantum Hall conductance as a topological invariant – the first Chern class of the $U(1)$ principal bundle over the two-dimensional torus. However, a satisfactory discussion of these ideas, for which we would need to introduce much more background in differential topology, would lead us too far astray. Instead, we shall head back to our prime subject, the discussion of topological concepts in low-energy field theories of condensed matter systems.

EXERCISE As preparation for the next section, refamiliarize yourself with the field theory approach to the disordered electron gas introduced in Section 6.5.

9.3.7 Field theory of the integer quantum Hall effect

Laughlin's gauge argument, and its subsequent refinement by others, helped to unravel many of the mysteries posed by the experimental discovery of the IQHE. Specifically it shed some light on the conspiracy of dissipationless edge currents, disorder induced localization of bulk states, and an exotic family of delocalized states in the formation of the effect. On the other hand, even with Laughlin's picture in store, we are still a long way from a more-than-schematic, quantitative understanding of the effect. A huge step towards a full microscopic theory of the IQHE – whose final structure still remains partly unknown – was taken by Pruisken when he adjusted the nonlinear σ-model of disordered fermion systems so as to account for the presence of a strong magnetic field. In the next two subsections, we will reconstruct Pruisken's field theory, and employ it to gain more insight into the long-range behavior of the quantum Hall system.

Pruisken s field theory: construction

As we have seen in earlier chapters of this book, much of the information about a non-interacting disordered electron system is contained in correlation functions involving the product of a retarded and an advanced single-particle Green function,

$$\langle \mathbf{x}_1 | \frac{1}{E_\mathrm{F} + \omega + i0 - \hat{H}} | \mathbf{x}_2 \rangle \langle \mathbf{x}_{2'} | \frac{1}{E_\mathrm{F} - \omega - i0 - \hat{H}} | \mathbf{x}_{1'} \rangle \Big|_{\mathrm{dis}} .$$

Depending on the choice of coordinates and the energy argument ω, these functions describe the conductance of the system, the statistics of its spectrum and many other characteristics.

[27] J. E. Avron and R. Seiler, Quantization of the Hall conductance for general, multiparticle Schrödinger Hamiltonians, *Phys. Rev. Lett.* **54** (1985), 259–62.

[28] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall conductance in a two-dimensional periodic potential, *Phys. Rev. Lett.* **49** (1982), 405–8.

We have also seen that the long-range behavior (scales $L \gg \ell$, much larger than the elastic scattering mean free path, ℓ) of this correlation function can be extracted from a field theory whose action in $d = 2$ is of nonlinear σ-model type,[29]

$$S[Q] = \frac{\pi\nu}{4} \int d^2x \left[D \,\mathrm{tr}\,(\hat{\partial}_\mu Q \hat{\partial}_\mu Q) - 2\omega \,\mathrm{tr}\,(Q\sigma_3^{\mathrm{ar}}) \right]. \tag{9.22}$$

For later convenience, let us recapitulate a few key features of the field theory defined by Eq. (9.22).

- ▷ The matrix fields $Q^{aa',ss'}(\mathbf{x}) \sim \bar{\psi}^{as}(\mathbf{x})\psi^{a's'}(\mathbf{x})$ describe the behavior of a product of two fermion field amplitudes ψ^{as}. The dynamics of the latter is controlled by the advanced/retarded single-particle Green function, $\langle \bar{\psi}^{as}(\mathbf{x})\psi^{a's'}(\mathbf{x}')\rangle \sim (E_{\mathrm{F}} + s(\omega + i0) - \hat{H})^{-1}(\mathbf{x},\mathbf{x}')\delta^{ss'}\delta^{aa'}$. Here, the two-component index $s = +/-$, and $a = 1, \ldots, R$ refers to the replica index.
- ▷ The fields Q take values in the coset space $\mathrm{U}(2R)/(\mathrm{U}(R) \times \mathrm{U}(R))$.[30] A concrete representation is given by $Q = T\sigma_3^{\mathrm{ar}}T^{-1}$, where $T \in \mathrm{U}(2R)$ and σ_3^{ar} is a Pauli matrix in two-dimensional advanced/retarded (ar) space.
- ▷ For $\omega \to 0$ (the limit we will concentrate on throughout) the action above is invariant under two distinct symmetry transformations: global transformations $T \to gT$, where $g \in G \equiv \mathrm{U}(2R)$ is constant, and local transformations $T \to Th(\mathbf{x})$, where $h(\mathbf{x}) \in H \equiv \mathrm{U}(R) \times \mathrm{U}(R)$, i.e. the group of all matrices fulfilling the condition $[h(\mathbf{x}), \sigma_3^{\mathrm{ar}}] = 0$.
- ▷ The microscopic parent action from which Eq. (9.22) was derived was, in fact, rotationally invariant under the full group $\mathrm{U}(2R)$. That our fields Q live in a smaller coset space signals the fact that, in a metallic system, this symmetry is spontaneously broken: the Qs are the Goldstone modes associated with the breakdown of the symmetry from $\mathrm{U}(2R)$ to $\mathrm{U}(R) \times \mathrm{U}(R)$.
- ▷ Under a generalized[31] gauge transformation, $\psi \to e^{i\phi}\psi$, the Qs transform as $Q \to e^{-i\phi}Qe^{i\phi}$. Gauge invariance then implies that the operators $\hat{\partial}_\mu$ appearing in the effective action must be interpreted as covariant derivatives,

  $$\hat{\partial}_\mu = \partial_\mu - i[A_\mu, \,],$$

  where A_μ transforms as a non-abelian vector potential, $A_\mu \to e^{-i\phi}A_\mu e^{i\phi} - ie^{-i\phi}\partial_\mu e^{i\phi}$. (Notice, however, that the "physical vector potential" generating the perpendicular magnetic field $A_{\mathrm{phys}} \sim \delta^{ab}\delta^{ss'}$ is diagonal in replica and ar spaces and, therefore, does not enter the covariant derivative.)

In a series of famous papers, Pruisken[32] extended the formalism above so as to account for the effect of a strong magnetic field. It turned out that a key player in the action of that

[29] Here we are using the reduced variant of the model (see Problem 8.8.4) suitable to compute the product of a retarded and an advanced Green function.

[30] Due to the presumed presence of a massive magnetic field, time reversal symmetry is broken and the introduction of a "time-reversal space" unnecessary.

[31] By "generalized" we mean that ϕ can be a matrix in ar space as well as in replica space.

[32] For a review, see A. M. M. Pruisken, Field theory, scaling and the localization problem, in *The Quantum Hall E ect*, ed. R. E. Prange and S. M. Girvin (Springer-Verlag, 1987).

generalized field theory was a certain variant of a θ-term. As with many other examples before, there are two ways to obtain this action: one may generalize the derivation of the σ-model discussed in Section 6.5 for the presence of a strong magnetic field and derive the θ-term from first principles. For the details of this not entirely straightforward program we refer to the original papers. Alternatively, one may guess the structure of the generalized action on the basis of symmetry arguments and fix the coupling constants by running some consistency checks. It is this second strategy that we shall pursue shortly.

As discussed in Chapter 7, elements of the conductivity tensor $\sigma_{\mu\nu}$ can be obtained by a two-fold differentiation $\frac{\delta^2}{\delta A_\mu\, \delta A_\nu} Z[A]$ of the partition or generating function with respect to some generalized vector potential. Referring below for a more detailed discussion, all we shall aim for presently is to understand what kind of action is needed to produce a finite Hall conductivity $\sigma_{12} \sim \frac{\delta^2}{\delta A_1\, \delta A_2} Z[A]$. By symmetry, the mixed derivative computed on the field theory defined by Eq. (9.22) vanishes (a point that warrants some consideration). Rather, we have to look out for an operator comprising the two long derivatives $\hat\partial_1 Q$ and $\hat\partial_2 Q$ in a single local expression. For example, we might contemplate a term like $\int \mathrm{tr}(\hat\partial_1 Q\, \hat\partial_2 Q)$. This expression, however, is again not permitted by symmetry. The reason is that (unlike the rotationally invariant two-derivative operator in Eq. (9.22)) it is not form invariant under rotation of the coordinate axes (again something to think about). However, the next obvious choice,

$$S_{\mathrm{top}}[Q] = \theta\epsilon^{\mu\nu} \int d^2x\ \mathrm{tr}(Q\hat\partial_\mu Q \hat\partial_\nu Q), \tag{9.23}$$

does the job. For one thing, this term *is* rotationally invariant. Secondly, the definition of the long derivatives $\hat\partial_\mu$ implies that it contains terms linear in the combination $A_1 A_2$, from where we conclude that the (as yet undetermined) coupling constant θ must have something to do with the Hall conductivity.

To be somewhat more specific, let us draw on Problem 7.6.4 where it has been shown that a source-vector potential suitable for the calculation of the conductance takes the form $A_\mu = U^{-1}\partial_\mu U$ with $U = \exp(i(x_1\kappa_1\sigma_1^{\mathrm{ar}} + x_2\kappa_2\sigma_2^{\mathrm{ar}}) \otimes E_{11}^{\mathrm{r}})$ (here, E_{11}^{r} is a projector onto the first replica channel and κ_μ are numbers). With this choice,

$$\sigma_{11} = \lim_{R\to 0} \frac{1}{4\pi L^2}\, \partial^2_{\kappa_1\kappa_1}\big|_{\kappa=0}\, \mathcal{Z}, \quad \sigma_{12} = \lim_{R\to 0} \frac{1}{4\pi i L^2}\, \partial^2_{\kappa_1\kappa_2}\big|_{\kappa=0}\, \mathcal{Z}.$$

We also know that the conventional Drude theory of a weakly disordered metal is obtained by setting $Q = \sigma_3^{\mathrm{ar}}$. (Remember that fluctuations around the origin of the field space, σ_3^{ar}, describe mechanisms of localization, i.e. quantum effects beyond the Drude picture.) We next use this information to determine the coupling constants of the theory. To this end, we go to the Drude level (set $Q = \sigma_3^{\mathrm{ar}}$), substitute the source A_μ into our long derivatives, and evaluate the action. This leads to (exercise)

$$S[Q = \sigma_3^{\mathrm{ar}}] = 2\pi L^2 D\nu(\kappa_1^2 + \kappa_2^2) + 16 i L^2 \theta \kappa_1\kappa_2.$$

Evaluation of the derivatives above on $Z[A]|_{\mathrm{Drude}} \simeq \exp(-S[\sigma_3^{\mathrm{ar}}])$ then readily leads to the result $\sigma_{11}^0 = 2\pi D\nu$ for the longitudinal Drude conductance σ_{11}^0 (which we knew anyway),

and to the suspected identification $\theta = -\sigma_{12}^0/8$ of the coupling constant θ with the (Drude) Hall conductance. Summarizing, we obtain **Pruisken's action of the IQHE**,

$$S[Q] = \frac{1}{8}\int d^2x \, \left[\sigma_{11}\,\mathrm{tr}(\hat{\partial}_\mu Q\ \hat{\partial}_\mu Q) - \sigma_{12}\epsilon_{\mu\nu}\mathrm{tr}(Q\hat{\partial}_\mu Q\ \hat{\partial}_\nu Q)\right]. \tag{9.24}$$

What makes the identification $\theta \sim \sigma_{12}$ more interesting is that θ (and therefore the Hall conductance) actually plays the role of a topological angle; the second term in Eq. (9.24) is a topological term. That such a term might be present in our field theory follows from the fact that

$$\pi_2(\mathrm{U}(2R)/(\mathrm{U}(R)\times\mathrm{U}(R))) = \mathbb{Z}.$$

Unfortunately, we do not have the mathematical background to prove this result.[33] We can, however, make it plausible; and we can demonstrate that S_{top} above is a representation of the corresponding θ-term.

To this end, let us temporarily focus on a single replica channel ($a = 1$, say) and consider the field configuration

$$\begin{aligned}\tilde{Q}^{11}(\mathbf{x}) &= U(\mathbf{x})\sigma_3^{\mathrm{ar}}U^{-1}(\mathbf{x}) \equiv \mathbf{n}(\mathbf{x})\cdot\sigma^{\mathrm{ar}},\\ \tilde{Q}^{ab}(\mathbf{x}) &= \delta^{ab}\sigma_3^{\mathrm{ar}}, \quad a \neq 1,\end{aligned}$$

where $U(\mathbf{x}) \in \mathrm{U}(2)$. The second equality in the first line defines a unit vector $\mathbf{n}(\mathbf{x})$. It expresses the fact that the projection of the field space onto a single replica channel is isomorphic to S^2.[33] Evaluating S_{top} on this particular field configuration, it is straightforward to verify that $S_{\mathrm{top}}[\tilde{Q}] = i\frac{\sigma_{12}}{2}\int d^2x\,\mathbf{n}\cdot(\partial_1\mathbf{n}\times\partial_2\mathbf{n})$, an expression we identified earlier (see Eq. (9.8)) as the topological term of a two-dimensional field theory on the sphere. We can now generalize from our particular $\tilde{Q}$ to field configurations $Q = T(\mathbf{x})\tilde{Q}(\mathbf{x})T^{-1}(\mathbf{x})$ where $T(\mathbf{x}) \in \mathrm{U}(2R)$. However, using arguments similar to those employed in Section 9.3.1, one may convince oneself that small variations T will not change the value of the action S_{top}. Indeed, S_{top} is the general θ-term on the coset space $\mathrm{U}(2r)/(\mathrm{U}(r)\times\mathrm{U}(r))$.

Pruisken s field theory: long-range physics

Now that we have "derived" the Pruisken action, the next question to ask is what to do with it. To begin with the bad news, the long-distance behavior of the model is still pretty much unknown and, in fact, a subject of ongoing research. (That this is the state of affairs some 20 years after its derivation signals the fact that we are dealing with a very rich field theory.) However, by investing one's physical insight in the quantum Hall problem (and with a little bit of good will) quite a few things about the model can, nonetheless, be said.

Suppose we were dealing with a system of annular geometry, similar to that discussed in the last section. Let us further assume that the Fermi energy lies in between the center of the nth and $(n+1)$th Landau bands, so that there are no delocalized states at E_{F}. As discussed in the previous section, the bulk of the sample is then pretty much impervious to external

[33] Notice, however, that in the special case $R = 1$, $\mathrm{U}(2)/(\mathrm{U}(1)\times\mathrm{U}(1)) \cong \mathrm{SU}(2)/\mathrm{U}(1) \cong S^2$ is the 2–sphere. What is non-trivial here is the generalization of $\pi_2(\mathrm{U}(2)/(\mathrm{U}(1)\times\mathrm{U}(1))) = \pi_2(S^2) = \mathbb{Z}$ to general R.

perturbations, and the interesting Fermi-energy physics takes place at the boundaries (the inner and outer perimeters).

Indeed, it is possible to rewrite the topological action as a pure boundary operator. A straightforward application of Stokes' theorem[34] shows that

$$S_{\rm top}[T] = \frac{\sigma_{12}}{2}\int_{\partial M} d\mathbf{s}\cdot {\rm tr}(T\sigma_3^{\rm ar}\nabla T^{-1}), \tag{9.25}$$

where the integral runs over the boundary ∂M of the annulus. Notice, however, that Eq. (9.25) is not represented in terms of the fundamental degrees of freedom of the theory (the Qs) but rather relies on a particular "coordinate" representation ($Q = T\sigma_3^{\rm ar}T^{-1}$). As discussed in Section 9.4, this is not accidental but rather reflects a fundamental property of topological terms. For future reference, we also anticipate that the boundary descendant of the bulk θ-term Eq. (9.23) is an example of a **Wess–Zumino term**.

We next show that, under the conditions stated above, the boundary representation of the theory raises a consistency problem unless σ_{12} = integer. To appreciate the problem, recall that the bulk representation of the theory is invariant under local transformations $T(\mathbf{x}) \to T(\mathbf{x})h(\mathbf{x})$, where $h \in H = {\rm U}(R)\times{\rm U}(R)$. This follows trivially from the fact that its degrees of freedom $Q = T\sigma_3^{\rm ar}T^{-1} \to Th\sigma_3^{\rm ar}h^{-1}T^{-1} = Q$ are invariant. Now, under our present working conditions – Fermi energy in a mobility gap – there are no Q-field excitations in the bulk of the system (the Qs describe mobile Fermi energy excitations), and the theory reduces to two decoupled boundary theories which must be separately invariant. In this respect, the only contribution that may potentially cause trouble is the topological boundary contribution Eq. (9.25) (since it is not a functional of the invariant degree of freedom, Q). Indeed, let us consider the specific choice $h = \exp(i\sigma_3^{\rm ar}\theta)$, where θ is the azimuthal coordinate of the system. Focusing on the effect of this transformation at the inner perimeter, say, we obtain

$$\begin{aligned} S_{\rm top}[T] &= \frac{\sigma_{12}}{2}\int_0^{2\pi} d\theta\,{\rm tr}(T\sigma_3^{\rm ar}\partial_\theta T^{-1}) \to \frac{\sigma_{12}}{2}\int_0^{2\pi} d\theta\,{\rm tr}(Th\sigma_3^{\rm ar}\partial_\theta(h^{-1}T^{-1})) \\ &= S_{\rm top}[T] + \frac{R\sigma_{12}}{2}\int_0^{2\pi} d\theta\,{\rm tr}(\sigma_3^{\rm ar}h\partial_\theta(h^{-1})) = S_{\rm top}[T] + \frac{R\sigma_{12}}{2}\int_0^{2\pi} d\theta\,\partial_\theta\,{\rm tr}(\sigma_3^{\rm ar}\ln h^{-1}) \\ &= S_{\rm top}[T] + \frac{R\sigma_{12}}{2}{\rm tr}(\sigma_3^{\rm ar}\ln h^{-1})\Big|_0^{2\pi} = S_{\rm top}[T] - 2\pi i\sigma_{12}R. \end{aligned} \tag{9.26}$$

The invariance of the exponentiated (!) boundary action requires that $\exp(2\pi iR\sigma_{12}) \stackrel{!}{=} 1$, from which we conclude that $\sigma_{12} \stackrel{!}{=}$ integer. Summarizing, we have seen that, for Fermi energies in between two Landau band centers (that is, conditions where QH plateaus are experimentally observed), the intrinsic consistency of the theory requires quantization of the Hall conductance. A refined variant of such arguments (see Pruisken's aforementioned article) shows that the value of the conductance indeed coincides with the number of Landau levels below the Fermi energy. Notice also that the reasoning above is again "topological." This time, the key players are the winding numbers of the mappings $S^1 \to {\rm U}(1)\times{\rm U}(1)$ from

[34] For a review of the general form of Stokes' theorem, see Section 9.4.2 below.

the boundary manifold (topologically, a circle) into the projection of the local transformation group to a single replica channel (U(1) × U(1)).

EXERCISE As an exercise in group manipulations, extend the argument given above to more general transformations at the boundary. To this end, consider a general (but single-valued!) boundary transformation $h(\theta) \in H$ and show that we will not obtain information beyond the quantization criterion discussed above. Hint: Use the facts that $\mathrm{U}(r) = \mathrm{U}(1) \times \mathrm{SU}(r)$, and that $\forall R \in \mathrm{SU}(r) : 0 = \ln\det R = \mathrm{tr}\,\ln R$.

INFO The argument above focuses on the boundaries of the system. As a brief digression, let us show how the same information can be obtained from a **bulk picture**. In the bulk, all Fermi energy states are localized. Technically, this implies that the longitudinal conductance renormalizes to zero, $\sigma_{11} \overset{L\ \xi}{\longrightarrow} 0$, where the notation is meant to indicate that, in a renormalization group sense, the conductance scales to zero on length scales $L \gg \xi$ much larger than the two-dimensional localization length. An alternative formulation of the same fact is to say that transformations $\psi(\mathbf{x}) \to T(\mathbf{x})\psi(\mathbf{x})$, where $T(\mathbf{x}) \in \mathrm{U}(2R)$ fluctuates on scales $\gg \xi$, must leave the theory invariant.[35] For $\sigma_{11} \to 0$, and frequency differences $\omega \to 0$, the non-topological sector of the action vanishes and is therefore trivially oblivious to fluctuations of the Qs. However, this is not the case with the topological action. No matter how slowly it fluctuates in space, a topologically non-trivial configuration $Q(\mathbf{x})$ will have topological action $S[Q] = 2\pi\sigma_{12}n$, where n is the corresponding winding number. The only way to make the theory generally impervious to Q-field fluctuations is to require $\sigma_{12} \overset{!}{=}$ integer – the bulk variant of the quantization criterion.

Quantum Hall transition

Having understood the basic quantization phenomenon, we might now ask what other features of the quantum Hall system can be extracted from the field theory approach. Given that the nonlinear σ-model arguably represents the most powerful approach to disordered electron systems in general, we might, for example, be ambitious enough to seek more information about the nature of the delocalized states expected to reside at the centers of the Landau bands. As we saw above, these states do play a vital role in the formation of the QHE. However, as long as they are deeply buried below the Fermi energy, they influence the system rather indirectly. We should expect them to become much more vivid as the Fermi energy sweeps through the center of a Landau band. Indeed, as $E_\mathrm{F} \to n\omega_\mathrm{c}$ approaches a band center, the system begins to build up long-ranged correlations. This is because the physics at the Fermi energy is now controlled by ever more extended, or delocalized, states. Formally, $E_\mathrm{F} \to (n+1/2)\omega_\mathrm{c}$ goes along with a diverging correlation length ξ, a phenomenon indicative of a second-order phase transition. Indeed, $E^* = (n+1/2)\omega_\mathrm{c}$ marks the position of a very peculiar (and still not fully understood) quantum phase transition, the **quantum Hall transition**.

[35] Indeed, we had argued earlier that metallic behavior goes along with a spontaneous breakdown of replica rotation symmetry. Our Qs are the Goldstone modes of this phenomenon. Conversely, localization is accompanied by a restoration of this symmetry; the Goldstone modes disappear and the theory will no longer respond to fluctuations of the Qs (or the Ts for that matter). Further, notice that we are now concentrating on the "large" transformation group of the model, $\mathrm{U}(2R)$, and not the "small" invariance group $\mathrm{U}(R) \times \mathrm{U}(R)$.

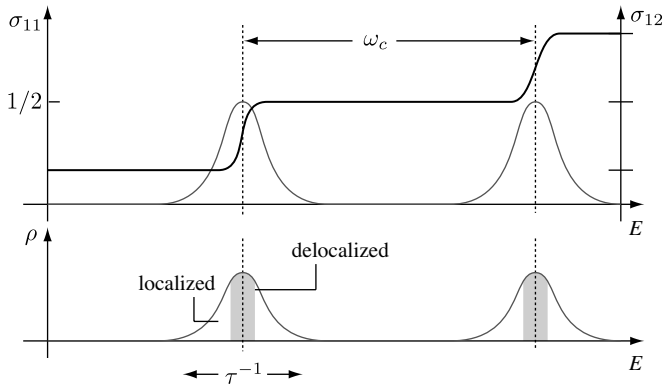


Figure 9.9 On the behavior of the longitudinal conductance, Hall conductance, and density of states, ρ, at the quantum Hall transition.

In Fig. 9.9, the behavior of the three most relevant players in the system, longitudinal conductance, Hall conductance, and density of states, is shown as functions of the Fermi energy. In the vicinity of the Landau band centers the system becomes critical. Right at a transition point, $E_{\rm F} \equiv E^*$, (a) the longitudinal conductance $\sigma_{11} = 1/2$, (b) the Hall conductance $\sigma_{12} = n+1/2$ is *half* integer, and (c) the correlation length ξ characterizing the spatial profile of wavefunctions at the Fermi energy has diverged. The latter implies that, for any system of finite size L, there is a whole range of energies around the critical value for which $\xi > L$. Within this energy range, the wavefunctions are effectively delocalized. Naturally, the width of the band of delocalized states shrinks upon increase of the system size and, in the thermodynamic limit, approaches zero. More precisely, upon approaching the band center, the correlation length diverges as

$$\xi \sim |E - E^*|^{-\nu},$$

where ν defines the correlation length exponent and we have introduced $\Delta E \equiv |E - E^*|$ as a relevant scaling variable. For a given system size, the width of the energy band, ΔE, of delocalized states is determined by the condition $L \sim \xi(\Delta E)$, or $\Delta E \sim L^{-1/\nu}$. The number of states within that window scales as $N \sim \frac{\Delta E}{\delta} \sim L^{-1/\nu+2}$, where $\delta \sim L^{-2}$ is the two-dimensional level spacing. Unfortunately, there is still no reliable analytical prediction for ν. (We shall see in a moment why this is so.) However, high-precision numerical analyses[36] have shown that $\xi = 2.35 \pm 0.08$. This implies that the number of states within the delocalized region diverges in the thermodynamic limit: even though the width of the delocalized energy window approaches zero, it hosts a continuum of extended states.

We shall next explore the extent to which these features can be understood from the **field theory approach**. To this end, let us imagine the partition function formally expanded in

[36] See, e.g., B. Huckestein, Scaling theory of the integer quantum Hall effect, *Rev. Mod. Phys.* **67** (1995), 357–96.

terms of the topological index of field configurations:

$$\mathcal{Z} = \sum_W e^{2\pi i W \sigma_{12}} \mathcal{Z}_W,$$

where $\mathcal{Z}_W$ is the partition function reduced to the sector of fields of winding number W, and $S_{\rm top} = 2\pi i W \sigma_{12}$ enters as a topological phase. In general, there is not much we can say about $\mathcal{Z}_W$, other than that it will be small: $|\mathcal{Z}_n| \overset{\sigma_{11}\gg 1}{\ll} 1$. Indeed, an estimate similar to the one employed on page 509 yields (exercise)

$$\begin{aligned} 0 &\leq \frac{1}{2}\int d^2x \,\mathrm{tr}\,[(\partial_\mu Q + i\epsilon_{\mu\nu} Q\partial_\nu Q)(\partial_\mu Q + i\epsilon_{\mu\lambda} Q\partial_\lambda Q)] \\ &= \int d^2x \,\mathrm{tr}\,[\partial_\mu Q\ \partial_\mu Q - i\epsilon_{\mu\nu} Q\ \partial_\mu Q\ \partial_\nu Q] = \int d^2x \,\mathrm{tr}(\partial_\mu Q\ \partial_\mu Q) - 16\pi W. \end{aligned}$$

From here, we conclude that the non-topological contribution to the action obeys the inequality

$$S_0[Q] = \frac{\sigma_{11}}{8}\int d^2x \,\mathrm{tr}(\hat\partial_\mu Q\ \hat\partial_\mu Q) \geq 2\pi W \sigma_{11},$$

and that $\mathcal{Z}_n \sim \exp(-2\pi W\sigma_{11})$ is weighted by a small "energetic" factor. We have had ample opportunity to see that such factors can, in principle, be compensated for by large "entropic" counterweights. However, Pruisken has shown that, in the present context, this does not happen.

Similarly, the functional expectation values for longitudinal and Hall conductance can be organized in an instanton series:

$$\sigma_{ij} = \sum_{W=0}^{\infty} e^{2\pi i W \sigma^0_{12}}\, C^{(W)}_{ij}(\sigma^0_{11}).$$

The coefficients $C^{(W)}_{ij}$ appearing in this series depend – in the exponentially sensitive way discussed above – on the longitudinal Drude conductance. Importantly, the right-hand side of the series depends on the Drude values σ^0_{ij} while the left-hand side sets the true renormalized conductance (i.e. the second-order derivative of the functional with respect to a generalized source-vector potential). There is not much more we can say about the structure of these series except for one important symmetry criterion: under a change of orientation of the coordinate system, σ_{12} changes sign while σ_{11} does not. Specifically,

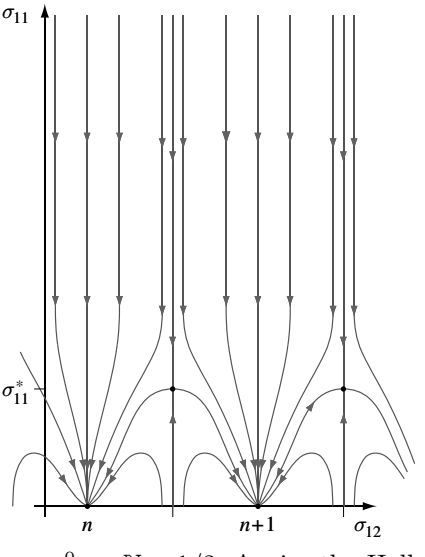
$$\sigma_{12}(B) = -\sigma_{21}(B), \quad \sigma_{11}(B) = +\sigma_{22}(B),$$

which is an example of an **Onsager relation**. Consistency with these relations requires that the topological series be of the form

$$\begin{aligned} \sigma_{11} &= \sigma^0_{11} + \delta\sigma_{11} + \sum_{W=1}^{\infty} \cos(2\pi W\sigma^0_{12})\ a^{(W)}(\sigma^0_{11}), \\ \sigma_{12} &= \sigma^0_{12} + \sum_{W=1}^{\infty} \sin(2\pi W\sigma^0_{12})\ b^{(W)}(\sigma^0_{11}), \end{aligned} \tag{9.27}$$

where $a^{(W)}$ and $b^{(W)}$ are expansion coefficients. The notation emphasizes that, in the topologically trivial sector, $W = 0$, the longitudinal conductance may be subject to renormalization while σ_{12} remains unrenormalized.[37]

In view of these structures, let us now speculate a little on the **renormalization characteristics** of the model. Here, renormalization means, as usual, that we consider the outcome of the theory on ever increasing length scales. We have to keep in mind, however, that this program must be carried out for each topological sector separately: renormalization, i.e. the successive elimination of fast fluctuations, cannot change the topological index of a field configuration. On the same footing a topological angle (presently, the coefficient σ_{12}^0 of the topological term) should not renormalize, at least not in a conventional sense. (For the cautious formulation, see below.)



Consider, then, the flow $(\sigma_{12}(\lambda), \sigma_{11}(\lambda))$ in the two-dimensional parameter plane defined by longitudinal and Hall conductance as we increase a reference length scale λ. From Eq. (9.27) it follows that there are two families of lines on which σ_{12} does not renormalize: $\sigma_{12} = \sigma_{12}^0 \in \mathbb{N}$ and $\sigma_{12} = \sigma_{12}^0 \in \mathbb{N} + 1/2$. As to the first line, we do not expect any critical behavior in the vicinity of integer σ_{12}. (Remember that the quantum Hall transition is observed at $(\sigma_{12}, \sigma_{11}) = (\mathbb{N} + 1/2, 1/2)$.) Rather we expect that, in the vicinity of integer Hall conductances, the system behaves pretty much like an ordinary two-dimensional electron gas. This implies (cf. our discussion above) that the Goldstone modes become gapped, σ_{11} scales to 0 and σ_{12} scales to an integer value: for $(\sigma_{12}, \sigma_{11})$ close to a line $\sigma_{12} = n \in \mathbb{N}$, the system will flow towards the fixed point $(n, 0)$. A special situation arises for $\sigma_{12} = \sigma_{12}^0 \in \mathbb{N} + 1/2$. Again, the Hall conductance does not renormalize, but now we are sitting on a critical surface (namely, $E = E^*$, corresponding to half-integer Hall conductance). At the transition point, the correlation length has diverged and the system has become metallic. Accordingly we should expect the longitudinal conductance to scale towards a finite fixed point value σ_{11}^*. Indeed, experimentally, one observes $\sigma_{11}^* = \mathcal{O}(1)$ at the transition points. The two-parameter flow diagram discussed above was proposed in a seminal paper by Khmelnitskii.[38]

To theoretically understand the transition behavior, Pruisken and collaborators have derived renormalization group equations that take the lowest two topological sectors, $W =$

[37] Renormalization cannot change the prefactor of the topological term, σ_{12}^0. The contribution of the topologically trivial sector to the conductance is just this coefficient.

[38] D. E. Khmelnitskii, Quantization of Hall conductivity, *JETP Lett.* **38** (1983), 552–6.

0, 1, into account:

$$\begin{aligned}\beta_{11} &\equiv \frac{\partial\sigma_{11}}{\partial \ln L} = -\frac{1}{2\pi^2\sigma_{11}} - c\sigma_{11}e^{-2\pi\sigma_{11}}\cos(2\pi\sigma_{12}),\\ \beta_{12} &\equiv \frac{\partial\sigma_{12}}{\partial \ln L} = c\sigma_{11}e^{-2\pi\sigma_{11}}\sin(2\pi\sigma_{12}),\end{aligned} \tag{9.28}$$

where $c > 0$ is a numerical constant.[39] These equations indeed have a family of fixed points $(\sigma_{12}^*, \sigma_{11}^*)$, where $\sigma^* = \mathcal{O}(1)$ and $\sigma_{12}^* \in \mathbb{N} + 1/2$. In the vicinity of these points, σ_{12} (σ_{11}) is a relevant (irrelevant) scaling variable.

Summarizing, Pruisken's approach appears to predict a parameter flow as shown in the **two-parameter phase diagram** above. This diagram nicely conforms with experimental observations but, alas, there are some problems. For one thing, the interesting physics takes place in the vicinity of a fixed point value $\sigma_{11}^* = \mathcal{O}(1)$, well outside the regime of applicability of the σ-model as such. (Remember that the derivation of this hinges on $\sigma_{11} \gg 1$.) One may trust in the principle that "good" models (and Pruisken's model of the QHE certainly is good!) usually produce meaningful results even in parameter regimes where they no longer stand on safe ground. However, with the current problem, the chances are that this principle does not apply. Indeed, we have come across a very similar situation before: for large spin $S \gg 1$, an antiferromagnetic spin chain is described by an O(3) nonlinear σ-model. For topological angles $\theta = \pi$ (corresponding to half-integer spin) the model is critical and flows towards some strong coupling fixed point. The important observation now is that, in the vicinity of this point, the system is described no longer by an O(3) nonlinear σ-model but by an altogether different model: a field theory on the group manifold SU(2) with a topological term of Wess–Zumino–Witten type. Intriguingly, on its journey towards the strong coupling fixed point, the model manages to *enlarge* its field manifold from the coset space $\mathrm{O}(3)/\mathrm{O}(2) \simeq S^2$ of the O(3) nonlinear σ-model to the larger space $\mathrm{SU}(2) \overset{\text{locally}}{\simeq} \mathrm{O}(3)$ of the fixed point theory.

Now, when projected onto a single replica channel, Pruisken's $\sigma_{12} = 1/2$ theory indeed reduces to an O(3) nonlinear σ-model with topological angle $\theta = \pi$. It is, therefore, quite conceivable that, at strong coupling, the general model also flows towards a target model with a larger field manifold. Although nothing rigorous is known, a scenario to this effect has been outlined in the literature. At any rate, the applicability of Pruisken's RG equations in the vicinity of their fixed points remains questionable.

INFO There is one **other problem** which should not be swept under the carpet: as discussed above, the absence of Goldstone modes in the localized phase requires the coupling constant of the topological term to be integer. This coupling constant is set by the Drude conductance σ_{12}^0 and should not renormalize. In a way, we have to require that, for Fermi energies which lie between Landau levels, the Drude Hall conductance must already be integer, lest the theory run into a consistency problem. On the other hand, topological criteria do not require the physical conductance σ_{12} to be integer. This is exactly the opposite of our physical picture. We should

[39] One might have expected the topological angle σ_{12}^0, and not the physical conductance σ_{12}, as an argument of the transcendental functions (sine and cosine). For a discussion of why the physical conductance appears there, we refer to Pruisken's article (Pruisken, in *The Quantum Hall E ect*).

expect that for any value of the Drude conductance (away from $\mathbb{N} + 1/2$) the system will flow towards an integer value of the physical conductance. (This is the flow illustrated in the two-parameter flow diagram above.) Essentially, the problem boils down to the fact that the coupling constant of the topological term does not renormalize. (If it did, it should flow towards an integer value which could then justly be interpreted as the physical conductance.) To deal with this difficulty, Pruisken subjects the fields to a transformation $Q \to UQU^{-1}$, where U are the source fields introduced above, and then discusses the structure of the theory after the fields Q have been integrated out. This produces consistent results, but the procedure is highly implicit and, therefore, remains somewhat mysterious.

Summarizing, Pruisken's field theory microscopically explains many of the intriguing aspects of the quantum Hall effect. Notably, it elucidates the interplay of disorder scattering and topological mechanisms (spectral flow!) in the formation of the effect. Contrary to early expectations, though, it may well be that the actual quantum Hall transition lies outside the scope of the model. At any rate, a rigorous identification of the universality class of this transition, not to mention a quantitative calculation of its critical exponents, remains an open problem.

This concludes our discussion of θ-terms in condensed matter field theory. In our brief survey, we were unable to discuss one of the most exciting applications of θ-terms in field theory in *general*: 'tHooft's concept of θ-vacua, and its relevance to understanding some of the most intriguing observed features of matter – CP- and T-violation. (However, the energetic reader is strongly encouraged to turn to a textbook on theoretical particle physics to learn more of this subject.) Rather, we proceed now to discuss another large and important family of topological field theories.

9.4 Wess–Zumino terms

Almost every time that we met with a θ-term in the previous chapter, a field theory with a Wess–Zumino (WZ) term[40] was just around the corner. Yet most condensed matter physicists appear to be only vaguely familiar with the ideas behind WZ field theory (in contrast to, say, the much more widely appreciated concept of θ-terms). Perhaps the most important reason for this lack of appreciation is that the general meaning of a WZ term is difficult to grasp in the "traditional" languages familiar to condensed matter physicists. On the other hand, the relevant concepts become quite transparent once we venture to reformulate a few elements of field theory in the language of modern differential geometry. This reformulation will be the subject of the first half of this section. Not assuming any background knowledge, we begin with a crash course in differentiation on manifolds and exterior calculus. (Readers familiar with differential forms are invited to skip this introduction.) In Section 9.4.2 we

[40] As to the terminology of Wess–Zumino terms, there is no generally accepted convention. Historically, these topological terms first appeared in the work of Wess and Zumino. However, owing to Witten's seminal analysis of SU(N)-invariant chiral fermion systems (see below), they are often referred to as Wess–Zumino–Witten (WZW) terms. Yet another designation (especially popular in the Russian community) is Wess–Zumino–Novikov–Witten (WZNW) terms. Except for the discussion of SU(N)-symmetric systems below, we stick here to the short variant of WZ terms.

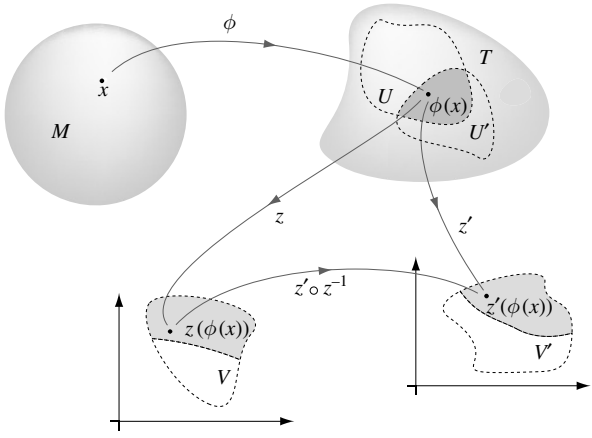


Figure 9.10 On the construction of coordinate representations of field manifolds. For a discussion, see the main text.

then explain the general ideas behind WZ field theory before we turn to the discussion of a number of interesting applications.

9.4.1 A crash-course in differential geometry

Figure 9.10 reiterates the mathematical backbone underlying most of our field theories: a field $\phi : M \to T, \mathbf{x} \mapsto \phi(\mathbf{x})$ is a mapping from a base-space M to a field space T. In practice, we mostly tend to nonchalantly identify field values $\phi \in T$ and their coordinate representations $\mathbf{z}(\phi) \in \mathbb{R}^n$ (n is the dimension of T). However, especially when it comes to the discussion of topological aspects, we must be very careful with such premature identifications. The point is that topologically non-trivial field spaces usually cannot be represented in terms of one globally defined system of coordinates. Take the sphere S^2 as an example. You may choose the standard representation in terms of two angles (θ, ϕ), a stereographic projection onto a single complex variable z, or any other parameterization. Inevitably, there will be regions on S^2 where the mapping "$S^2 \to$ coordinates" becomes ill-defined. One may object that this ambiguity, manifesting itself only at a set of "measure zero,"[41] cannot be of much practical significance. Yet, with regard to topology this is not the case.

Coordinate representations

Given the importance of these singularities in the context of topological field theory, it is pertinent this time to discuss the construction of proper coordinate representations with mathematical rigor: the basis of each coordinate system is formed by a system of open

[41] For example, in a system of polar coordinates, the problematic region is a *line* connecting the north and the south pole of the sphere.

subsets $U_i \subset T$ chosen so that the union $\bigcup_i U_i = T$ covers T.[42] One next defines coordinate mappings $z_i : U_i \to V_i \subset \mathbb{R}^n$ from the patches U_i onto some open subsets V_i of $\mathbb{R}^n$. The value $z_i(p) \in \mathbb{R}^n$ is a coordinate representation of the point $p \in T$. To make this representation unique, we require the z_is to have an inverse.

The central point is that, on our manifold, there will be non-vanishing overlaps $U_i \cap U_j \neq \{\}$, i.e. points $p \in T$ that have more than one coordinate representation (see Fig. 9.10). Ambiguities between the different systems are excluded by requiring that the maps $z_i \circ z_j^{-1} : z_j(U_i \cap U_j) \to z_i(U_i \cap U_j)$ be diffeomorphisms (invertible and differentiable). Conceptually, the functions $z_i \circ z_j^{-1}$ mediate the change between different coordinates. This being so, they must be as benign (differentiable) as possible. Notice, however, that it would be senseless to require the z_is themselves to be differentiable, simply because in general $T \not\subset \mathbb{R}^n$ and the notion of differentiability need not even exist on T.

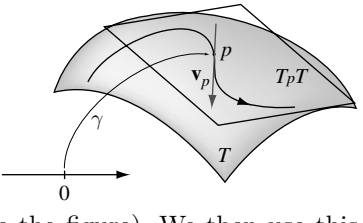
In the jargon of differential geometry, the coordinate maps z_i are called **charts** of T while a fully covering collection $\{z_i\}$ is called an **atlas**. The existence of a proper atlas is, by definition, equivalent to the statement that T is a differentiable manifold.

Example Consider $T = S^1$, the unit circle. The one-dimensional manifold S^1 has a natural embedding into $\mathbb{R}^2 : S^1 = \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{x}^2 = 1\}$. We need a minimum of two charts to cover S^1. For example, one may use

$$\begin{aligned} z^{-1} : (0, 2\pi) &\to S^1, \quad z^{-1}(\phi) = (\cos(\phi), \sin(\phi)), \\ z'^{-1} : (0, 2\pi) &\to S^1, \quad z'^{-1}(\phi) = (\cos(\phi + \pi), \sin(\phi + \pi)); \end{aligned}$$

z/z' cover all of S^1 except for the points $(1, 0)/(-1, 0)$.

Tangent space



Having discussed the coordinate representations of (field) manifolds, we are now in a position to lift elements of standard calculus (differentiation, integration, etc.) from $\mathbb{R}^n$ to manifolds. (Later, we identify topological terms as integrals over certain differentials on T.) We begin by introducing the tangent space T_pT as a locally flat approximation to the manifold T at a point $p \in T$ (see the figure). We then use this planar approximation to describe how functions defined on T vary in the neighborhood of p. To construct the tangent space, consider a curve $\gamma : \mathbb{R} \to T$ with $\gamma(0) = p$. It is tempting (see the figure) to define a vector $\mathbf{v}_p$ tangent to T at p by setting $\mathbf{v}_p \stackrel{?}{\equiv} d_s\gamma(s)|_{s=0}$. However, this "definition" is problematic because, in general, $T \not\subset \mathbb{R}^m$, so that $\mathbf{v}_p$ is not a decent vector. Nonetheless, the idea above is not far from the truth. To make it suitable, let us consider some function $f \in \mathcal{C}(U)$ defined on an open neighborhood $U \ni p$. (Here, $\mathcal{C}(X)$

[42] Of course, one can exercise much freedom in the choice of the U_is. For example, for the sphere, one might define $U_{1,2}$ to be any two overlapping "caps" whose union covers T. Notice that overlaps are, in fact, unavoidable as we want to cover a compact set T by open subsets U_i.

denotes the space of smooth, real-valued functions defined on X.) We may then employ our curve γ to compute the directional derivative

$$\mathbf{v}_p^\gamma(f) \equiv d_s|_{s=0} f(\gamma(s)).$$

The notation on the right-hand side indicates that we have constructed a mapping that takes functions as arguments and produces a number.[43] We next define this mapping to be a **tangent vector** at point p. (Notice that the assignment "curve $\mapsto$ tangent vector" constructed in this way is not unique: two curves γ_1 and γ_2 tangent[44] to each other at p will produce the same directional derivative, $\mathbf{v}_p^{\gamma_1} = \mathbf{v}_p^{\gamma_2}$.) The set of all directional derivatives formed in this way defines the **tangent (vector) space** T_pT at point p.[45]

The definition of "vectors" given above may seem strange. (If you wait a while, its utility will become apparent!) However, given a coordinate function z we can meet the standard identification "vector $\leftrightarrow$ n-component object" familiar from linear algebra. This is achieved by writing

$$\mathbf{v}^\gamma(f) = d_s|_{s=0} f(z^{-1} \circ z \circ \gamma(s)) \equiv \sum_{i=1}^{n} \partial_i f \, v_i^\gamma,$$

where $\partial_i f \equiv \partial_i (f \circ z^{-1})$ is the ordinary partial derivative of the function $f \circ z^{-1} : V \subset \mathbb{R}^n \to \mathbb{R}$ and $v_i^\gamma \equiv d_s|_{s=0} z^i(\gamma(s))$. We define $v_i^\gamma \in \mathbb{R}$ to be the ith component of $\mathbf{v}^\gamma$ (in the coordinate representation defined by z). Notice that the **components of a tangent vector** v_i can alternatively be obtained as

$$\boxed{v_i = \mathbf{v}(z^i),}$$

i.e. as the directional derivative of the ith component of the coordinate function. (Here, we have simplified the notation by omitting the superscript reference to the curve γ.) Relatedly, a coordinate system induces a natural **basis of the tangent space**, T_p. This, is defined by

$$\boxed{\forall f \in \mathcal{C}(U):\ \mathbf{e}_i(f) \equiv \partial_i f.}$$

Suppose that the reference point p is represented by two coordinate functions z and z'. It is then straightforward to verify (exercise!) that the components of the tangent vector transform as $v_i^\gamma = \sum_j \frac{\partial z^i}{\partial z'_j} v_j'^\gamma$.

EXERCISE Compute the basis vectors $\mathbf{e}$ and $\mathbf{e}'$ corresponding to the two charts forming the atlas of S^1 discussed on page 538. Show that $\mathbf{e} = \mathbf{e}' = \mathbf{e}_\phi$ where $\mathbf{e}_\phi$ is the azimuthal vector of a two-dimensional polar coordinate system, and the natural embedding of S^1 into $\mathbb{R}^2$ is understood.

[43] A purist might object that $\mathbf{v}_p^\gamma(f)$ is just what we had earlier defined to be a function*al*, so that we should use square brackets $[f]$ for the argument. However, following standard convention in differential geometry, we shall here stick to (f).

[44] In differential geometry, one uses this criterion to *de ne* the notion of tangency: two curves $\gamma_{1,2}$ through p are tangent to each other (with the same tangent velocity) if, for all functions f, $\mathbf{v}_p^{\gamma_1}(f) = \mathbf{v}_p^{\gamma_2}(f)$.

[45] Mathematically, T_pT is isomorphic to the space of all equivalence classes of curves through p, where two curves are called "equivalent" if they are tangent to each other.

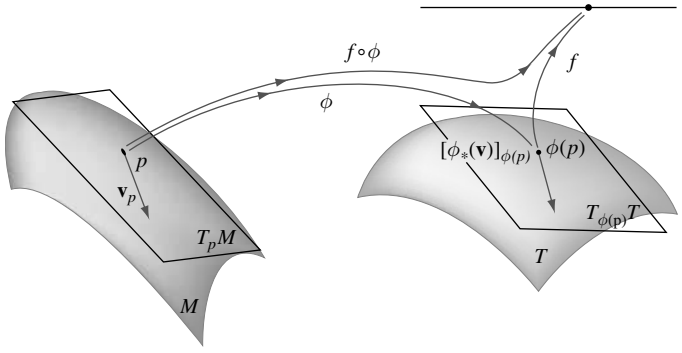


Figure 9.11 On the definition of the tangent mapping. For a discussion, see the main text.

The union $TT \equiv \bigcup_{p\in T} T_pT$ of all local tangent spaces is called the **tangent bundle** of the manifold T.[46] In fact, the tangent bundle is a differentiable manifold by itself. Its elements are given by $(p, \mathbf{v}_p) \in TT$, where $p \in T$ and $\mathbf{v}_p \in T_pT$. (The dimension of TT is given by $2n$, twice the dimension of T.) A mapping

$$\begin{aligned} \mathbf{v} : T &\rightarrow TT, \\ p &\mapsto (p, \mathbf{v}_p), \end{aligned}$$

smoothly assigning to each point of T a tangent vector is called a **vector field** on T.

Now, suppose we are given two manifolds M and T. (Later on, the role of M will be played by the base space of field theory, a differential manifold by itself.) Further, suppose that there is a mapping (the field!) $\phi : M \rightarrow T$. The definition of tangent spaces above then implies the existence of an induced mapping, the so-called **tangent mapping**,

$$\begin{aligned} \phi_* : TM &\rightarrow TT, \\ \mathbf{v} &\rightarrow \phi_*(\mathbf{v}). \end{aligned}$$

The image vector $(\phi_*(\mathbf{v}))_p$ is defined by setting $(\phi_*(\mathbf{v}))_p(f) \equiv \mathbf{v}_{\phi^{-1}(p)}(f \circ \phi)$, for any function defined in a neighborhood of $p \in T$ (see Fig. 9.11). Suppose we are given a system of coordinates w around $p \in M$ and z around $\phi(p) \in T$. It is then straightforward to show that the coordinate representation of the vector $(\phi_*(\mathbf{v}))_p$ is given by

$$\boxed{(\phi_*(\mathbf{v}))^i_p = \frac{\partial \phi^i}{\partial w^j}\, \mathbf{v}^j_{\phi^{-1}(p)},} \tag{9.29}$$

where $\phi^i = z^i(\phi)$ is a shorthand for the coordinate representation of ϕ. The formula above explains why the mapping ϕ_* is sometimes referred to as the differential of the mapping ϕ. It illustrates the general rule that everything taking place on the tangent spaces is a measure of local (or "infinitesimal") variations.

[46] The authors hate the double-T notation TT, too. However, the prefix T for tangent is a ubiquitous standard, and we want to keep emphasizing that the apparatus introduced here will be later applied to the target manifolds of field theory, T.

Di erential forms

A **1-form** ω_p is a linear mapping $\omega_p : T_pT \to \mathbb{R}$ (i.e. an element of the dual space of the vector space T_pT). Smoothly extending ω_p to a map ω globally defined on TT, we obtain a so-called **differential 1-form** (or, for brevity, just the 1-form). We will denote the space of 1-forms on T by $\Lambda^1(T)$. A number of important remarks on these definitions are in order:

▷ Most 1-forms that we shall encounter in practice are realized as **differentials of functions**: for $f \in \mathcal{C}(T)$ we define the differential df by

$$\boxed{df_p(\mathbf{v}_p) \equiv \mathbf{v}_p(f).}$$

(Exercise: Convince yourself that, for manifolds $T \subset \mathbb{R}^n$, this reduces to the standard definition of the differential with which you are familiar.)

▷ However, not every 1-form is a differential of a function. Consider, for example, the tangent basis of $T = S^1$ constructed in the exercise on page 539. Let us define a 1-form by setting $\omega(\mathbf{e}) = \omega(\mathbf{e}') = 1$, where the equality holds in the domain of overlap of the two charts. This 1-form cannot be represented as the differential of a single-valued function on S^1 (exercise: why?).

▷ Finally, let us make a formal remark: for reasons that will become clear in a moment, functions $f \in \mathcal{C}(T) \equiv \Lambda^0(T)$ are sometimes referred to as 0-**forms**. Technically, a 1-form is a mapping

$$\begin{aligned} \omega : TT &\to \Lambda^0(T), \\ \mathbf{v} &\to \omega(\mathbf{v}), \end{aligned}$$

that maps vector fields onto 0-forms. The value of the function $\omega(\mathbf{v})$ at a point p is given by $\omega(\mathbf{v})(p) = \omega_p(\mathbf{v}_p)$. Alternatively, we can say that the insertion of a vector field into a 1-form lowers the degree of the form from 1 down to 0.

Given a system of local coordinates, z, each 1-form can be represented as

$$\omega = f_i \, dz^i, \tag{9.30}$$

where the coefficient-functions are given by $f_i = \omega(\mathbf{e}_i)$ (a result that is instructive to check).

A 1-form maps a single vector field onto a function. However, to describe the geometric structure of a manifold (distances, surface and volume elements, etc.) we need mappings that take more than one vector field as input. Mappings of this type are called tensors and defined as follows: a **covariant tensor** of rank r is a multi-linear mapping

$$\begin{aligned} \omega : \underbrace{TT \times \cdots \times TT}_{r} &\to \Lambda^0(T), \\ (\mathbf{v}_1, \ldots, \mathbf{v}_r) &\mapsto \omega(\mathbf{v}_1, \ldots, \mathbf{v}_r). \end{aligned}$$

For example, some manifolds admit the definition of a **metric**. A metric g is a tensor of rank 2 that is positive ($\forall p \in T : g_p(\mathbf{v}_p, \mathbf{v}_p) > 0$) and non-degenerate ($\forall \mathbf{w}_p : g_p(\mathbf{v}_p, \mathbf{w}_p) = 0 \Rightarrow \mathbf{v}_p = 0$). A manifold with a metric is called a **Riemannian manifold**. We call $\sqrt{g_p(\mathbf{v}_p, \mathbf{v}_p)}$ the **length** of the tangent vector $\mathbf{v}_p$. For later reference, we note that, in a system of local

coordinates, g has the representation $g = g_{ij} dz^i \otimes dz^j$. Here, $g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j)$ is the **metric tensor** of the manifold.

EXERCISE Show that, in polar coordinates (r, θ, ϕ), the standard metric of $\mathbb{R}^3$, $g = dx_1 \quad dx_1 + dx_2 \quad dx_2 + dx_3 \quad dx_3$, assumes the form

$$g = dr \quad dr + r^2 d\theta \quad d\theta + r^2 \sin^2\theta \; d\phi \quad d\phi. \tag{9.31}$$

However, by far most important in practical applications are tensors that are fully antisymmetric in their arguments: a p-**form** ω is a tensor that changes sign under odd permutations of its arguments: $\omega(\mathbf{v}_{S(1)}, \ldots, \mathbf{v}_{S(p)}) = \mathrm{sgn}(S)\omega(\mathbf{v}_1, \ldots, \mathbf{v}_p)$, where S is an element of the permutation group. The space of all p-forms on T is denoted by $\Lambda^p(T)$. Given a p-form ω and a q-form ξ we can produce a $(p+q)$-form $\omega \wedge \xi$ by the following rule:

$$\boxed{(\omega \wedge \xi)(\mathbf{v}_1, \ldots, \mathbf{v}_{p+q}) \equiv \frac{1}{p!q!} \sum_S \mathrm{sgn}(S)\, \omega(\mathbf{v}_{S(1)}, \ldots, \mathbf{v}_{S(p)})\, \xi(\mathbf{v}_{S(p+1)}, \ldots, \mathbf{v}_{S(p+q)}).}$$

The operation $\wedge : \Lambda^p(T) \times \Lambda^q(T) \to \Lambda^{p+q}(T)$ is called the **exterior product** of forms. (In passing we note that $\wedge$ defines the product of the **Grassmann algebra** $\Lambda(T) \equiv \bigoplus_{p=0}^{\infty} \Lambda^p(T)$ which we met already briefly in Section 4.1.)

EXERCISE Verify the following features: (a) $\omega \quad \omega = 0$ if $\omega \in \Lambda^p$ and p is odd, (b) $\omega \quad \xi = (-)^{pq} \xi \quad \omega$, and (c) $\omega \quad (\xi \quad \eta) = (\omega \quad \xi) \quad \eta$, permitting us to write just $\omega \quad \xi \quad \eta$ without brackets.

Given a coordinate function z, each p-form has the unique **coordinate representation**

$$\boxed{\omega = \frac{1}{p!} \omega_{i_1, \cdots, i_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p},} \tag{9.32}$$

where $\omega_{i_1, \ldots, i_p} = \omega(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_p})$. To see this, notice that our definition of the exterior product above implies that $(df_1 \wedge \cdots \wedge df_p)(\mathbf{v}_1, \ldots, \mathbf{v}_p) = \sum_S \mathrm{sgn}(S)\, df_1(\mathbf{v}_{S(1)}) \cdots df_p(\mathbf{v}_{S(p)})$. Also notice that the coefficients $\omega_{i_1, \ldots, i_p}$ are antisymmetric under odd exchange of the indices i_j. This being so, Eq. (9.32) can alternatively be written as

$$\omega = \omega_{I_1, \ldots, I_p} dz^{I_1} \wedge \cdots \wedge dz^{I_p},$$

where, by convention, summations over capitalized indices are ordered, $I_1 < I_2 < \cdots < I_p$.

Above we have seen that, computing the differential of a 0-form $f \in \Lambda^0(T)$, we are led to a 1-form $df \in \Lambda^1(T)$. This principle can be generalized to forms of arbitrary degree: let us define the **exterior derivative** d by setting

$$\boxed{d\omega \equiv \partial_j \omega_{I_1, \ldots, I_p} dz^j \wedge dz^{I_1} \wedge \cdots \wedge dz^{I_p},} \tag{9.33}$$

where the coordinate representation Eq. (9.32) is understood. Obviously, $d\omega \in \Lambda^{p+1}(T)$, i.e. d can be interpreted as an operator on $\Lambda(T)$ that raises the degree of forms by one. However, mathematically inclined people will object that it is unclear whether Eq. (9.33), an equation based on a specific coordinate representation, really represents a proper definition (i.e., is

it clear that the same rule, applied to a different coordinate representation of ω, leads to the same form $d\omega$?). These worries are addressed in the following exercise.

EXERCISE Show that (a) the definition Eq. (9.33) does not depend on the choice of coordinates, (b) $d^2 = 0$, and (c) $d(\omega \quad \xi) = d\omega \quad \xi + (-)^p \omega \quad d\xi$, where p is the degree of ω.

In the following, two subspecies of differential forms will be of particular interest for us: we call a differential form $\omega \in \Lambda^p(T)$ **closed** if $d\omega = 0$. Conversely, ω is called **exact** if $\omega = d\xi$, i.e. if ω is obtained as the exterior derivative of some $(p-1)$-form ξ. Now, owing to the identity $d^2 = 0$, every exact form is closed. Yet not every closed form is exact.

EXERCISE Consider the 1-form ω on S^1 discussed on page 541. Since $\omega(\mathbf{e}) = \omega(\mathbf{e}') = 1$, ω has the local (i.e. restricted to individual charts) representation $\omega = d\phi$ or $\omega = d\phi'$. (The coordinates are those introduced on page 538.) Obviously, ω is closed. On the other hand, we have seen that there is no 0-form (function) f such that $\omega = df$; ω is not exact.

The classification of forms that are closed but not exact is a deep mathematical problem (the subject of **cohomology theory**). We shall return to this issue below when we discuss the geometry of topological terms. However, before doing so, we need to introduce one last concept of basic differential geometry. Previously, we have seen that a mapping $\phi : M \to T$ between two manifolds induced a mapping ϕ_* from the vector fields on M to those on T. In a very similar manner, ϕ gives rise to a mapping ϕ^* between forms on the two manifolds. This so-called **pullback** is defined by

$$\begin{aligned} \phi^* : \Lambda(T) &\to \Lambda(M), \\ \omega &\mapsto \phi^*(\omega) \equiv \omega \circ \phi_*. \end{aligned}$$

For example, for a 1-form $\omega \in \Lambda(T)$ and $\mathbf{v} \in TM$, we have $[\phi^*(\omega)](\mathbf{v}) = \omega(\phi_*(\mathbf{v}))$, etc. Notice that ϕ^* maps in a direction opposite to that of ϕ, hence the name "pullback."

In a system of local coordinates z and w on M and T, respectively, the components of the 1-form $\phi^*(\omega)$ are given by

$$\phi^*(\omega)_i = [\phi^*(\omega)](\mathbf{e}_i) = \omega(\phi_*(\mathbf{e}'_i)) \overset{(9.29)}{=} \omega\left(\frac{\partial \phi^k}{\partial z^i}\mathbf{e}_k\right) = \frac{\partial \phi^k}{\partial z^i}\omega_k,$$

where $\{\mathbf{e}'_i\}$ is a basis of T.

EXERCISE Check that the **coordinate representation of the pullback** of a p-form is given by

$$\phi^*(\omega)_{I_1,\ldots,I_p} = \det\left(\frac{\partial \phi^{J_1}, \ldots, \partial \phi^{J_p}}{\partial z^{I_1}, \ldots, \partial z^{I_p}}\right) \omega_{J_1,\ldots,J_p}.$$

Also verify the useful formula $\phi^*(\omega \quad \xi) = \phi^*\omega \quad \phi^*\xi$, and the commutativity $\phi^* \circ d = d \circ \phi^*$ of a pullback on an exterior derivative. (If you feel exhausted, just verify that $\phi^* d\omega = d(\phi^*\omega)$ holds when applied to forms $\omega \in \Lambda^{0,1}(T)$. The general proof proceeds along similar lines but is a bit more cumbersome.)

Indeed, the pullback of forms is a very important operation since it enables us to define ...

Integration on manifolds

Consider a **top-dimensional form** $\omega \in \Lambda^n(T)$, i.e. a form whose rank is equal to the dimensionality of T. To begin with, let us assume that $\mathrm{supp}(\omega) \subset U$ (i.e. $\omega_{p \notin U} = 0$), where $U \subset T$ is the domain of definition of a chart z. We then define the **integral** of ω over U as

$$\boxed{\int_U \omega \equiv \int_V z^{-1}\omega.} \tag{9.34}$$

To make this definition somewhat less abstract, notice that ω can be represented as

$$\omega = f dz_1 \wedge \cdots \wedge dz_n,$$

where f is a function on T (exercise: why?). The definition above then assumes the form

$$\int_U f(x) dz^1 \wedge \cdots \wedge dz^n = \int_V f(z) dz^1 \wedge \cdots \wedge dz^n \equiv \int_{V,\text{Riemann}} f(z) dz^1 \cdots dz^n.$$

Here, we have used the fact that $z^{-1}(f(x)dz^1 \wedge \cdots \wedge dz^n) = f(x(z))dz^1 \wedge \cdots \wedge dz^n$.[47] In the (crucial) second equality, we declare the integral over the product of forms $dz^1 \wedge \cdots \wedge dz^n$ in $V \subset \mathbb{R}^n$ to be the ordinary Riemann integral. This identification is meaningful because $(dz^1 \wedge \cdots \wedge dz^n)(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ measures the volume of the parallelepiped spanned by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (in the standard metric of $\mathbb{R}^n$), i.e. it represents a sensible "volume element."[48]

To make the definition of the integral complete, we would have to discuss its extension to a **global integral over the manifold**. However, for practical reasons we shall not do so: for one thing, we are often enough dealing with manifolds T that admit a 1-chart atlas, in which case the definition above is sufficient. Only slightly worse is the situation where T can be covered by a single chart *except* for isolated singular points. (This is the case with, e.g., $T = S^1, S^2$ – exercise: why?) Since the integral is oblivious to "sets of measure 0," an integral over the domain of integration of such "nearly complete" charts is as good as an integral over all of T; again, the definition above does the job. In the rare cases where one is dealing with an unpleasant manifold which does not belong to the two species above, one has to work harder and split up the support of the integrand by means of a so-called "partition of unity." However, in view of the relative rarity of such cases, and the fact that they are dealt with in every textbook on differential geometry, we limit attention to the "local" definition above.

[47] Do not be confused by the appearance of the same symbols $dz^1 \wedge \cdots$ on both sides of the equations. On the left-hand side, dz^i is referring to the differential of the ith coordinate function $z^i : T \to \mathbb{R}$. On the right-hand side, $z^i : \mathbb{R}^n \to \mathbb{R}$ simply projects a vector onto its ith component.

[48] However, the discussion above sweeps one subtlety under the carpet: under a change of coordinates, $z \to w$, $\omega = f(z)dz^1 \wedge \cdots \wedge dz^n \to f(z(w)) \det(\partial z/\partial w)\ dw^1 \wedge \cdots \wedge dw^n$. Compatibility with the transition behavior of integration volume elements, $dz^1 \cdots dz^n \to |\det(\partial z/\partial W)|\, dw_1 \cdots dw_n$, requires $\det(\partial z/\partial W) > 0$. Coordinate systems with this property are said to have the same **orientation**. A manifold with an atlas of identically oriented charts is said to be **orientable**. (A prominent counterexample is presented by the Möbius strip.) The definition of the integral above implies that we have chosen a definite orientation.

For future reference, we remark that, if $\phi : T_1 \supset U_1 \to U_2 \subset T_2$ is a diffeomorphism between open subsets U_1 and U_2 of two manifolds T_1 and T_2, then

$$\int_{U_2} \omega = \int_{U_1} \phi^* \omega. \tag{9.35}$$

This is the generalization of the transformation law familiar from calculus to the integration on manifolds. (To understand this equation, notice that, if $z : U_2 \to V$ is a chart of U_2, then $z \circ \phi : U_1 \to V$ is one of U_1. The statement made by Eq. (9.35) then follows from Eq. (9.34).)

With the phalanx of definitions above, we are now – at last! – in a position to discuss the utility of differential forms in topological field theory.

9.4.2 From θ- to Wess–Zumino terms

In Section 9.3, we have seen that, more often than not, the topological phase associated with non-vanishing "winding numbers" could be given a representation in terms of an action S_{top}. However, no guidelines as to existence or non-existence of such representations have been given. We begin by discussing some principles behind the formulation of topological Lagrangian densities. This will bring us to a position where the connection to Wess–Zumino terms can be established.

The geometry of θ-terms

Let us first observe that most target manifolds T of topologically non-trivial field theories are Riemannian, i.e. come with a natural metric g. This may be the case because $T \subset \mathbb{R}^m$ is embedded into some $\mathbb{R}^m$ and inherits the natural metric of the latter (e.g. $S^2 \subset \mathbb{R}^3$, etc.); or because $T = G/H$ is a coset space of Lie groups[49] (a situation characteristic of problems with spontaneous symmetry breaking); or, indeed, for some other reason.

Most important is the fact that a metric g entails the existence of a canonical top-dimensional form ω on T. Here, the attribute "canonical" means the following: locally, each top-dimensional form can be represented as $\omega = f(z) dz^1 \wedge \cdots \wedge dz^n$, where $z^i, i = 1, \ldots, n$, are coordinate functions, n is the dimension of T, and f is some function. In general, the form of f depends on the choice of coordinates and cannot be globally specified. However, on a Riemannian manifold, a canonical n-form with coordinate-invariant definition exists. To see this, let $g_{ij} dz^i \otimes dz^j$ be the metric. Further, let us define

$$\omega = \sqrt{g} dz^1 \wedge \cdots \wedge dz^n, \tag{9.36}$$

where $g = \det\{g_{ij}\}$ is the determinant of the metric tensor. What makes ω special is that it has the same representation Eq. (9.36) in every coordinate system.[50]

[49] Remember that a Lie group is a manifold with the additional structure of a group. As shown in textbooks of group theory, the group structure induces a metric.

[50] To see this, recall that, under a change of coordinates $z \to W$, $dz^1 \wedge \cdots \wedge dz^n \to \det A^{-1}\, dw^1 \wedge \cdots \wedge dw^n$, where $A = (\partial w / \partial z)$. At the same time, $g_{ij} \to (A^T g A)_{ij}$, i.e. $g^{1/2} \to \det A g^{1/2}$. The two determinants cancel each other, so that ω remains form invariant.

The n-form ω is called the **volume element** of the manifold[51] and $V_T \equiv \int_T \omega$ is the volume of the (compact) Riemannian manifold T. Without loss of generality, we can set our unit of length so that $V_T = 1$.

EXERCISE For $d = 3$ and the metric Eq. (9.31), show that ω assumes the familiar form $\omega = \frac{1}{4\pi} r^2 \sin\theta \, dr \quad d\theta \quad d\phi$ proportional to the three-dimensional volume element in polar coordinates.

We now claim that, for base and field manifolds of equal dimensionality $\dim M = \dim T = d$,

$$\boxed{S_{\text{top}}[\phi] = i\theta \int_M \phi^* \omega,} \tag{9.37}$$

defines a **coordinate invariant representation of the -term**. To get warmed up to this abstract representation, we first note that $\phi^*\omega$ is a top-dimensional form on M, i.e. we really have something to integrate and the notation makes sense. Now, let us consider a trivial field configuration $\phi_0(x \in M) = \text{const}$. In this case, $\phi_0^*\omega = 0$ (exercise: why?) and $S_{\text{top}}[\phi_0] = 0$, as one should expect. Next, let us assume that $\phi = \phi_1$ is a diffeomorphic (1-1 and differentiable) covering of T. (The presumed existence of such a mapping amounts to the statement that we are dealing with a topologically non-trivial field theory.) In this case, the transformation law Eq. (9.35) holds and we get

$$S_{\text{top}}[\phi_1] = i\theta \int_M \phi_1^* \omega \overset{(9.35)}{=} i\theta \int_T \omega = 1 \times i\theta.$$

Now consider a mapping ϕ_n that covers T W times (i.e. every point $p \in T$ is the image of W points $x_{1,\ldots,W} \in M$). One can show that $S_{\text{top}}[\phi_W] = iW\theta$, i.e. S_{top} indeed counts the winding number of fields ϕ. Rather than giving the proof of this statement for general W, let us consider a simple example. Let $M = T = S^2$ and $\phi_W(\phi, \theta) = (W\phi, \theta)$.[52] Then, $\omega = \sin\theta \; d\theta \wedge d\phi$, and (check!) $\phi_W^*\omega = W \sin\theta \; d\theta \wedge d\phi$. We thus obtain $V_{S_2} = 4\pi$ and $i\theta \int_M \phi_W^*\omega = iW\theta$, in agreement with the general rule.

EXERCISE Let $n_i : S^2 \to \mathbb{R}, i = 1, 2, 3$, be the ith component of the unit vector defining a point $p \in S^2$. We consider the 2-form $\omega = \mathbf{n} \cdot (d\mathbf{n} \quad d\mathbf{n}) \equiv \epsilon^{ijk} n_i dn_j \quad dn_k$. Using the standard polar coordinate representation $n_i = n_i(\theta, \phi)$, show that ω can alternatively be represented as $\omega = \sin\theta \; d\theta \quad d\phi$, which is the familiar volume element on the sphere (upon choosing an orientation so that $z^{-1*}(d\theta \quad d\phi) = d\theta \, d\phi$). Considering a field $\mathbf{n} : (x, y) \mapsto \mathbf{n}(x, y)$ show that

$$\int_M \mathbf{n}^* \omega = \int_M \mathbf{n} \cdot (\partial_x \mathbf{n} \times \partial_y \mathbf{n}) dx \quad dy, \tag{9.38}$$

which we identify (again, for a definite orientation) with our earlier representation of the θ-term of a field theory with S^2-valued fields.

[51] To motivate this terminology, consider a basis in which the metric tensor $g_{ij} = g_i \delta_{ij}$ is diagonal. The volume spanned by the (mutually orthogonal) tangent vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is then simply given by the product of their lengths, i.e. by $\prod_{i=1}^{N} \sqrt{g}_i = \sqrt{g}$. This is precisely what we get when we evaluate $\omega(\mathbf{e}_1, \ldots, \mathbf{e}_n)$, i.e. the form ω measures the volume of the domain spanned by its arguments.

[52] Here we have adopted the usual abuse of notation; in principle we should write $(z \circ \phi_W \circ z'^{-1})(\phi, \theta) = (W\phi, \theta)$, where z and z' are polar coordinate charts on M and T respectively.

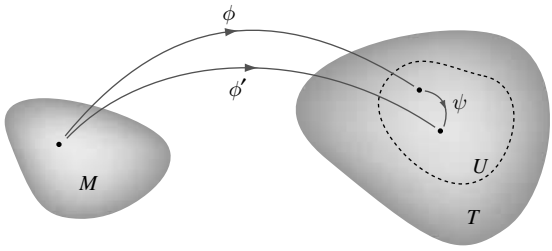


Figure 9.12 On the insensitivity of the integral representation Eq. (9.37) under field variations. For a discussion, see the main text.

The discussion above shows that, when evaluated on certain reference configurations ϕ_W, the integral Eq. (9.37) yields the winding number W. To complete the identification with our earlier representations of the θ-term, we have to show that Eq. (9.37) does not change under continuous distortions of ϕ_W (i.e. that it responds to the topological sector, and nothing else). Readers not content with our assertion that this is the case may wish to navigate through the following argument (which, as a byproduct, nicely illustrates the power of geometric methods in topological field theory).

INFO To show the invariance of Eq. (9.37) under continuous field deformations, let us consider two field configurations ϕ and ϕ' which can be continuously deformed into each other. More specifically, we set $\phi' = \psi \circ \phi$, where $\psi : T \to T$ is different from unity only inside the domain $U \supset T$ of some chart, see Fig. 9.12. (This is no serious restriction as, by iterative deformations of this type, any field configuration continuously deformable into ϕ can be reached.)

We then have

$$S_{\rm top}[\phi] - S_{\rm top}[\phi'] = i\theta \int_M (\phi^*\omega - (\psi\circ\phi)^*\omega) = i\theta \int_M \phi^*(\omega - \psi^*\omega).$$

Now $\omega - \psi^*\omega \in \Lambda^d(T)$ is a d-form on T, different from unity only locally (inside U). This implies its exactness, i.e. the existence of a representation $\omega - \psi^*\omega = d\kappa$, where $\kappa \in \Lambda^{d-1}(T)$.[53] We thus find

$$S_{\rm top}[\phi] - S_{\rm top}[\phi'] = i\theta \int_M \phi^* d\kappa = i\theta \int_M d(\phi^*\kappa) = i\theta \int_{\partial M} \phi^*\kappa = 0,$$

where, in the second equality, we have used the general commutativity of the exterior derivative and pullback and, in the third equality, Stokes' theorem (for a reminder, see below). In the fourth equality we have assumed that, on the boundary ∂M of the base manifold (physically, infinity), the fields $\phi|_{\partial M}$ approach a constant, so that $\phi^*\kappa = 0$.

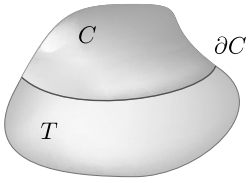
The representation Eq. (9.37) sheds some light on the structure of θ-terms on Riemannian[54] field manifolds. However, this was, of course, not the only reason for maneuvering

[53] The reason is that, locally, $\omega - \psi^*\omega$ has a coordinate representation $\omega - \psi^*\omega = f dz_1 \wedge \cdots \wedge dz_n$. With the *ansatz* $\kappa = g dz_2 \wedge \cdots \wedge dz_n$, the equation $d\kappa = \omega - \psi^*\omega$ reduces to the differential equation $\partial_1 g = f$ (if the special role ascribed to the coordinate direction 1 is irritating, notice that κ is not uniquely defined; every $\kappa' = \kappa + \eta$ with a closed ($d\eta = 0$) form η will do the job just as well), defined on some open interval of x_1 (for fixed $x_2, \ldots, x_n$). This is an ordinary differential equation which can be solved.

[54] In fact, all we need to formulate the construction above is a canonical top-dimensional form, i.e. none of our arguments above actually relied on the fact that, in Riemannian geometry, this form happens to be the

through the geometric constructions above. Our prime reason for introducing the invariant formulation is that it contains the key to understanding the connection to Wess–Zumino terms.

The geometry of Wess–Zumino terms



In this section, we discuss the general geometric principle common to all Wess–Zumino (WZ) terms. We then explore how our previous sporadic encounters with WZ field theories fit into this scheme and discuss a few more applications. Note that θ-terms appear in theories whose field manifold and base manifold are of equal dimensionality: $\dim M = \dim T$. In contrast, WZ terms are at home in field theories with $\dim M = \dim T - 1$.[55] How can a topological term be constructed that relies on this dimensional relation? As we shall see in a moment, the key to the answer lies in **Stokes' theorem**:

$$\boxed{\int_C d\omega = \int_{\partial C} \omega.} \tag{9.39}$$

Here, $C \subset M$ is a benign (smooth, orientable, etc.[56]) subset of a differentiable manifold M, ∂C is its boundary, and $d\omega \in \Lambda^n(M)$ is top-dimensional on M (see figure).

INFO The proof of Eq. (9.39) is beyond the scope of our present discussion and we have to refer to textbooks on differential geometry. Nonetheless, a few **remarks for readers not familiar with Stokes theorem** may be helpful. First note that ∂C is a manifold by itself, with $\dim \partial C = n-1$. Thus, ω is top-dimensional on ∂C and can be integrated. To gain some insight into the substance of Eq. (9.39) let us consider a few special cases. For example, let M be a three-dimensional manifold, $n = 3$. Consider the two-form $\omega = v_1 dx_2 \quad dx_3 + v_2 dx_3 \quad dx_1 + v_3 dx_1 \quad dx_2$ and interpret the coefficients $v_i = v_i(\mathbf{x})$ as the components of a three-dimensional vector field $\mathbf{v} = (v_1, v_2, v_3)$. With $d\omega = \operatorname{div} \mathbf{v}\, dx_1 \quad dx_2 \quad dx_3 \equiv \operatorname{div} \mathbf{v}\, dV$ and the identification $\omega = \mathbf{v} \cdot dS$, where dS is the two-dimensional "surface element," we obtain the standard representation of **Gauss law**

$$\int_M dV \operatorname{div} \mathbf{v} = \int_{\partial M} dS \cdot \mathbf{v}.$$

Similarly, let M be some two-dimensional surface, $n = 2$, and $\omega = v_1 dx_1 + v_2 dx_2 + v_3 dx_3$. Identifying the surface element as above, we then have $d\omega = \operatorname{curl} \mathbf{v} \cdot dS$ and

$$\int_M dS \cdot \operatorname{curl} \mathbf{v} = \int_{\partial M} d\mathbf{s} \cdot \mathbf{v},$$

where $\mathbf{v} \cdot d\mathbf{s} \equiv \omega$, $d\mathbf{s}$ is commonly known as a "line element," and $\int_{\partial M}$ is a line integral: this is called **Ampère s law**.

volume element. For example, some field theories with $\dim(T) = 2$ live on manifolds with a so-called symplectic structure (which is nothing but a skew-symmetric, non-degenerate 2-form). This form may then take over the role of our ω above and, in perfect analogy to our discussion above, gives rise to a topological Lagrangian density.

[55] More generally, WZ terms can be constructed if, in dimensions $\dim M + 1$, the manifold T possesses a closed differential form. (In the special case $\dim M + 1 = \dim T$, this form will be the volume form of T.)

[56] More precisely, C must be a so-called **chain** on M. However, for brevity, we shall keep the sloppy characterization above.

So suppose that we are dealing with a field theory defined on a d-dimensional base manifold M with $(d+1)$-dimensional target manifold T. Without (too much) loss of generality, we may think of $T = S^{d+1}$ and $M = S^d$ as unit spheres.

INFO Although we have seen that spheres frequently appear as target spaces in field theory, the statement above may seem to be too restrictive – for example, what about the many field theories with group-valued target manifolds? The rationale behind emphasizing spherical target manifolds lies in an important statement of homotopy theory: the integration spaces we encounter in field theories with continuous symmetries are usually embedded in some sufficiently large $\mathrm{U}(N)$. However, it can be shown that the topological structure of the group $\mathrm{U}(N \geq 2)$ essentially reduces to that of its subgroup $\mathrm{SU}(2) \subset \mathrm{U}(N)$, i.e. mappings into $\mathrm{U}(N)$ can be continuously deformed into mappings into $\mathrm{SU}(2)$. (If you find this difficult to visualize, consider a two-dimensional plane with a circular hole as an example. Any curve in the plane can be continuously deformed to a curve on the circle surrounding the hole. In this sense, the circle preempts the topological content of the plane. Similar reduction mechanisms exist for higher-dimensional manifolds.) However, $\mathrm{SU}(2) \simeq S^3$ is isomorphic to the 3-sphere. (Representing $\mathrm{SU}(2)$ as the group of matrices $g = x_0\sigma_0 + i\sum_{j=1}^3 x_j\sigma_j$, $x_\mu \in \mathbb{R}$, the condition $\det(g) = 1$ boils down to the relation $\sum_{\mu=0}^3 x_\mu^2 = 0$.) This argument shows that, as far as topology is concerned, the spaces $S^{1,2,3}$ basically exhaust the list of relevant target spaces.

As to the base manifold, the identification $M = S^d$ is motivated by the compactification scheme discussed in Section 9.3 above.

Notice that the target manifold $T = S^{d+1}$ is Riemannian and, therefore, comes with a canonical $(d+1)$-form ω.

Given these prerequisites, our strategy will be to utilize, as much as possible, our previous understanding of the geometry of the θ-term. To this end, let us interpret $\phi(M)$ as a d-dimensional submanifold in T. Further, let $\Gamma_+ \quad \phi(M)$ be a subset of T on which the T volume form $\omega = d\kappa_+$ can be represented as the exterior derivative of some d-form κ_+.

EXAMPLE In fact, we may even assume that $\Gamma_+ = S^{d+1} - \{p\}$ covers all of our sphere except for one point p. For example, on the 2-sphere S^2, the volume form $\omega = \sin\theta\, d\theta \quad d\phi$ is obtained from $\kappa_+ = (1-\cos\theta)d\phi$ as $\omega = d\kappa_+$ everywhere except for the south pole. Similarly, with $\kappa_- = -(1+\cos\theta)d\phi$, $\omega = d\kappa_-$ everywhere except for the north pole. Of course, the choice of the exclusion point is quite arbitrary. Notice, however, that there is no global representation $\omega = d\kappa$. If such a representation existed, $\mathrm{Vol}(T) = \int_T \omega = \int_T d\kappa = \int_{\partial T} \kappa = 0$, because T is boundaryless, i.e. the volume of the sphere would vanish.

We now dimensionally extend the field $\phi : M = S^d \to \Gamma_N \subset S^{d+1} = T$ to a mapping $\tilde{\phi} : S^{d+1}_{\mathrm{N}} \to S^{d+1}$ defined on the entire northern hemisphere, S^{d+1}_{N}, of S^{d+1}. This is achieved in a series of steps: firstly, identify $M = S^d$ as the equator of S^{d+1} (see Fig. 9.13). We then introduce a $(d+1)$-dimensional coordinate representation (s, x), $s \in [0,1]$, of S^{d+1}_{N}. These coordinates are defined in such a way that $(s = 1, x)$ parameterizes the equator $\simeq S^d = M$ while $\lim_{s\to 0}(s,x) = p_N$ is the north pole; otherwise, their choice is arbitrary. We finally extend our field to a mapping $\tilde{\phi} : S^{d+1}_{\mathrm{N}} \to \Gamma_N, (s,x) \to \tilde{\phi}(s,x)$. Apart from the obvious consistency condition $\tilde{\phi}(s=1,x) = \phi(x)$, the choice of this extension is, again, arbitrary.

Given this setup, let us define

$$\boxed{S_{\rm WZ}[\phi] = iC \int_{S^{d+1}_{\rm N}} \tilde{\phi}^* \omega,} \tag{9.40}$$

as a trial candidate of a topological action ($C = \text{const.}$). Constructed in manifest analogy to the θ-term Eq. (9.37), this expression is "topological" in nature (it is scale and reparameterization invariant, etc.). Nonetheless, the definition does not look quite trustworthy. Notably, we have written "$S_{\rm WZ}[\phi]$" while the right-hand side of the definition involves the extension $\tilde{\phi}$. To see that the integral is, in fact, independent of the particular choice of the extension, we use the fact that, on $\tilde{\phi}(S^{d+1}_{\rm N}) \subset \Gamma_N$, the volume form can be written as $\omega = d\kappa_N$. This implies

$$S_{\rm WZ}[\phi] = iC \int_{S^{d+1}_{\rm N}} \tilde{\phi}^* d\kappa_N = iC \int_{S^{d+1}_{\rm N}} d(\tilde{\phi}^* \kappa_{\rm N}) = iC \int_{M=S^d} \tilde{\phi}^* \kappa_{\rm N} = iC \int_M \phi^* \kappa_{\rm N}.$$

Here we have made use of (i), in the second equality, the commutativity of the pullback and exterior derivative, (ii) in the third equality, Stokes' theorem, and, (iii) in the crucial fourth equality, that the integral over $\partial S^{d+1}_{\rm N} = S^d = M$ depends only on the value of the boundary field $\tilde{\phi}(s = 1, x) = \phi(x)$. This proves the independence of the action $S_{\rm WZ}$ on the extension scheme. At the same time, we have obtained the alternative representation

$$\boxed{S_{\rm WZ}[\phi] = iC \int_M \phi^* \kappa_N,} \tag{9.41}$$

of the topological action. This form makes the independence of the action $S_{\rm WZ}$ of the field extension manifest. For this, however, a price has had to be paid: Eq. (9.41) involves the d-form $\kappa_{\rm N}$ which we saw is tied to a certain coordinate representation (owing to the absence of a global representation $\omega = d\kappa$) on the field manifold. In other words, the extension-independent representation Eq. (9.41) necessarily involves the choice of a specific coordinate system of the field.

INFO As an example, consider $M = S^1$ and $T = S^2$. Think of the latter as the space of unit-vectors $\mathbf{n}$. Our field is a mapping $\mathbf{n} : S^1 \to S^2, t \to \mathbf{n}(t)$, where $t \in [0, 1]$ and periodic boundary

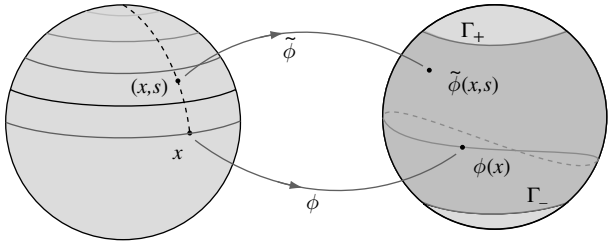


Figure 9.13 On the extension of a field theory defined on a d-dimensional base manifold $M \simeq S^d$ to a $(d + 1)$-dimensional theory.

conditions $\mathbf{n}(0) = \mathbf{n}(1)$ are understood. Extension to a field $\mathbf{n}(s,t)$ then allows us to represent the action $S_{\rm WZ}[\mathbf{n}]$ as (see Eq. (9.38))

$$S_{\rm WZ}[\mathbf{n}] = i\frac{C}{4\pi}\int_0^1 ds \int_0^1 dt\, \mathbf{n}\cdot(\partial_s\mathbf{n}\times\partial_t\mathbf{n}).$$

Now, except for the south pole, we can write the volume (or better to say area) form on S^2 as $\omega = d[(1-\cos\theta)d\phi]$. Then, using the fact that $\mathbf{n}^*[(1-\cos\theta)d\phi] = (1-\cos\theta(t))d\phi(t) = (1-\cos\theta(t))\partial_t\phi(t)dt$, we obtain the alternative representation

$$S_{\rm WZ}[\mathbf{n}] = i\frac{C}{4\pi}\int_0^1 dt\,(1-\cos\theta(t))\partial_t\phi(t). \tag{9.42}$$

Notice that this form explicitly uses the coordinate representation $\mathbf{n} \quad (\phi,\theta)$.

Equations (9.40) and (9.41) are two different representations of the **Wess–Zumino action**. As discussed above, both have their advantages and disadvantages – the need to artificially extend the field vs. lack of representation invariance – which is why they are used interchangeably in the literature.

Finally, we need to discuss one further important point; the value of the coupling constant C. As discussed above, the coupling constant of the θ-term, the topological angle, was quite arbitrary. This is not the case with the WZ term. In fact, we shall see that the constant C is subject to quantization conditions. To understand why, recall that above we have chosen the northern hemisphere $S^{d+1}_{\rm N}$ as the domain of integration of our extended field theory. Of course, the southern hemisphere $S^{d+1}_{\rm S}$ would have provided just as good a choice. In this case, we would have defined

$$S'_{\rm WZ}[\phi] \equiv -iC\int_{S^{d+1}_{\rm S}}\tilde{\phi}^*\omega,$$

where $\tilde{\phi}$ is a field extension to the southern hemisphere, and the extra minus sign takes care of the fact that the equator is the boundary of the oppositely oriented southern domain. Of course, our definition of the WZ action would be senseless if this ambiguity mattered. Therefore,

$$S_{\rm WZ}[\phi] - S'_{\rm WZ}[\phi] = iC\int_{S^{d+1}_{\rm S}}\tilde{\phi}^*\omega + iC\int_{S^{d+1}_{\rm N}}\tilde{\phi}^*\omega = iC\int_{S^{d+1}}\tilde{\phi}^*\omega = iCW,$$

where W is integer. At first sight, the non-vanishing of this expression looks worrysome. However, since the action appears in the exponent no harm is done as long as the difference is equal to (i times) an integer multiple of 2π, i.e. the the coupling constant C obeys the **quantization condition**[57]

$$\boxed{C = 2\pi k, \quad k\in\mathbb{Z}.} \tag{9.43}$$

The number k is called the **level** of the WZ theory. To summarize the main results of this section, we have found that:

[57] Recall that the volume form ω underlying the construction has been normalized in such a way that $\int_{S^{d+1}}\omega = 1$. For different choices of the normalization, the value of C will change accordingly.

▷ The WZ action is closely allied to the θ-term. It affords two different representations where:
▷ Equation (9.40) involves a dimensional extension of the field, while
▷ Equation (9.41) relies on an explicit coordinate representation of the field.
▷ The coupling constant of the WZ term is quantized according to Eq. (9.43).

Having discussed the structure and geometry of WZ theories, we now return to physics. Using a simple prototype system as an example, we begin by exploring how WZ terms enter low-energy theories of many-body quantum systems.

9.4.3 Example: magnetic moment coupled to fermions

Consider a single energy level ϵ of a spinful fermion system. (One may think, for example, of a discrete level of an atom.) Let us assume that fermions inhabiting the level are coupled to a classical magnetic moment $\mathbf{n}$. The coherent state action of this system is given by

$$S[\psi, \mathbf{n}] = \int_0^\beta d\tau\, \bar{\psi}(\partial_\tau + \xi + \gamma \mathbf{n} \cdot \sigma)\psi,$$

where γ is a coupling constant and, as usual, $\xi = \epsilon - \mu$. A complete specification of the problem would have to include a term $S[\mathbf{n}]$ controlling the dynamics of the uncoupled magnetic moment. However, for the purposes of the present discussion, it is sufficient to consider the moment–fermion coupling in isolation.

INFO Actions of this type appear as building blocks of larger systems. For example, once generalized to a *set* of levels ϵ_a, our model might describe a system of atomic shell electrons subject to **Hund s rule** coupling to a spin or orbital magnetic moment. Alternatively, the magnetic moment $\mathbf{n}$ might describe the Hubbard–Stratonovich decoupling of some electron–electron interaction in a spinful channel.

Integration over the fermion degrees of freedom brings us to the reduced action

$$S[\mathbf{n}] = -\mathrm{tr} \ln(\partial_\tau + \xi + \gamma \mathbf{n} \cdot \sigma).$$

To proceed, let us write $\mathbf{n} = R\mathbf{e}_3$, where $R \in \mathrm{SO}(3)$ is a rotation matrix, and then use the fact that a matrix $U \in \mathrm{SU}(2)$ can be found[58] such that

$$\mathbf{n} \cdot \sigma = (R\mathbf{e}_3) \cdot \sigma = U \sigma_3 U^{-1}. \tag{9.44}$$

For example, it is straightforward to verify that, with the standard polar representation, $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T$, the choice

$$U = e^{-i\frac{\phi}{2}\sigma_3} e^{-i\frac{\theta}{2}\sigma_1} e^{-i\frac{\theta}{2}\sigma_3}, \tag{9.45}$$

[58] This is, in fact, just a concrete realization of the **correspondence between SU(2) and SO(3)** familiar from group theory.

will suffice. However, for the moment we shall not need an explicit representation of the rotation matrix. Rather, substituting the general expression Eq. (9.44) into the action, we obtain

$$\begin{aligned} S[\mathbf{n}] \to S[U] &= -\mathrm{tr}\ln(\partial_\tau + \xi + \gamma U\sigma_3 U^{-1}) = -\mathrm{tr}\ln(U^{-1}\partial_\tau U + \xi + \gamma\sigma_3) \\ &= -\mathrm{tr}\ln(\partial_\tau + \xi + \gamma\sigma_3 + U^{-1}\dot{U}), \end{aligned}$$

where, in the first equality, we have used the cyclic invariance of the trace and, in the last equality, we have defined $\partial_\tau U \equiv \dot{U}$. To proceed, we assume that the two energies $|\xi \pm \gamma| \gg T$ are well separated from the chemical potential, $|\epsilon \pm \gamma| \gg T$. We also assume that the frequency scales $\tilde{\omega}$ on which the fields $\mathbf{n} \leftrightarrow U$ vary are so slow that they do not change the occupation of the fermionic levels: $\tilde{\omega}/\gamma \ll 1$. In these circumstances, we may proceed by a straightforward expansion of the logarithm: $S[U] = \mathrm{tr}(\hat{G}\hat{U}^{-1}\dot{U}) + \mathcal{O}(\tilde{\omega}/\gamma)^2$, where $\hat{G} = (-\partial_\tau - \xi - \gamma\sigma_3)^{-1}$ and we have anticipated (check!) that higher-order terms of the expansion will be small in the parameter $\tilde{\omega}/\gamma$. Switching to a frequency representation,

$$\begin{aligned} S[U] &= \sum_n \mathrm{tr}(\hat{G}_n(\hat{U}^{-1}\dot{U})_{m=0}) = -\int d\tau\, \mathrm{tr}(n_{\mathrm{F}}(\xi + \gamma\sigma_3)U^{-1}\partial_\tau U) \\ &= -\int d\tau\, \mathrm{tr}\left(\left[n_{\mathrm{F}}(\xi + \gamma\sigma_3) - \frac{1}{2}\right] U^{-1}\partial_\tau U\right), \end{aligned}$$

where we have used the fact that the frequency sum of a fermionic Green function introduces the Fermi distribution function, n_{F}. In the last line, we have shifted n_{F} by $1/2$, which is permissible because $\mathrm{tr}(U^{-1}\dot{U})$ is a boundary term:

$$\int_0^\beta d\tau\, \mathrm{tr}(U^{-1}\dot{U}) = \int_0^\beta d\tau\ \partial_\tau\, \mathrm{tr}(\ln U) = \mathrm{tr}\ln(U)|_0^\beta = 0. \tag{9.46}$$

Now, if both levels are either occupied ($n_{\mathrm{F}}(\xi \pm \gamma) - 1/2 \approx 1/2$) or unoccupied ($n_{\mathrm{F}}(\xi \pm \gamma) - 1/2 \approx -1/2$), the action vanishes[59] since $\mathrm{tr}(\left[n_{\mathrm{F}}(\xi + \gamma\sigma_3) - \frac{1}{2}\right] U^{-1}\partial_\tau U) \propto \mathrm{tr}(U^{-1}\partial_\tau U)$ is again a boundary term.

Assuming, however, that $\xi + \gamma$ is empty while $\xi - \gamma$ is occupied we obtain

$$S[U]|_{\xi+\gamma \gg 0 \gg \xi-\gamma} \equiv S_{\mathrm{WZ}}[U] = -\frac{1}{2}\int_0^\beta d\tau\, \mathrm{tr}(\sigma_3 U^{-1}\dot{U}). \tag{9.47}$$

We now run a few tests on S_{WZ} to identify it as a topological term of Wess–Zumino type. For one thing, our base manifold $M \simeq S^1$ is one-dimensional and S_{WZ} involves a first-order derivative. This exemplifies the scale invariance characteristic of topological terms. The suspicion that S_{WZ} is topological is corroborated by the fact that it is purely imaginary. (The time-derivative acts on a unitary matrix, i.e. something like a generalized "phase" – for more details, see below.) But what type of topological term are we dealing with? To get an answer to this question, we first need to determine the dimensionality of the target manifold T. Clearly, $T \neq \mathrm{SU}(2)$ in spite of the fact that we are temporarily using $U \in \mathrm{SU}(2)$ for our

[59] Up to corrections of $\mathcal{O}(\exp(-|\xi \pm \gamma|/T))$ which, in any case, are beyond the scope of the first-order expansion of the "tr ln" above.

fields: our original theory was defined for $\mathbf{n} \in S^2$. Indeed, the representation Eq. (9.44) is invariant under the "gauge transformation" $U \to U \exp(i\psi\sigma_3)$. This means that the "true" field manifold is $\mathrm{SU}(2)/\mathrm{U}(1) \simeq S^2$ as one should expect. Now, $\dim S^2 = 2 = \dim M + 1$, so that S_{WZ} is likely to be a Wess–Zumino term.

Indeed, we observe that, under the gauge transformation above, $\mathrm{tr}(\sigma_3 U^{-1}\dot{U}) \to \mathrm{tr}(\sigma_3 U^{-1}\dot{U}) + 2i\dot{\psi}$, i.e. the topological density is not invariant. This indicates that we will not be able to find a coordinate-independent representation of S_{WZ} in terms of $\mathbf{n}$. As discussed in the previous chapter, the lack of parameterization invariance is a hallmark of WZ terms. (There is no need to worry, though, about the gauge invariance of the *theory*: the periodic boundary conditions imposed on U imply that $\psi(\beta) = \psi(0) + 2\pi n$, where n is some winding number. Under the gauge transformation, the action thus changes by $\delta S = -2\pi i n$, so that $\exp(-\delta S) = 1$ remains invariant.)

To present S_{WZ} in a less abstract form, we can substitute the angular representation Eq. (9.45) into Eq. (9.47). The straightforward evaluation of the derivative then gives

$$S_{\mathrm{WZ}}[\phi, \theta] = -\frac{i}{2} \int d\tau \; (1 - \cos\theta)\partial_\tau \phi. \tag{9.48}$$

We identify this expression as the coordinate representation Eq. (9.42) of the WZ term arising in theories with base S^1 and target S^2. Recall that S_{WZ} evaluates to $-i/2$ times the area swept out by the closed curve (ϕ, θ) on the sphere.

In fact, we have already met with the action Eq. (9.48) in Section 3.3 when we discussed the path integral of a single spin. That the core contribution to the action of that problem reappears here should not come as too much of a surprise: the classical magnetic moment $\propto \mathbf{n}$ "enslaves" the electron spin. In the limit of perfect alignment – realized under the conditions assumed above – we are left with the dynamics of a quantum spin whose dynamics are tied to that of $\mathbf{n}$. The resulting action therefore coincides with that of a free spin described by the unit vector $\mathbf{n}$.

In the literature, the action Eq. (9.48) is frequently referred to as the **Berry phase action**. To understand the rationale behind this terminology, let us briefly recall (see the Info block below) a few facts about the quantum mechanical Berry phase. Consider a quantum particle subject to the Hamiltonian $\hat{H}(x(t))$, where the D-component vector $x(t) \equiv \{x_i(t)\}$ parameterizes a weakly time-dependent contribution to $\hat{H}$. As shown by Berry, the dynamical phase acquired during the evolution of the particle can be expressed as $\exp(-i \int dt' \; \epsilon_0(t') + i\gamma(t))$, where $\epsilon_0(t)$ is the energy of the instantaneous ground state of $\hat{H}(t)$. The first contribution to the exponent is the usual dynamical phase of quantum evolution. (Here, it is assumed that the time-dependence of $\hat{H}$ is sufficiently weak for us to neglect transitions into excited levels; for a static Hamiltonian, this contribution reduces to

Sir Michael Berry 1941–

Theoretical physicist who has made groundbreaking contributions to the field of quantum nonlinear dynamics and optics. Berry introduced the concept of the Berry phase (or geometric phase as he himself prefers to call it) and explored its manifestations in various physical contexts. (Figure courtesy of Sir Michael Berry.)

the familiar phase $\epsilon_0 t$.) The second contribution, $\gamma(t)$, is of geometric origin, inasmuch as it depends on the path traced out by the vector x in parameter space, but not on dynamical details. (In particular, it is independent of the velocity at which the parameter path is traversed.) The **geometric phase** assumes a particularly simple form if the parameter dependence of $\hat{H}$ is periodic in time, $x(0) = x(t)$. In this case, $x(t)$ defines a closed curve and γ can be expressed as an integral over any surface in parameter space bounded by that curve. Specifically, for a spin S particle Zeeman-coupled to a weakly time-dependent magnetic field $\mathbf{B}(t) \equiv \mathbf{n}(t)B$ of constant magnitude, B,

$$\gamma = \frac{S}{2} \int d^2x \ \mathbf{n} \cdot \partial_1 \mathbf{n} \times \partial_2 \mathbf{n}, \tag{9.49}$$

is the area bounded by the curve $\mathbf{n}(t)$ on the 2-sphere (which, in this case, defines the parameter space). Comparison with Eq. (9.48) and the example discussed on page 538 indeed identifies the WZ-term as the Berry phase of the spin problem.[60]

INFO A brief reminder of the **Berry phase in quantum mechanics**: Consider a particle governed by a Hamiltonian $\hat{H}(x(t))$. It is assumed that the time-dependence of the parameter vector x is adiabatic, which means that a particle initially prepared in the ground state $|0(t=0)\rangle$ of the Hamiltonian $\hat{H}(x(0))$ will remain in the instantaneous ground state $|0(t)\rangle$ (i.e. the lowest eigenstate $\hat{H}(x(t))|0(t)\rangle = \epsilon(t)|0(t)\rangle$ of the operator $\hat{H}(x(t))$) throughout the entire observation time. (In practice, this means that the rate ω at which the parameters change in time must be much smaller than the excitation gap of the system.)

We are interested in computing the dynamical phase corresponding to the time evolution of $|0(t)\rangle$. To this end, let us represent the wavefunction of the particle as $|\psi(t)\rangle = e^{-i\phi(t)}|0(t)\rangle$ and consider the time-dependent Schrödinger equation

$$\hat{H}(x(t))|\psi(t)\rangle = i\partial_t|\psi(t)\rangle.$$

Substitution of the representation above and multiplication by $\langle 0(t)|$ then leads to the equation $\partial_t\phi = \epsilon(t) - i\langle 0(t)|\partial_t|0(t)\rangle$. Integrating over time and comparing with our discussion above, we are led to the identification $\gamma(t) = i\int_0^t dt' \ \langle 0(t')|\partial_{t'}|0(t')\rangle$ of the Berry phase (exercise: why is γ real?). Now, the instantaneous ground state inherits its time dependence from the parameters $x(t)$. We may thus write

$$\gamma(t) = i\int_0^t dt' \ \langle 0(x(t'))|\partial_{x_i(t')}|0(x(t'))\rangle \partial_t x_i(t') = i\int_{\mathrm{c}} dx \ \langle 0(x)|\partial_x|0(x)\rangle = i\int_{\mathrm{c}} \langle 0|d0\rangle.$$

Here, the second integral has to be interpreted as a line integral in parameter space. It is taken along a curve c which starts at $x(0)$, follows the evolution of the parameter vector, and ends at $x(t)$. Importantly, the line integral depends only on the choice of γ but not on the velocity at which this curve is traversed (the dynamics of the process). In this sense, we are dealing with a phase of geometric origin. The third integral representation above emphasizes the geometric

[60] In the discussion above, we considered a particle coupled to a weakly time-dependent magnetic moment – a canonical setup for the appearance of Berry phases. But why did the Berry phase/WZ-action also appear in our earlier discussion of a spin coupled to a fixed magnetic field? To answer this question, consider the world from the point of view of the spin. In the reference frame of the spin, the magnetic field is dynamical and a Berry phase term will be generated. Moreover, the area traced out by the field vector (i.e. the area determining the geometric phase) is equal to the area traced out by the spin (now we are back in the fixed-field reference frame) in its motion around the field axis.

nature of the phase even more strongly: for any value of x, we have a state $|0(x)\rangle$. We may then construct the differential 1-form $\langle 0(x)|d0(x)\rangle$.[61] The geometric phase is obtained by evaluating the integral of this form along the curve c.

The advantage of the third representation above is that it suggests yet another formulation of the geometric phase, at least in cases where a closed path in parameter space is traversed. For a closed loop c, application of Stokes' theorem gives

$$\boxed{\gamma = i \oint_c \langle 0|d0\rangle = i \int_S \langle d0| \quad |d0\rangle,} \tag{9.50}$$

where S may be any surface in parameter space that is bounded by γ. This last representation is aesthetic, and easy to memorize, but also a bit too compact to be of real computational use. To give it a more concrete meaning, we insert a spectral decomposition in terms of instantaneous eigenstates,

$$\gamma = i \int_S \sum_{m=0} \langle d0|m\rangle \quad \langle m|d0\rangle.$$

(Exercise: Why does the $m = 0$ term vanish, i.e. $\langle d0|0\rangle \quad \langle 0|d0\rangle = 0$? Hint: Make use of the fact that $d\langle 0|0\rangle = \langle d0|0\rangle + \langle 0|d0\rangle = 0$ and the skew-symmetry of the -product.) We now evaluate the equation $0 = \langle m|d[(\hat{H} - \epsilon_0)|0\rangle]$ to obtain $\langle m|d0\rangle = (\epsilon_0 - \epsilon_m)^{-1}\langle m|dH|0\rangle$ or

$$\boxed{\gamma = i \int_S \sum_{m=0} \frac{\langle 0|dH|m\rangle \quad \langle m|dH|0\rangle}{(\epsilon_m - \epsilon_0)^2}.} \tag{9.51}$$

This is about as far as we get in general terms. We have established the geometric nature of the Berry phase. However, inasmuch as it requires explicit knowledge of the spectrum of $\hat{H}(x(t))$, the actual calculation of the phase remains a difficult problem.

There are cases, however, where the calculation of the geometric phase reduces to a straightforward surface integral in parameter space. One such **example** is provided by a spin subject to a weakly time-dependent magnetic field. Consider the Hamiltonian $\hat{H} = \mu \mathbf{n}(x) \cdot \sigma \equiv \mu U(x)\sigma_3 U^{-1}(x)$, where U is the rotation matrix introduced in Eq. (9.44). The $2S + 1$ instantaneous eigenstates of $\hat{H}$ are given by $U|S_3\rangle$, where $\sigma_3|S_3\rangle = S_3|S_3\rangle$ and $S_3 = -S, \ldots, S$ is the azimuthal spin quantum number. To compute the Berry phase, we can consider the first of the two representations in Eq. (9.50). Noting that the ground state is given by $|0\rangle = U|S_3 = -S\rangle$ (we assume that the magnetic moment $\gamma > 0$) and parameterizing the rotation matrix as in Eq. (9.45), one verifies that $\langle 0|d0\rangle = \langle -S|U^{-1}dU| - S\rangle = iS(1 - \cos\theta)d\phi$. We thus obtain $\gamma = S \int_c d\phi (1 - \cos\theta) = S \int dt\ (1 - \cos\theta)\phi$, which coincides (up to a Wick rotation $t \to -i\tau$, which we know does not affect topological terms) with the $S = 1/2$ WZ-action Eq. (9.48).

9.4.4 Spin chains: beyond the semi–classical limit

In Section 9.3.3, we began to explore the physics of one-dimensional spin chains. Much of our analysis was limited to the case $S \gg 1$, a semi-classical regime where quantum fluctuations

[61] Should you find the representation too abstract, choose any basis $\{|\lambda\rangle\}$ and write $\langle 0(x)|d0(x)\rangle \equiv \langle 0(x)|\lambda\rangle d\langle \lambda|0(x)\rangle$, where $d\langle \lambda|0(x)\rangle$ is the exterior derivative of the function $\langle \lambda|0(x)\rangle$.

are weak. In this limit, the spin chain is described by an O(3) nonlinear σ-model with a θ-term. This model, however, does not stay invariant under renormalization. It flows towards a "strong coupling" regime where fluctuations are large and the effective value of the spin (formally: the coupling constant of the gradient term in the model) becomes weak. Apart from the conjecture that, for half-integer/integer initial spin, the model flows towards an ordered/disordered phase with gapless/gapped excitations, there was nothing we could say about the large-distance behavior of the model (or about the physics of chains consisting of small spins $S = \mathcal{O}(1)$).

In this section, we invest quite some effort in developing a fresh attack on the physics of the small S spin chain. This is motivated in part by the enormous amount of recent experimental activity on quasi-one-dimensional spin compounds. ("Real" spin chains are often realized as structural sub-units of transition metal compounds. In these systems, the spin – which is carried by Hund's rule coupled inner shell electrons of transition metal atoms – may reach as high as 7/2.) Another, and more theory-related, motivation for our study is that the low-energy physics of the spin chain is governed by a fascinating interplay of different topologically non-trivial quantum field theories.

Following Affleck and Haldane,[62] the principal idea of our approach will be to exploit the equivalence of the antiferromagnetic spin chain to a one-dimensional Hubbard model at half filling.[63] The advantage gained by this digression is that the fermionic model is amenable to various analytical tools which (at least not in any known sense) do not apply to the spin-chain *per se*. We will find that the effective low-energy model describing the fermion system is the WZ field theory conjectured to be the fixed point theory of the σ-model in Section 9.3.3.

Fermion representation of the antiferromagnetic spin chain

Consider a one-dimensional chain of equidistant (spacing a) lattice sites i. Each site hosts n_c degenerate fermion states, or "orbitals." Consider a half-filled situation, i.e. on average, each orbital is occupied by a single spin-(1/2) fermion. We now introduce some dynamics so that the low-energy physics of the system is equal to that of an antiferromagnetic spin chain with spin $S = n_c/2$. (Notice that this construction is not quite as artificial as it may seem: the effective moments observed in "real" crystals are usually composite objects, composed of more than one elementary electron spin and stabilized by electron correlations.)

To align the n_c spins at each site, we introduce a strong ($U \gg T$) Hund's rule coupling

$$\begin{aligned}\hat{H}_{\rm int} &= -U \sum_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_i = -\frac{U}{4} \sum_i \left(\psi_{ia}^{\alpha\,\dagger} \sigma^{\alpha\beta} \psi_{ia}^{\beta} \right) \cdot \left(\psi_{ia'}^{\alpha'\,\dagger} \sigma^{\alpha'\beta'} \psi_{ia'}^{\beta'} \right) \\ &= \frac{U}{4} \sum_i \left[2 \left(\psi_{ia}^{\alpha\,\dagger} \psi_{ib}^{\alpha} - \delta_{ab} \right) \left(\psi_{ib}^{\beta\,\dagger} \psi_{ia}^{\beta} - \delta_{ba} \right) + \left(\psi_{ia}^{\alpha\,\dagger} \psi_{i\alpha}^{\alpha} - n_c \right)^2 \right] + \text{const.} \end{aligned} \tag{9.52}$$

[62] I. Affleck and F. D. M. Haldane, Critical theory of quantum spin chains, *Phys. Rev. B* **36** (1987), 5291–300.

[63] Recall that, for strong interaction, the one-dimensional Hubbard model maps onto the so-called $(t - J)$-model which (at half filling) reduces to the antiferromagnetic Heisenberg model (the spin chain).

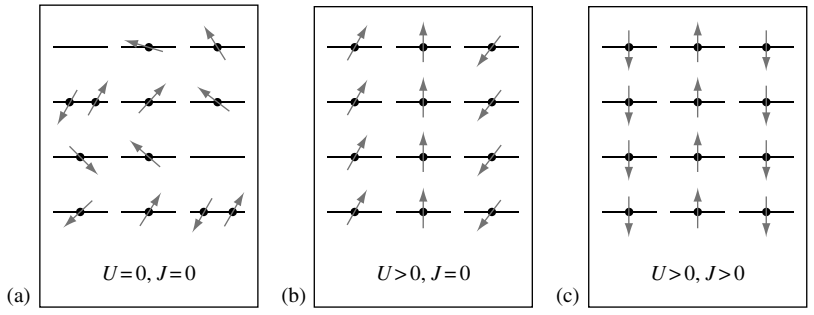


Figure 9.14 (a) A chain of sites, each containing n_c fermions on average. (b) A strong Hund's rule coupling maximizes the spin carried by each state to $S = n_c/2$. (c) Upon the switching on of a finite nearest neighbor hopping matrix element, the system becomes a spin S antiferromagnet.

Here, $a = 1, \ldots, n_c$ is the orbital index, α, β are spin indices, and, to get from the first to the second line, we have used the identity $\sigma^{\alpha\beta} \cdot \sigma^{\alpha'\beta'} = 2\delta^{\alpha\beta'}\delta^{\beta\alpha'} - \delta^{\alpha\beta}\delta^{\alpha'\beta'}$. Of special interest to our further discussion will be the last term in the second line. This Hubbard-type interaction tells us that each site favors a site occupancy of n_c electrons (i.e half-filled). Indeed, n_c electrons are needed to manufacture a net spin of maximum weight $S = n_c/2$ (see Fig. 9.14) and, thus, to optimize the Heisenberg interaction Eq. (9.52).

Let us now introduce a small amount of inter-site hopping:

$$\hat{H}_0 = -\frac{1}{2}(n_c JU)^{1/2} \sum_i \left[\psi_{ia}^{\alpha\,\dagger} \psi_{i+1a}^{\alpha} + \text{h.c} \right], \tag{9.53}$$

where the constant J determines the hopping strength. In the limit $J/U \to 0$, the half-filled system becomes equivalent to the spin S antiferromagnetic chain,

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \overset{\langle \hat{n}_i \rangle = n_c}{\longrightarrow} H_{\text{af}} = J \sum_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+1} + \mathcal{O}(J/U). \tag{9.54}$$

The easiest way to see this is to recall the situation in the standard Hubbard model (the $n_c = 1$ variant of our present model) at half filling (cf. Section 2.2). There, virtual deviations from half filling led to an effective antiferromagnetic exchange coupling between the $S = 1/2$ spins carried by neighboring sites. The effective strength of this interaction was $J \sim t^2/U$, where t is the strength of the hopping term. Formally, the generalization of this mechanism to the case $n_c > 1$ can be shown, for example, by subjecting the Hamiltonian $\hat{H}$ to a canonical transformation eliminating the hopping term (all in complete analogy to the $n_c = 1$ canonical transformation discussed in Section 2.2).

We have, thus, established the equivalence between the strongly interacting $J \ll U$ fermion Hamiltonian and the spin chain. Now there comes a major conceptual jump – and admittedly one that is not backed up by quantitative reasoning: we postulate that the equivalence between the two systems pertains to the case $J > xU$, at least as far as the relevant long-range excitations are concerned. If this were not the case, there should be some kind of abrupt change (a phase transition) in the behavior of the system as the interaction

is increased. Although this scenario cannot be rigorously excluded, it seems unlikely to be taken too seriously.

Non-abelian bosonization

Let us, then, consider the low-energy physics of the weakly interacting ($J > U$) fermion system. In fact, we shall begin by considering the totally non-interacting case.[64] Switching from a lattice to a continuum description and linearizing around the two Fermi points (for details, see Section 2.2), we describe this prototypical system in terms of the action

$$S[\psi^\dagger, \psi] = \sum_{s=\pm 1} \int dx\, d\tau\, \psi_s^{\dagger r} \left(-isv_{\mathrm{F}}\partial_x + \partial_\tau\right) \psi_s^r = \int d^2x\; \bar{\psi}^r \not{\partial} \psi^r, \tag{9.55}$$

where $\psi_{s=\pm 1}$ are the left- and right-moving fermion fields. In the latter equality we have set $v_{\mathrm{F}} = 1$ and switched to the Dirac notation (for details, see Section 4.3), and $r = (a, \alpha) = 1, \ldots, 2n_{\mathrm{c}} \equiv N$ is a composite index comprising spin and orbital components of the fermion field. In previous chapters, we have seen that the one-component ($N = 1$) variant of this model could be equivalently described in terms of a free bosonic action,

$$S[\bar{\psi}, \psi] = \int d^2x\; \bar{\psi} \not{\partial} \psi \quad \leftrightarrow \quad S[\theta] = \frac{1}{2\pi} \int d^2x\; \partial_\mu \theta\; \partial_\mu \theta, \tag{9.56}$$

where the double arrow $\leftrightarrow$ indicates that all fermion operators $\mathcal{O}[\bar{\psi}, \psi]$ (currents, densities, etc.) can be expressed in terms of boson operators $\tilde{\mathcal{O}}[\phi]$, and correlation functions $\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle_\psi$ can be identically rewritten as $\langle \tilde{\mathcal{O}}_1 \tilde{\mathcal{O}}_2 \cdots \rangle_\phi$.

Edward Witten, 1951–
Mathematical physicist and string theorist. Awarded the 1990 Fields Medal for his ground breaking work in differential geometry. Witten contributed massively to the success of string theory. (Photo by Randall Hagadorn. Courtesy of the Institute of Advanced Study.)

To what extent does this picture survive generalization to the many-channel case? The answer to this question was given in a seminal paper by Witten. Witten found that[65]

> The free fermion action Eq. (9.55) can be equivalently described in terms of a two-dimensional nonlinear σ-model with a Wess–Zumino term.

More precisely, he showed that

$$S[\bar{\psi}, \psi] = \int d^2x\; \bar{\psi}^r \not{\partial} \psi^r \quad \leftrightarrow \quad S_{\mathrm{WZW}}[g] = \frac{1}{8\pi} \int_{S^2} d^2x\, \mathrm{tr}(\partial_\mu g \partial_\mu g^{-1}) + \Gamma\,[g], \tag{9.57}$$

[64] As in our previous discussion of the single-channel case, it will turn out that the inclusion of interactions is straightforward once the effective bosonic degrees of freedom have been identified.

[65] E. Witten, Nonabelian bosonization in two dimensions, *Commun. Math. Phys.* **92** (1984), 455–72.

where $g \in \mathrm{U}(N)$ and

$$\Gamma[g] = -\frac{i}{12\pi} \int_{B^3} d^3x \; \epsilon^{ijk} \, \mathrm{tr}(g^{-1}\partial_i g \, g^{-1}\partial_j g \, g^{-1}\partial_k g), \tag{9.58}$$

denotes the **WZ action**.[66] On the right-hand side of Eq. (9.57), two-dimensional space-time has been compactified to a 2-sphere S^2. This sphere is then understood as the boundary of a three-dimensional unit ball, B^3, which serves as the integration domain of the WZ functional. As in the one-component case, the double arrow in Eq. (9.57) implies equality of all correlation functions upon suitable identification of operators (see the Info block below).

Equation (9.58) is the multi-component or "non-abelian" generalization of the prototypical bosonization identity (9.56). To understand the connection between the two equations, consider the restriction of $\mathrm{U}(N)$ to its maximal abelian subgroup, i.e. the group of all diagonal matrices $g = \mathrm{diag}(e^{i2\theta^1}, \ldots, e^{i2\theta^N})$. Evaluated on such configurations, the WZ term vanishes (exercise: show this) while the gradient term,

$$S[\theta^r] = \sum_{r=1}^{N} \frac{1}{2\pi} \int d^2x \; (\partial_\mu \theta^r)^2, \tag{9.59}$$

collapses to the sum of N free boson actions. This is the description we would have obtained had we applied Eq. (9.56) to each of the N fermion components individually (which, after all, is a perfectly legitimate thing to do!). But what, then, is the advantage of the generalized variant of bosonization Eq. (9.57)? Referring for a more detailed discussion to the Info block below, we here merely note that the action Eq. (9.57) possesses a huge symmetry group: transformations $\psi_s \to g_s \psi_s, g_s \in \mathrm{U}(N)$, leave the action invariant, i.e. the symmetry group of the problem is given by $\mathrm{U}(N) \times \mathrm{U}(N)$. Suppose we had bosonized each fermion in the standard abelian manner, thus arriving at the action Eq. (9.59). We might now ask how the symmetries of the problem – which of course must survive a change of representation – act in the θ-language. Frustratingly, there is no answer to this question;[67] the symmetries are no longer manifestly present and there is no direct way to benefit from their existence. With the non-abelian generalization Eq. (9.57), this is not so. As we shall see in a moment, the symmetry group $\mathrm{U}(N) \times \mathrm{U}(N)$ acts on the g-degrees of freedom by left–right multiplication, $g \to g_+ g g_-^{-1}$.[68] In previous chapters we have emphasized time and again the tremendous importance of symmetries and the resulting conservation laws. Indeed it turns out that, in the present problem, the comparative complexity of the action Eq. (9.57) is far outweighed by the manifest presence of the symmetries.

[66] Deviating from our earlier conventions, we will denote the WZ-action by $\Gamma[g]$ throughout (instead of the notation $S_{\mathrm{WZ}}[g]$ used above). In doing so, we follow a standard literature convention. Also, the *full* action (including the gradient term) is commonly denoted by S_{WZW} (where the last "W" credits Witten's contribution to two-dimensional WZ field theory), and the resemblance between the two symbols S_{WZ} and S_{WZW} may cause confusion.

[67] One may object that a theory of free bosons is so simple that one need not care about the conservation laws introduced by symmetries. However, this argument is too shortsighted. It ignores the fact that most operators of interest are transcendental in the θs, i.e. the full theory is not quite as simple as Eq. (9.59) would suggest.

[68] This provides another explanation of why the θ-description is too narrow to accommodate the symmetry: a diagonal matrix g will not remain diagonal when acted upon by the transformation matrices.

INFO The proof of the bosonization identity (9.57) relies on an extension of the considerations summarized in Section 4.3 to the case of non-abelian symmetries. Although a detailed discussion of the construction (for which we refer to Witten's original, yet highly pedagogical, paper) would be beyond the scope of this text, let us briefly summarize some of the **principal ideas behind the method of non-abelian bosonization.**

Let us begin by recalling that, in the abelian case $N = 1$, we had two basic symmetries, $\psi_s \to g_s\psi_s$, where $g_s = \exp(i2\phi_s)$, ϕ_s are constant phases and $\psi_{s=\pm1}$ the left- and right-moving fermion fields. These symmetries express the independence of the left- and right-moving states or, equivalently, the chirality of the problem. A straightforward application of Noether's theorem (or, equivalently, a direct variation of the action) shows that the conservation laws induced by the chiral symmetry are $\partial_z\bar{j} = \partial_{\bar{z}}j = 0$, where $j = \psi_-^\dagger\psi_-$, $\bar{j} = \psi_+^\dagger\psi_+$ and we have introduced complex coordinates $z \equiv \frac{1}{\sqrt{2}}(x_0 + ix_1)$, $\partial_z = \frac{1}{\sqrt{2}}(\partial_0 - i\partial_1)$.[69]

In Section 4.3 we saw that, on the bosonic level, the chiral symmetries act as $\theta \to \theta + \phi_+ - \phi_-$. The conservation laws corresponding to these transformations read as $\partial_z\partial_{\bar{z}}\theta = 0$ (exercise: check this). At the same time we know that the transformation generated by ϕ_+ (ϕ_-) generates the conservation law $\partial_z\bar{j} = 0$ ($\partial_{\bar{z}}j = 0$). Comparison with the two equations above leads to the identification $\bar{j} = \psi_+^\dagger\psi_+ \quad -\frac{i}{\sqrt{2}\pi}\partial_{\bar{z}}\theta$ and $j = \psi_-^\dagger\psi_- \quad \frac{i}{\sqrt{2}\pi}\partial_z\theta$, where the factor of $i/\sqrt{2}\pi$ has been included to obtain consistency with the definition of the vectorial current $j_{v,0} = \rho \sim \partial_1\theta \sim j + \bar{j}$.[69]

We now reformulate these results in a manner amenable to generalization: introducing $g \equiv \exp(i2\theta) \in \mathrm{U}(1)$, the bosonic action of Eq. (9.56) assumes the form $S[g] = \frac{1}{8\pi}\int d^2x\, \partial_\mu g\ \partial_\mu g^{-1}$, while the symmetries now act by conjugation: $g \to g_+ g g_-^{-1}$. Here, $e^{i2\phi_\pm} \equiv g_\pm$. Finally, the conserved currents can be defined as

$$j = \frac{1}{\sqrt{8\pi}} g^{-1}\partial_z g, \quad \bar{j} = -\frac{1}{\sqrt{8\pi}}(\partial_{\bar{z}}g)g^{-1}. \tag{9.60}$$

At this stage, the ordering of the – abelian – factors g is of course arbitrary. However, this will change once we proceed to the multi-channel case. (Notice that the equations (9.60) are solved by $g = g_+(\bar{z})g_-(z)$, where the two independent factors g_+ and g_- describe the right/left-moving fermion states. This solution does not rely on the commutativity of the phases g and generalizes to the non-abelian case.)

For $N > 1$, the phases g_s generalize to two independent unitary matrices $g_\pm \in \mathrm{U}(N)$. The enlarged symmetry entails the conservation laws $\partial_z\partial_{\bar{z}}j^{rr'} = 0, r, r' = 1, \ldots, N$, where the currents are given by $j^{rr'} = \psi_+^{r\dagger}\psi_+^{r'}$ and $\bar{j}^{rr'} = \psi_-^{r\dagger}\psi_-^{r'}$.

EXERCISE To verify this last statement, consider the infinitesimal unitary transformation $U_s = \exp(i\eta W)$ and expand the action Eq. (9.55) to first order in the Hermitian generator matrices W.

What are the bosonic counterparts of these expressions? It is natural to generalize the phases $g \in \mathrm{U}(1)$ of the abelian case to unitary matrices $g \in \mathrm{U}(N)$. As before, the symmetry group will act by left–right-multiplication, i.e. $g \to g_+ g g_-^{-1}$ and we expect the conserved currents to be given by Eq. (9.60).[70]

[69] Throughout, it will be preferable to use this complex notation. Notice that the two conserved currents j and $\bar{j}$ are related to the components of the vectorial currents discussed in Section 4.3 by $j_{v0} \sim j + \bar{j}$ and $j_{v1} \sim \frac{1}{i}(j - \bar{j})$.

[70] Notice, however, that the currents are now matrices $j_s = \{j_s^{rr'}\}$ and that – unlike in the abelian case – the ordering of the matrices g on the right-hand side of the definition is crucial. The factors in Eq. (9.60) are ordered in such a way that $\partial_{\bar{z}}j = 0$ and $\partial_z\bar{j} = 0$ are compatible with each other.

We now need an action that (a) is invariant under the multiplicative action of the symmetry group and (b) for $N = 1$ reduces to $\int \partial_\mu g \partial_\mu g^{-1}$. An obvious candidate would be

$$S_0[g] = \frac{1}{8\pi} \int d^2x \, \mathrm{tr}(\partial_\mu g \partial_\mu g^{-1}). \tag{9.61}$$

Yet, for a number of reasons, Eq. (9.61) does not suffice. Firstly, for this action, the conservation laws derived from the symmetry of the problem, read $\partial_\mu(g^{-1}\partial_\mu g) = 0$, which is inconsistent with our result above, $\partial_{\bar z}(g^{-1}\partial_z g) = 0$. Secondly, we have seen in the previous chapter (cf. Section 8.5) that the nonlinear σ-model Eq. (9.61) renormalizes at large length scales to smaller values of the coupling constant. At the same time, it is supposed to describe the fermionic action (9.55), which obviously does not renormalize. This tells us that $S[g]$ alone does not suffice to establish the boson–fermion correspondence.

But let us now inspect the second term in the action proposed by Witten, the two-dimensional WZ functional. The first thing we have to understand is why the second term of the action indeed represents a WZ functional in the sense of the discussion of Section 9.4.2.[71] To construct a WZ term, we need a differential form ω on the target manifold that is closed, $d\omega = 0$, but only locally exact ($\omega = d\kappa$ only locally). In Problem 9.7.2 it is shown that, on a group-valued target manifold, these criteria are met by the form $\omega = \mathrm{tr}(g^{-1}dg \quad g^{-1}dg \quad g^{-1}dg)$. (Do not be confused by the notation. What it really means is $\omega = \sum_{ijklmno}((g^{-1})_{ij}dg_{jk} \quad (g^{-1})_{lm}dg_{mn} \quad (g^{-1})_{no}dg_{oi})$, where dg_{jk} is the differential form of the function g_{jk} assigning to each element $g \in \mathrm{U}(N)$ its matrix component g_{jk}.) The general theory developed in Section 9.4.2 then tells us that $iC\int_{B^3} g^*\omega$ is a WZ term where g is a smooth extension of the field $g : S^2 \to \mathrm{U}(N)$ to a field defined on the entire ball B^3. Expressed in terms of some coordinate functions (x_1, x_2, x_3) of B^3, this expression becomes identical to Eq. (9.58). (As to the quantization of the coupling constant, see Problem 9.7.2.) In passing we note that the WZ functional is manifestly invariant under the action of the chiral symmetry group, as is required by the general structure of the theory.

It is instructive to inspect the equations of motion obtained from the WZ term. In Problem 9.7.2 we show that, upon variation $g \to e^W g \simeq (1 + W)g$ and expansion to first order in W, we obtain

$$\Gamma[(1 + W)g] - \Gamma[g] = \frac{i}{4\pi} \int d^2x \, \epsilon_\mu \ \mathrm{tr}(W \partial_\mu g \partial \ g^{-1}) + \mathcal{O}(W^2). \tag{9.62}$$

A straightforward calculation shows that the variation of the gradient term S_0 is given by

$$S_0[(1 + W)g] - S_0[g] = -\frac{1}{8\pi} \int d^2x \, \mathrm{tr}(W(g\partial_\mu^2 g^{-1} - (\partial_\mu^2 g)g^{-1})) + \mathcal{O}(W^2).$$

Combining these two results, we obtain the equations of motion

$$-2i\epsilon_\mu \ \partial_\mu g \partial \ g^{-1} + g\partial_\mu^2 g^{-1} - (\partial_\mu^2 g)g^{-1} = 0.$$

A straightforward calculation shows that these equations are indeed equivalent to the relation $\partial_z \bar j \propto \partial_z((\partial_{\bar z} g)g^{-1}) = 0$.

Summarizing, we have succeeded in finding an action that (a) is manifestly chirally invariant and (b) produces the same conservation laws as the free fermion theory. Notice, however, that we have not "proven" the analogy Eq. (9.57); rather, our analysis was mostly based on drawing

[71] Notice that, in Section 9.4.2, we focused on the case $\dim T = \dim M + 1$. Presently, however, $\dim M = 2$ while $\dim T = \dim \mathrm{U}(N)$ can become arbitrarily large.

illustrative analogies to the one-dimensional case. (For a more rigorous discussion, we refer to Gogolin *et al.*[72])

We conclude our preliminary survey of non-abelian bosonization by noting two crucial differences to the abelian case:

In the abelian case, a free fermion action was mapped onto an equally free boson action. Being quadratic in the fields, the two theories manifestly do not renormalize. In the non-abelian case, the situation is different: although it is not obvious, the right–hand side of Eq. (9.57) – a highly nonlinear functional of the group-valued fields g – does not renormalize. For a proof of this feature we refer to Problem 9.7.3.

Above, we have seen that bilinears $\sim \psi_\pm^\dagger \psi_\pm$ composed of left- or right-moving fermions afford a representation in terms of the Bose fields. Without proof, we mention that even bilinears involving fermions of different chirality can be expressed in terms of bosonic fields:

$$\psi_+^{r\dagger} \psi_-^{r'} \sim g^{rr'}. \tag{9.63}$$

However, unlike in the abelian case (see Eq. (4.46)), no boson representation of individual fermion operators is known. This is quite unfortunate as it excludes the applicability of the formalism to several interesting fields of investigation, notably the physics of fermions in the presence of disorder.

In fact, the U(N) action introduced in Eq. (9.57) defines two independent field theories at once: every matrix $g \in \mathrm{U}(N)$ can be decomposed as $g = e^{i2\phi} g'$ into a matrix $g' \in \mathrm{SU}(N)$ and a phase factor $e^{i2N\phi} = \det(g) \in \mathrm{U}(1)$. Substituting this decomposition into the action, we obtain

$$S[g] = S[g'] + \frac{N}{2\pi} \int_{S^2} d^2x \, (\partial_\mu \phi)^2. \tag{9.64}$$

Equation (9.64) tells us that the action decomposes into a WZW action for an SU(N)-valued field variable and an independent second action for the phase degree of freedom. Recalling that the invariance of the action under a homogeneous phase (gauge) transformation corresponds to the conservation of electric charge, we identify the phase action as that of the collective charge degrees of freedom (the charge density waves), while the SU(N)-action describes the spin degrees of freedom.

Renormalization group "ow of the WZW model

The action (9.57) describes a free fermion fixed point, if not in an obvious manner. To understand better the behavior of the WZW model under renormalization, let us generalize the free fermion action by introducing an arbitrary coupling constant λ^{-1} in front of the gradient term. We thus consider the action

$$S = \frac{1}{\lambda} \int_{S^2} d^2x \, \mathrm{tr}(\partial_\mu g \, \partial_\mu g^{-1}) - \frac{i}{12\pi} \int_{B^3} d^3x \, \epsilon^{ijk} \mathrm{tr}(g^{-1}\partial_i g \, g^{-1}\partial_j g \, g^{-1}\partial_k g), \tag{9.65}$$

where $g \in \mathrm{SU}(N)$ (and the complementary U(1) action is trivially free). Except for the presence of the WZ term, the model defined by this action is equivalent to the SU(N) nonlinear σ-model studied in Section 8.5. However, the latter is known *not* to have a fixed

[72] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, 1998).

point at finite values of λ. This tells us that the WZ term must have a crucial impact on the RG flow. Suppose, then, we started renormalizing the model at small values of λ. In this regime, the gradient term dominantly suppresses field fluctuations and we expect the WZ term to be of little significance. Consequently, the RG flow will initially resemble that of the standard SU(N) model – towards larger values of λ. Eventually, however, λ and the coupling constant $i/12\pi$ of the WZ term will become of the same order. At this point, at last, the coupling constant will interfere with the flow. As we know that $\lambda = 8\pi$ defines a fixed point, we expect that it will simply truncate the flow of the coupling constant.

To confirm this expectation, we need to go through the RG program, at least to one-loop order. Fortunately, however, the RG analysis of the WZW model (see Problem 9.7.3) almost exactly parallels that of the standard SU(N) model. As a result, we can infer the scaling equation

$$\frac{d\lambda}{d\ln b} = \frac{N\lambda}{4\pi}\left[1 - \left(\frac{\lambda}{8\pi}\right)^2\right]. \tag{9.66}$$

This result confirms our qualitative expectation: the value $\lambda^* = 8\pi$ defines an (attractive) fixed point at which the upwards flow of λ comes to an end. Using the methods of conformal field theory, one can indeed show (see Witten's original paper[65]) that λ^* defines an exactly solvable reference point. (This is important additional information inasmuch as our one-loop analysis does not rigorously prove that λ^* is a fixed point.)

WZW model of interacting fermions

All we have accomplished so far is a highly complicated reformulation of the trivial free fermion problem. However, as we shall see in a moment, this exercise has been far from useless: as with the abelian case, it will turn out that the boson language is of unsurpassed efficiency when it comes to the discussion of particle interactions.

As usual, the most relevant particle interactions are mediated by certain four-fermion operators. Important constraints on the structure of these operators follow, once again, from the symmetries of the model. Above we have seen that the non-interacting model is invariant under transformations by $\mathrm{U}(2N_\mathrm{c}) \times \mathrm{U}(2N_\mathrm{c})$, where the first/second factor acts on spin and color indices of the left-/right-moving fermions. However, the Hubbard–type interaction Eq. (9.52) reduces this symmetry. Inspection of the interaction operator shows that chiral symmetry gets lost (i.e. only transformations that act identically on the left-/right-moving sector are permitted, $\mathrm{U}(2N_\mathrm{c}) \times \mathrm{U}(2N_\mathrm{c}) \to \mathrm{U}(2N_\mathrm{c})$), and the remaining $\mathrm{U}(2N_\mathrm{c}) \to \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(N_\mathrm{c})$ gets reduced to symmetry transformations that act on the charge/spin/color sector separately.

There are a number of continuum interaction operators that are compatible with the symmetries of the lattice system. Consider, for example, the bilinears $j^q_s \equiv \psi^{\dagger\alpha}_{sa}\psi^{\alpha}_{sa}$, $j^{s,i}_s \equiv \psi^{\dagger\alpha}_{sa}\sigma^{i,\alpha\beta}\psi^{\beta}_{sa}$ and $j^c_{s,j} \equiv \psi^{\dagger\alpha}_{sa}T_{aa'}\psi^{\alpha}_{sa'}$, where $T^j \in \mathrm{u}(N_c)$, $j = 1, \cdots, n_c^2 - 1$ are the generators of $\mathrm{U}(N_c)$ transformations, and $s = +/-$. These are the left-/right-moving components of the conserved currents $j^{q,s,c} = j^{q,s,c}_+ + j^{q,s,c}_-$, generated by U(1), SU(2), and SU(N_c)

transformations, respectively. As in the abelian case, interactions solely between left- or right-moving fermions are largely inessential. However, the operators

$$\lambda_q = j^q_+ j^q_-, \quad \lambda_s = \mathrm{tr}(j^s_+ j^s_-), \quad \lambda_c = \mathrm{tr}(j^c_+ j^c_-), \tag{9.67}$$

are physically relevant and compatible with the symmetries of the model. Another relevant player is the "**umklapp operator**" $\lambda_{\rm uk}(\psi^{\alpha\dagger}_{+a}\sigma_2^{\alpha\beta}\psi^{\dagger\beta}_{+a})\,(\psi^{\alpha'}_{-a'}\sigma_2^{\alpha'\beta'}\psi^{\beta'}_{-a'}) + \text{h.c.}$[73] For the discussion of a few more allowed interaction operators, we refer to the original reference of Affleck and Haldane.[62]

EXERCISE Show that the umklapp operator is invariant under the action of all three symmetry groups.

Expressed in terms of the continuum fields, the Hund's rules coupling Eq. (9.52) translates to a sum of the interaction operators listed above. The question we now have to answer is how these operators – which at sufficiently large strength will turn the free fermion model into the Heisenberg model in which we are interested – affect the long-distance dynamics of the model. Naturally, we shall address this question in the bosonized language developed in the previous section. As a warm-up to the case of arbitrary spin, we shall begin with the discussion of the colorless case, $N_{\rm c} = 1$, i.e. the **spin-1/2 chain**.

Writing $g = e^{i2\phi}g'$, where $g' \in \mathrm{SU}(2)$ acts on the spin indices, and using Eq. (9.60) for the bosonic representation of the currents, we obtain $j^q = -\frac{i}{\sqrt{2}\pi}\partial_z\phi$ and $j^s = \frac{1}{\sqrt{8}\pi}g'^{-1}\partial_z g'$ for the spin current. Reflecting its invariance under spin transformations, the bosonic representation of the umklapp operator reads $\sim \lambda_{\rm uk}\cos 4\phi$.

EXERCISE To obtain the bosonic representation of the umklapp operator, substitute Eq. (9.63) into its definition and obtain $\sim \lambda_{\rm uk}e^{i4\phi}\mathrm{tr}(g'\sigma_2 g'^T\sigma_2) + \text{h.c.} \sim \lambda_{\rm uk}\cos(4\phi)$. Here the last equality is best proven by using the representation $g' = \exp(i\mathbf{v}\cdot\sigma)$.

The action generalized for the presence of these interaction operators takes the form $S[\phi, g'] = S[\phi] + S[g']$, where

$$\begin{aligned} S[\phi] &= \frac{1}{2\pi}(1-\lambda_q)\int d^2x\left((\partial\phi)^2 + C\lambda_{\rm uk}\cos(\sqrt{4}\phi)\right), \\ S[g'] &= S_{\rm WZW}[g'] + \lambda_s\int d^2x\ \mathrm{tr}(\partial g'\partial g'^{-1}). \end{aligned} \tag{9.68}$$

Notably, the action still decouples into a spin and a charge sector (the separation of spin and charge characteristic of one-dimensional systems). Further, both $S[\phi]$ and $S[g']$ are old acquaintances: $S[\phi]$ is the action of the two-dimensional sine–Gordon model. In Section 8.6 we have seen that, for $\lambda_q < 0$ (repulsive interactions), it flows towards a phase with a mass gap. This flow is driven by the umklapp operator. What this tells us is that umklapp scattering leads to the presence of an excitation gap for charge density waves. This gap

[73] Recall that **umklapp scattering** is the scattering of two fermion states of opposite spin from one point of the Fermi surface ($\pm k_{\rm F}$) to the other ($\mp k_{\rm F}$). At half filling, $k_{\rm F} = \pi/2a$, the momentum transferred in this process is of magnitude $2(\pi/2a - (-\pi/2a)) = 2\pi/a = G$, where $G = 2\pi/a$ is the reciprocal lattice vector. Since lattice momentum is conserved only up to multiples of G, umklapp scattering (at half filling) is a permissible process.

is nothing but the **Mott–Hubbard gap** present in the spectrum of interacting fermions at half filling. (Recall that the presence of umklapp processes is tied to the case of half filling.) Turning to the action $S[g']$, we note that the λ_s-perturbation merely renormalizes the coupling constant of the gradient term of $S_{\rm WZW}$, $1/8\pi \to (1/8\pi) + \lambda_s$. However, in the previous section, we have seen that this change does not alter the long-range behavior of the system: irrespective of the value of the coupling constant, the model will flow back towards the free fermion fixed point at $\lambda^* = 1/8\pi$.

Summarizing, we have found that the $N_{\rm c} = 1$ interacting fermion system builds up a Mott–Hubbard gap for its charge excitations while the spin excitations are described by a critical WZW theory. Now, our entire analysis was based on the presumed equivalence ($N_{\rm c} = 1$ interaction fermion system) $\quad$ ($S = 1/2$ Heisenberg chain). The existence of long-range excitations in the former then implies that the latter must be in an ordered phase (since a disordered phase would be *defined* by the absence of long-range excitations). This, however, is a result with which we are familiar.[74] The truly interesting question is what happens for larger values of the spin (i.e. larger values of $N_{\rm c}$).

At $N_{\rm c} > 1$, the structure of the theory gets significantly more involved. Referring for a more detailed analysis to the original paper by Affleck and Haldane,[62] we restrict ourselves here to a qualitative discussion of the most relevant aspects.

- ▷ Consider a matrix field $g' \in {\rm SU}(2) \times {\rm SU}(N_{\rm c})$.[75] Among the various interaction operators there is one that drives excitations of the color sector into a massive phase. This means that the soft excitations of the model can be parameterized as $g' = \hat{g} \times \mathbf{1}_{N_{\rm c}}$, where $\hat{g}$ acts in the spin sector. Substituting these configurations into the free fermion reference action, we obtain $S_{\rm WZW}[g'] = N_{\rm c} S_{\rm WZW}[\hat{g}]$, where the prefactor $N_{\rm c}$ arises from the tracing out of the color sector. Remembering our discussion of the quantization of the WZ coupling constant in Section 9.4.2, we identify this model as a **WZW model of level** $k = 2N_{\rm c}$.
- ▷ Among the plethora of conceivable perturbations of the critical theory, one operator family deserves special attention: ${\rm tr}(g^n)$, where $g \in {\rm SU}(2)$. (Here, we are writing g instead of $\hat{g}$ for notational simplicity and n is some integer.) For two reasons, these are interesting operators: firstly, they represent the most relevant perturbations of the theory (due to the absence of derivatives); secondly, they relate to an important discrete symmetry of the model. To understand why, let us return to the lattice version of the theory and consider the fermion representation of the spin operator $S_l^j = \psi_l^{\alpha\dagger} \sigma^{\alpha\beta} \psi_l^\beta$. Decomposing the fermion into left- and right-moving components, $\psi_l = e^{ik_{\rm F} l} \psi_+(x_l) + e^{-ik_{\rm F} l} \psi_-(x_l)$, where $x_l = la$, we obtain $S_l^j = \sum_s \psi_s^\dagger(x_l) \sigma^j \psi_s(x_l) + [(-)^l \psi_+(x_l)^\dagger \sigma^j \psi_-(x_l) + {\rm h.c.}] = {\rm tr}([c(j_+^s + j_-^s) + c'(-)^l (g + g^{-1})]\sigma^j)$, where c and c' are numerical constants. Here, we have used the relation $e^{ilk_f x} = e^{i(\pi/a)la} = (-)^l$ as well as the bosonization identity Eq. (9.63). The second term under the trace is of particular interest. It tells us that translation by one site, $l \to l + 1$, corresponds to a sign change of the field g. This observation can be read in different ways. For example, contributions to the action that are not invariant

[74] Recall that the $S = 1/2$ chain can be subjected to a Jordan–Wigner transformation whereupon it becomes a model of spinless interacting fermions. The existence of gapless excitations (charge density waves) thus proves the existence of long-range order in the spin chain.

[75] We will ignore the gapped charge sector throughout.

under $g \to -g$ explicitly break **translational invariance on the lattice**. Similarly, a ground state that is not sign invariant cannot be translationally invariant, etc.

▷ After these preparatory remarks, let us return to the discussion of the operators $\mathrm{tr}(g^n)$. As we are not interested in modeling situations where the Hamiltonian explicitly breaks translational invariance, only contributions with n even will be allowed. Physically, these operators correspond to products of n left-moving and n right-moving fermion states. In the case considered previously, $N_\mathrm{c} = 1$, there is only one such contribution, namely the backscattering operator discussed above. However, for $N_\mathrm{c} > 1$, terms $\mathrm{tr}(g^{2n\ \ N_\mathrm{c}})$ are physically allowed. Let us, then, consider the action $S[g] = N_\mathrm{c} S_\mathrm{WZW} + \lambda \int d^2x\ \mathrm{tr}(g^2)$.[76] Void of derivatives, the new contribution acts as a potential contribution to the action (although it is still invariant under the SU(2) symmetry transformation $g \to h^{-1}gh$). This being so, the physical behavior of the model crucially depends on the sign of the coupling constant λ.

▷ For $\lambda < 0$, the term $\lambda \int \mathrm{tr} g^2$ favors the mean-field configurations $\bar{g} = 1$ or $\bar{g} = -1$. Both ground states break the discrete sign inversion symmetry $g \to -g$ and, therefore, they cannot be translationally invariant. Before identifying the physical meaning of these ground states, let us briefly discuss the issue of fluctuations. Writing $g = e^{i\phi^j \sigma^j/2}\bar{g}$ and expanding to quadratic order in ϕ, we obtain a mass term $\frac{\lambda}{2} \int d^2x\ (\phi^j)^2$. This means that spin fluctuations around $\bar{g}$ are gapped. There is, indeed, one well known low-energy state of the spin chain that displays these features, the **dimer phase**. In this phase (see Fig. 9.15), spins at neighboring sites form spin singlets. Obviously, this state is not invariant under translation by one site (while translation by *two* sites is a symmetry). Further, there is no room for massless spin fluctuations, i.e. the system indeed shows an excitation gap.

▷ We next discuss the complementary case, $\lambda > 0$. In this case, configurations $\bar{g}$ with eigenvalues $(i, -i)$ are energetically favored. There is a continuous family of such states, namely $\bar{g} = \exp(i\pi \mathbf{n} \cdot \boldsymbol{\sigma}/2)$, where $|\mathbf{n}| = 1$. Again, these states break translational invariance $(g \to -g)$. However, they also break SU(2) symmetry $g \to hgh^{-1}$. This means that, unlike the dimerized case, the continuous spin rotation symmetry is spontaneously broken. Its breaking of both translational and spin rotational invariance identifies $\bar{g}$ as the **Néel state** of the chain.

▷ To describe the physics of the Goldstone modes corresponding to the symmetry broken state, we note that $h \exp(i\pi \mathbf{n} \cdot \boldsymbol{\sigma}) h^{-1} = \exp(i\pi \mathbf{n}' \cdot \boldsymbol{\sigma})$, where the unit vector $\mathbf{n}' = R\mathbf{n}$ and $R \in \mathrm{SO}(3)$ is a rotation matrix canonically corresponding to the SU(2)-matrix h.[77] Each $\mathbf{n}'$ defines a new ground state, i.e. we have identified the 2-sphere as the Goldstone mode manifold.

EXERCISE Try to guess what the action of the Goldstone modes might be!

To identify the Goldstone mode action, we substitute the soft field configurations $\exp(i\pi \mathbf{n}(x) \cdot \boldsymbol{\sigma}/2) = i\mathbf{n} \cdot \boldsymbol{\sigma}$ into the action. It is straightforward to verify that the gradient

[76] The operator $\mathrm{tr}(g^2)$ can be shown to be the most relevant of the family.

[77] To any element $h \in \mathrm{SU}(2)$, the prescription above assigns an element $R \in \mathrm{SO}(3)$. The correspondence is 2–1 and not 1–1 because both h and $-h$ map onto the same R.

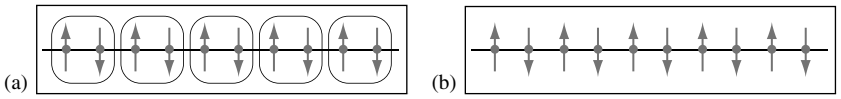


Figure 9.15 Dimerized phase of the spin chain (a) vs. Néel phase (b).

term of the WZW action simply becomes $S_0[i\mathbf{n}\cdot\boldsymbol{\sigma}] = \frac{N_c}{8\pi}\int d^2x\ (\partial\mathbf{n})^2$. As to the WZ functional, we show in Problem 9.7.2 that $N_c\Gamma[i\mathbf{n}\cdot\sigma] = \frac{N_c}{4}\int_{S^2}\mathbf{n}\cdot\partial_1\mathbf{n}\times\partial_2\mathbf{n} = \pi N_c S_{\rm top}[\mathbf{n}]$, i.e. the θ-term of the 2-sphere. Recalling that $N_c = 2S$, we conclude that, for large values of S (this is where the massive fluctuations around the Goldstone mode manifold can be neglected), the system is controlled by the action of the O(3) nonlinear σ-model with topological angle $2\pi S$,

$$S[\mathbf{n}] = \frac{1}{2\lambda}\int d^2x\ (\partial\mathbf{n})^2 + 2\pi S\, S_{\rm top}[\mathbf{n}], \tag{9.69}$$

the same $S[\mathbf{n}]$ identified earlier on semi-classical grounds as the low-energy action of the spin chain.

This is now a good point to pause and consider what we have obtained. At first sight, it seems as if nothing much has been achieved: as a result of a long series of derivations we have arrived back at the semi-classical representation of the chain, the σ-model. However, it turns out that it is the *interplay* between the WZW and the σ-model that enables us to really understand the physics of the problem. To see this, let us consider the $S = 1/2$ chain in the presence of a next-nearest neighbor ferromagnetic coupling – admittedly an artificial model but, for the sake of the present argument, this does not matter. One can show that the long-range physics of this system is described by the O(3)-model with topological angle $2\pi S = \pi$. On the other hand, we might have approached the problem via the WZW route discussed above. Within that context, the ferromagnetic perturbation turns out to be *irrelevant* (see Affleck and Haldane[62]), i.e. the long-range physics of the system is described by the critical free fermion WZW action. Comparing these two findings we can conclude that

> The O(3) nonlinear σ-model with topological angle $\pi \times$ (odd integer) is equivalent to the SU(2) WZW action at critical coupling,

(although no explicit field theoretical proof of this equivalence is known). This was the last missing piece of information needed to understand the long-range behavior of the spin chain. To summarize, we have found that (see Fig. 9.16) the antiferromagnetic spin S chain can be described in terms of a perturbed WZW action of level $k = 2S$. Depending on the sign of the most relevant perturbation $\sim \lambda\int\ {\rm tr}(g^2)$ (which is set by the material parameters of the problem), this model can be either in a globally gapped phase – the dimer phase of the chain – or in a Néel phase. The fluctuations superimposed on the Néel phase are described by an O(3) model with topological angle $2\pi S$. For integer spin, this model flows towards a strong coupling phase, i.e. the spin chain is in a disordered state. However, for S half-integer,

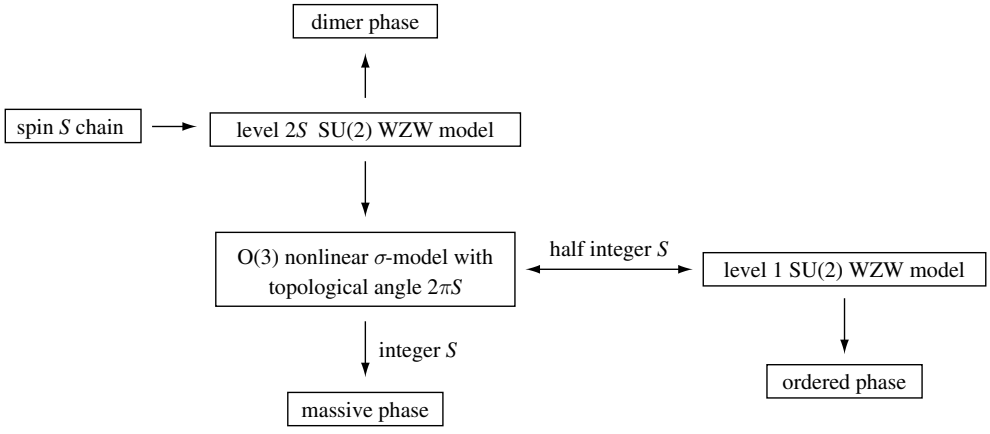


Figure 9.16 On the long-range physics of the antiferromagnetic spin chain. For a discussion, see the main text.

it becomes equivalent to the level 1 WZW model at criticality, implying ordered behavior of the chain.

Notice that none of the conclusions summarized above was established rigorously. More often than not we had to *trust* in the principle of adiabatic continuity, i.e. the belief that the physics of a system does not change qualitatively upon interpolating between the regime of weak interactions (where various approximation schemes work) to the regime of strong interactions (in which we are actually interested). Similarly, connections between different theories were constructed on the basis of symmetry arguments and indirect reasoning (rather than by "hard-boiled" calculations). In this way various pieces of evidence were pieced together to form a network that was intrinsically consistent and made physical sense. In recent years, this type of semi-quantitative research has become more and more prevalent in various areas of condensed matter physics. This development is driven by the increasing complexity of the questions and, relatedly, the absence of straightforward perturbative schemes. In the next section, we shall turn to another problem field where such "detective work" has been successful, the fractional quantum Hall effect.

9.5 Chern–Simons terms

As our third and last example of topological field theories with relevance to condensed matter physics, we now turn to the discussion of Chern–Simons (CS) theories. However, deviating from the strategy pursued in previous sections, this time we do not begin with a formal analysis of the underlying geometrical framework. Instead, we directly turn to a review of *the* master application of CS field theory in condensed matter physics, the fractional quantum Hall effect. It will then turn out that, once we have left the qualitative level and turned to the field integral description, we readily wind up in the basin of attraction of CS field theory.

9.5.1 Fractional quantum Hall effect (FQHE)

In Section 9.3.4 we interpreted the QHE as a topological phenomenon. Depending on the chosen perspective, the integer number setting the Hall conductance can be interpreted as the number of occupied edge channels or, more formally, as the number of instanton excitations in Pruisken's field theory. Both figures appear to be largely impervious to changes in the parameters of the system.

In essence, the basic mechanism behind the formation of integer-valued Hall conductance had been understood shortly after the discovery of the effect. It thus came as a surprise when Tsui *et al.*[78] discovered a sequence of plateaus at fractional values of the Hall conductance. More specifically, it turned out that:

- ▷ Fractional values $\sigma_{xy} = \nu \frac{e^2}{h}$, $\nu \equiv \frac{n}{m}$, of the Hall conductance are observed only in the purest samples. This indicates that, unlike with the integer effect, disorder does not stabilize the FQHE.
- ▷ Not every rational n/m qualifies as a plateau value. The most prominent plateaus are observed for the "principal sequence" $1/m$, where m is odd. More generally, plateaus have been observed for $\frac{n}{m} = \frac{p}{2sp+1}$, $s, p \in \mathbb{N}$.[79]
- ▷ Curiously, it has turned out that, at certain even-denominator fractions (formally, the limit $p \to \infty$ in the hierarchy) the system largely behaves as if no magnetic field were present at all! For instance, in the vicinity of $\nu = 1/2$, pronounced Shubnikov–de Haas oscillations (otherwise shown by Fermi liquids subject to a weak field) are observed.

At first sight, the coexistence of phenomena of that degree of complexity with the (seemingly so robust) integer QHE may cause some consternation. But, then, let us recall that we are considering an isolated Landau level at fractional filling! In other words, we are considering a hugely degenerate quantum state that is only partially populated. Within that environment, a macroscopic restructuring of the electron gas can be afforded at little cost (which will be set by "residual" mechanisms such as electron–electron interactions and/or disorder scattering). On the same footing, it is clear that perturbative expansions around any given trial ground state will be pretty fragile. (Perturbations enjoy a huge phase volume while "energy denominators" are small.)

Keeping these things in mind, it is perhaps no longer surprising that dramatic things happen in the fractionally filled Landau level. The considerations above suggest that elementary electrons – subject to the full strength of Coulomb interactions, at fully "quenched" kinetic energy – will hardly qualify as stable elementary excitations of the system. At the same time, the appearance of Fermi-liquid-like states at least for some filling fractions suggests that the dominant players in the game *are* elementary fermionic particles.

[78] D.C. Tsui, H.L. Stormer, and A.C. Gossard, Two-dimensional magnetotransport in the extreme quantum limit, *Phys. Rev. Lett.* **48** (1982), 1559–62.

[79] In fact, the two-parameter "hierarchy" defined by the right-hand side of this equation still is not general enough to account for all experimentally observed values. (The prominent exception is a fragile plateau at $\nu = 5/2$.) However, to explain these anomalous values of the Hall conductance, one has to keep track of the polarization of the electron spin, an extra level of complexity which we would here rather avoid.

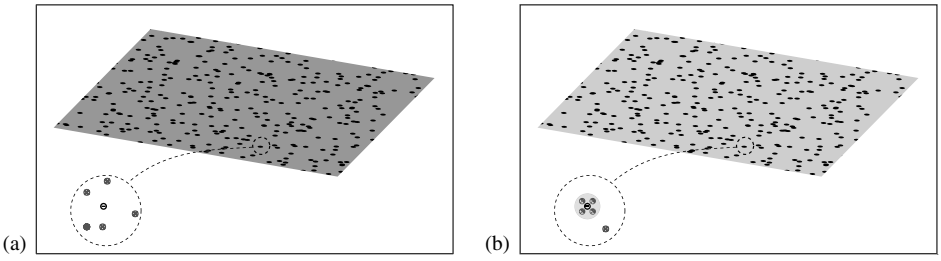


Figure 9.17 Illustrating the idea of the composite fermion approach. Imagine that an even number of flux quanta constituting the magnetic field get tied to the electrons in the system. The new composite particles ("electron + $(2n)$ flux quanta") continue to be fermions. They only see the remaining "free" flux quanta, i.e. a reduced external field (indicated by the lighter shading in part (b)).

Remarkably, it is not at all difficult[80] to conceive of a fermionic quasi-particle picture wherein – at least at a mean-field level – the plethora of phenomena above admits a straightforward explanation: imagine the external magnetic field as a large number N_ϕ of flux tubes piercing the plane of the two–dimensional electron gas (see Fig. 9.17). The ratio $\nu \equiv N/N_\phi$, where N denotes the number of electrons, defines the filling fraction of the system. Since we are working under "fractional" conditions, $0 < \nu < 1$, the number of flux quanta exceeds the number of charge carriers.[81] Now assume that, by some mechanism, each electron matches up with an even number of flux quanta to form a composite particle. (By way of example, consider $\nu = 1/3$, in which case there would be enough flux quanta around to let every electron pair with two flux tubes.) What can be said about the properties of these composite objects?

- ▷ Firstly, they would still be fermions. To understand why, recall that the statistics of particles can be probed by exchanging their position in space. However, our composite particles and the original electrons differ only in the presence of an even number of integer flux tubes. The flux tubes give rise to additional phase factors, so that our new particles are fermionic: one speaks of **composite fermions (CFs)**.
- ▷ The CFs see an effectively reduced external field. For example, for $\nu = 1/3$, each electron has absorbed two flux quanta. Thus, the residual field seen by the CFs is three times lower than the original field. In other words, the number of remaining flux quanta, $N_\phi - 2N = N$, is equal to the number of CFs. Forgetting for a moment about the origin of the composite particles, we are considering a large number N of (composite) fermions subject to N flux quanta, i.e. a situation where the *integer* QHE should arise. This picture suggests an interpretation of the FQHE as an IQHE of composite fermions.[82] In the specific case of a half-filled ($\nu = 1/2$)

[80] However, our discussion should not deceive the reader about the fact that the composite fermion picture represents the outgrowth of years of most intensive research!

[81] If $\nu > 1$, $\nu - [\nu]$, where $[\nu]$ is the largest integer smaller than ν, sets the filling fraction of the highest occupied Landau level and our discussion applies to that level.

[82] More generally, let us assume that every fermion binds $2s$ flux quanta to it. Further suppose that $N/(N_\phi - 2nN) = p$, i.e. that p Landau levels of the *residual* field are occupied. In this case, the CFs will also display the IQHE. This

band, $N_\phi - 2N = 0$, i.e. the CFs experience a mean-field of vanishing strength. This nicely conforms with the experimental observation of Fermi-liquid-like behavior (no QHE) close to half filling.

▷ We have seen in Section 9.3.4 that, when a flux quantum is adiabatically pushed through an annular quantum Hall geometry, an electron charge flows from the inner to the outer perimeter of the sample. In a way, the phase vortex created by the addition of the gauge flux leads to an expulsion of electronic charge. Similarly, the flux tubes involved in the construction of the CF picture effectively carry a positive charge ν. These "screen" the charge carried by the bare electron, so that the CF is not exposed to the full strength of the unit-charge Coulomb interaction.

In the history of the FQHE, the introduction of the CF picture by Jain[83] was preceded by a number of other important developments. Shortly after the experimental discovery of the effect, Laughlin[84] proposed a trial wavefunction which, in a close-to-optimal way,[85] minimizes the Coulomb repulsion between the quasi-particles in the lowest Landau level. Most of the concepts central to the subsequent introduction of the CF picture – fractionally charged quasi-particles, incompressibility of the QH state, the importance of correlations, etc. – have effectively been motivated by this trial wavefunction. Later, the theory of the (many-body) CF system was formulated in terms of a Chern–Simons type field theory.[86] This effective field theory has become the basis of many subsequent analyses of FQHE phenomena.

In this text, we shall turn the sequence of historical developments upside down: starting from a field theoretical description, we shall identify the CF degrees of freedom and then rediscover Laughlin's wavefunction. Once these structures are in place, the computation of the fractional Hall conductance will be little more than a straightforward exercise.

9.5.2 Chern–Simons field theory: construction

Consider the Hamiltonian of two-dimensional interacting electrons subject to a perpendicular magnetic field of strength B: $\hat{H} = \hat{H}_0 + \hat{H}_{\rm int}$ where

$$\begin{aligned} \hat{H}_0 &= \int d^2x\, a^\dagger(\mathbf{x}) \left[\frac{1}{2m}(-i\partial_{\mathbf{x}} + \mathbf{A}_{\rm ext})^2 + V(\mathbf{x}) \right] a(\mathbf{x}), \\ H_{\rm int} &= \frac{1}{2} \int d^2x\; d^2x'\; (\hat{\rho}(\mathbf{x}) - \hat{\rho}_0) V(\mathbf{x} - \mathbf{x}') (\hat{\rho}(\mathbf{x}') - \hat{\rho}_0). \end{aligned}$$

happens for filling fractions $\nu = N/N_\phi = p/(2sp+1)$. In essence, this simple picture explains the structure of the rationals where the FQHE is observed.

[83] J. K. Jain, Composite-fermion approach for the fractional quantum Hall effect, *Phys. Rev. Lett.* **63** (1989), 199–202.

[84] R. B. Laughlin, Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations, *Phys. Rev. Lett.* **50** (1983), 1395–8.

[85] For certain short-range correlated model interactions, the Laughlin wavefunction can even be shown to be an exact ground state, see F. D. M. Haldane, Fractional quantization of the Hall effect: a hierarchy of incompressible quantum fluid states, *Phys. Rev. Lett.* **51** (1983), 605–8.

[86] A. López and E. Fradkin, Fractional quantum Hall effect and Chern–Simons gauge theories, *Phys. Rev. B* **44** (1991), 5246–62.

Here, $\mathbf{A}_{\rm ext} = \frac{B_{\rm ext}}{2}(y,-x)^T$ is the vector potential of the external magnetic field (symmetric gauge), V is a single-particle potential created by the presence of, for example, impurities, and $\hat{H}_{\rm int}$ describes the particle interaction on the background of a constant counter-density ρ_0.

Singular gauge transformation

After our discussion in the previous section, we do not expect bare electrons to be a useful reference for the construction of a low-energy field theory. Certainly, it will be more promising to start out from composite fermions as discussed above. As we have seen, a CF is an electron with $2s$ integer flux tubes attached to it. Other fermions moving around the CF along a circular contour will acquire the winding phase $2s\phi$ indicative of the presence of $2s$ flux quanta. Within a first-quantized framework, these phase vortices can be attached[87] to the position of each fermion by virtue of the "gauge transformation"

$$\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N) \to \Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)\exp\left[-2is\sum_{i<j}\arg(\mathbf{x}_i-\mathbf{x}_j)\right], \tag{9.70}$$

where Ψ is the many-body wavefunction, and $\arg(\mathbf{x}) = \tan^{-1}(x_2/x_1)$ is the angle enclosed between $\mathbf{x}\in\mathbb{R}^2$ and the positive real axis. The transformation Eq. (9.70) becomes singular whenever two coordinates $\mathbf{x}_i \to \mathbf{x}_j$ approach each other, i.e. it does not represent an orthodox gauge transformation. In fact, the vector potential corresponding to the phase factor above, $\mathbf{a} = -2s\partial_{\mathbf{x}}\sum_i \arg(\mathbf{x}-\mathbf{x}_i) = -2s\sum_i \frac{(x_1-x_{i,1})\mathbf{e}_2-(x_2-x_{i,2})\mathbf{e}_1}{|\mathbf{x}-\mathbf{x}_i|^2}$, creates a perpendicular magnetic field[88] of strength

$$b = \epsilon^{ij}\partial_{\mathbf{x}_i}a_j = -4\pi s\sum_i \delta(\mathbf{x}-\mathbf{x}_i), \tag{9.71}$$

i.e. the field corresponding to $2sN$ flux tubes centered at the coordinates of the fermions. Summarizing, the singular gauge transformation above converts the N fermions into a system of CFs (fermions with $2s$ flux lines attached).

Derivation of the Chern–Simons action

Within the framework of the second quantization, the transformation Eq. (9.70) amounts to the replacement

$$a^\dagger(\mathbf{x}) \to a^\dagger(\mathbf{x})\exp\left[-2is\int d^2x' \arg\,(\mathbf{x}-\mathbf{x}')\hat{\rho}(\mathbf{x}')\right],$$

where, as usual, $\rho = a^\dagger a$.

EXERCISE Check that, with this definition, a quantum many-body wavefunction $|\ \rangle \equiv \prod_{i=1}^{N} a^\dagger{}_i|0\rangle$ transforms according to Eq. (9.70). (The index λ refers to the states $|\lambda\rangle$ of a suitably chosen single-particle basis [e.g. a basis of Landau states].)

[87] In fact, the attachment of these phases is *equivalent* to placing flux quanta at the winding center.
[88] This would, of course, not be possible for a non-singular gauge transformation.

Substituting the transformed operators into the Hamiltonian, one obtains

$$\hat{H}_0 \to \int d^2x\, a^\dagger(\mathbf{x}) \left[\frac{1}{2m}(-i\partial_{\mathbf{x}} + \hat{\mathbf{A}})^2 + V(\mathbf{x})\right] a(\mathbf{x}),$$

where $\hat{\mathbf{A}} = \mathbf{A}_{\rm ext} + \hat{\mathbf{a}}$ and

$$\hat{\mathbf{a}}(\mathbf{x}) = -2s \int d^2x'\ \frac{(x_1 - x_1')\mathbf{e}_2 - (x_2 - x_2')\mathbf{e}_1}{|\mathbf{x} - \mathbf{x}'|^2} \hat{\rho}(\mathbf{x}'). \tag{9.72}$$

At this stage it is convenient to switch to a real-time field integral representation. We thus introduce the partition function $\mathcal{Z} = \mathcal{N} \int D(\bar\psi, \psi)\, e^{iS[\bar\psi,\psi]}$, where the action $S[\bar\psi, \psi] = S_0[\bar\psi, \psi] + S_{\rm int}[\bar\psi, \psi]$, with

$$\begin{aligned} S_0[\bar\psi, \psi] &= \int dt\ d^2x\ \bar\psi \left[i\partial_t + \mu - \frac{1}{2m}(-i\partial_{\mathbf{x}} + \mathbf{A}[\bar\psi, \psi])^2 - V(\mathbf{x})\right] \psi, \\ S_{\rm int}[\bar\psi, \psi] &= -\frac{1}{2} \int dt \int d^2x\ d^2x'\ (\rho(\mathbf{x}) - \rho_0) V(\mathbf{x} - \mathbf{x}')(\rho(\mathbf{x}') - \rho_0), \end{aligned}$$

is obtained in the usual way by trading field operators for coherent state amplitudes (specifically, $\rho = \bar\psi\psi$ and $\mathbf{A}[\bar\psi, \psi] \equiv \hat{\mathbf{A}}_{(a,a^\dagger)\to(\psi,\bar\psi)}$). Thanks to the presence of the vector potential $A[\bar\psi, \psi]$, the kinetic energy operator has become a pretty unpleasant object, depending non-locally on up to six field amplitudes $\psi, \bar\psi$. To avoid this complication[89] let us shift the nonlinearities implied by the singular gauge transformation to some other place in the action. This can be achieved by promoting the vector potential to an integration variable whose value is set so as to generate the flux pattern: i.e. multiply the partition function by $1 = \mathcal{N} \int D\mathbf{a}_\perp \prod_{\mathbf{x},t} \delta\left(b(\mathbf{x}, t) + 4\pi s\rho(\mathbf{x}, t)\right)$, where $b = \epsilon^{ij}\partial_i a_{\perp,j}$ and the subscript " " indicates that the integration extends only over transversal configurations of the vector potential (that is, configurations obeying $\partial_i a_i = 0$).[90] As a result, we obtain the double functional integral,

$$\begin{aligned} \mathcal{Z} &= \mathcal{N} \int D(\bar\psi, \psi)\, D\mathbf{a}_\perp \prod_{\mathbf{x},t} \delta\left(b(\mathbf{x}, t) + 4\pi s\rho(\mathbf{x}, t)\right) \exp\left(-S[\bar\psi, \psi, \mathbf{a}_\perp]\right) \\ &= \mathcal{N} \int D(\bar\psi, \psi)\, D\mathbf{a}_\perp\, D\phi \exp\left(iS[\bar\psi, \psi, \mathbf{a}_\perp] - i \int d^2x\, dt\, \phi \left(\frac{b}{4\pi s} + \rho\right)\right) \\ &\equiv \mathcal{N} \int D(\bar\psi, \psi)\, D\mathbf{a}_\perp\, D\phi \exp\left(iS_{\rm CF}[\psi, \bar\psi, \mathbf{a}_\perp, \phi] + i\frac{\theta}{2} S_{\rm CS}[\mathbf{a}_\perp, \phi]\right), \end{aligned}$$

where we have introduced the common shorthand notation $\theta \equiv 1/2\pi s$, the action $S[\bar\psi, \psi, \mathbf{a}_\perp] \equiv S[\bar\psi, \psi]\big|_{\mathbf{A}[\bar\psi,\psi]\to\mathbf{A}_{\rm ext}+\mathbf{a}_\perp}$ is obtained by replacing the fixed vector potential

[89] Technically, it is not advisable to disturb the structure of the most basic operator of the theory.

[90] This latter condition is necessary because the δ-distribution does not fix the longitudinal, or gauge, freedoms of the potential. In two dimensions, the decomposition of the vector potential into longitudinal (φ) and transverse (θ) components is achieved by setting $a_i = \partial_i\varphi + \epsilon_{ij}\partial_j\theta$.

$\mathbf{a}[\bar{\psi},\psi]$ by the integration variable $\mathbf{a}_\perp$,[91]

$$S_{\rm CF}[\psi,\bar{\psi},a_\perp] = \int d^2x\ dt\,\bar{\psi}\left(i\partial_t + \mu - \phi + \frac{1}{2m}(-i\partial_{\mathbf{x}} + \mathbf{A})^2 - V\right)\psi + S_{\rm int}[\bar{\psi},\psi], \quad (9.73)$$

$$S_{\rm CS}[a_\perp] = -\int d^2x\ dt\,\phi\epsilon_{ij}\partial_i a_{\perp,j}, \quad (9.74)$$

and $\mathbf{A} = \mathbf{A}_{\rm ext} + \mathbf{a}_\perp$. At this stage, we have fulfilled our intermediate goal; a field integral representation has been derived wherein particles are tightly bound to fluxes. There is, however, something unsatisfactory about the present representation of the theory: the way the variables $\mathbf{a}_\perp$ and ϕ enter the action (9.74) suggests an interpretation of the theory as one of fermions coupled to a $(2+1)$-dimensional electromagnetic gauge potential $a_\perp \equiv (\phi,\mathbf{a}_\perp)$. However, the action $S_{\rm CF} + S_{\rm CS}$ falls short of the two standard criteria any decent theory of electromagnetism should obey: (i) in its present representation, it has nothing to say about the longitudinal degrees of freedom $\mathbf{a}_\parallel$ of the vector potential and, relatedly, (ii) the contribution $S_{\rm CS}$ is not gauge invariant (which means that, using a covariant representation $x = \{x_\mu\} \equiv (x_0 \equiv t, x_1, x_2)$, under a transformation $a_\mu \to a_\mu + \partial_\mu f$, $\mu = 0,1,2$, it changes value).

We now claim that the action $S_{\rm CS}$ possesses a natural gauge-invariant extension, namely the well-studied **Chern–Simons action**[92]

$$\boxed{S_{\rm CS}[a] = -\int d^3x\ \epsilon_{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma.} \quad (9.75)$$

Firstly, a straightforward integration by parts shows that $S_{\rm CS}$ is gauge invariant. Secondly, one verifies (exercise!) that, for a purely transverse configuration $a_\perp = (\phi,\partial_2\theta,-\partial_1\theta)$, $S_{\rm CS}[a_\perp]$ reduces to the form given in Eq. (9.74). Put differently, the prototypical action (9.74) is but the gauge-invariant Chern–Simons action evaluated in a particular gauge, namely the **Coulomb** or **radiation gauge** $a_\parallel = 0$. The gauge-invariant extension of the theory is obtained by integration over all gauge sectors,

$$\mathcal{Z} = \mathcal{N}\int D(\bar{\psi},\psi)\ Da\ \exp\left(iS_{\rm CF}[\bar{\psi},\psi,a] + i\frac{\theta}{4}S_{\rm CS}[a]\right), \quad (9.76)$$

where

$$S_{\rm CF}[\psi,\bar{\psi},a] = \int d^3x\ \bar{\psi}\left(i\partial_0 + \mu - \phi + \frac{1}{2m}(-i\partial_{\mathbf{x}} + \mathbf{A}_{\rm ext} - \mathbf{a})^2 - V\right)\psi + S_{\rm int}[\bar{\psi},\psi], \quad (9.77)$$

[91] Notice that Eq. (9.72) is purely transversal, so that the replacement $\hat{\mathbf{a}} \to \mathbf{a}_\perp$ makes sense.
[92] Recall that space-time vectorial components are defined as $x_\mu = (x_0,x_1,x_2)$, while $\partial_\mu = (-\partial_0,\partial_1,\partial_2)$.

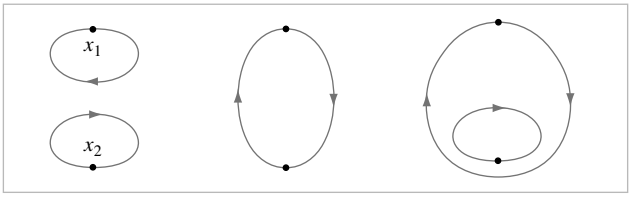


Figure 9.18 Different ways of realizing constructive particle exchange along transmutation paths.

is but the action (9.74) generalized to arbitrary gauge field configurations.[93] In the following we will investigate what information can be obtained from the functional integral Eq. (9.76) about the physical behavior of the FQH system.

Particle exchange in two dimensions

Before embarking on this program, it may be of interest to discuss a few general aspects of Chern–Simons field theory and **particle transmutation statistics**. In undergraduate quantum mechanics courses we learn to discriminate between particles with bosonic and fermionic statistics. On the formal level, the distinction between the two is met by considering the behavior of the wavefunction under an "exchange of particles," namely

$$\Psi(\ldots, x^1, \ldots, x^2, \ldots) = \pm\Psi(\ldots, x^2, \ldots, x^1, \ldots), \tag{9.78}$$

for bosons/fermions respectively. In fact, however, this definition should leave one with a certain feeling of uneasiness: what is actually meant by the phrase "exchange of particles"? Surely, the definition above does not imply a concrete physical prescription, i.e. strictly speaking it does not make sense.

To appreciate the fact that we are not just discussing a formal subtlety, let us try to give the exchange of particles a more physical meaning. (The construction below follows closely an argument from chapter I.2 of Wilczek.[94]) Consider two quantum particles occupying positions x^1, x^2 in a two-dimensional system. Suppose we were interested in computing the amplitude for these particles to re-occupy the positions x^1, x^2 after some time t. Clearly, these amplitudes receive two distinct contributions, (i) $x^1 \to x^1, x^2 \to x^2$ and (ii) $x^1 \to x^2, x^2 \to x^1$ where $x \to y$ denotes the single-particle amplitude for propagation from $x \to y$. Interpreting the second process as an operation of particle exchange, we are interested in identifying a fixed relative phase between (i) and (ii). Thinking about the total transition process in terms of a coherent double sum over single-particle paths, it is clear that contributions from (i) cannot be continuously deformed into those of type (ii). The path-sum falls into disconnected pieces, implying that there will be no variational (or classical) principles telling us about the relative phase. Yet, quantum mechanics itself provides us with

[93] As usual with gauge theories, the integral Eq. (9.76) must be interpreted in a qualified sense: the very fact that the action is gauge invariant implies that the integration over all the different gauge realizations yields the – infinite – volume of the gauge sector. To give the functional integral some meaning, a **gauge xing** contribution $S_{\rm fix}[a]$ has to be added to the action, i.e. a contribution that restricts the integration to a specific reference gauge. Mostly, however, the presence of the gauge fixing action is not indicated explicitly.

[94] F. Wilczek, *Frictional Statistics and Anyons Superconductivity* (World Scientific Publishing, 1990).

an important clue: from the point of view of the first of the two particles, the net result of a (i)-process will be a rotation of particle 2 (around the position of particle 1) by an angle $\phi \in 2\pi\mathbb{Z}$. This is illustrated for $\phi = 0$ (left) and $\phi = 2\pi$ (right) in Fig. 9.18. Conversely, a (ii)-process corresponds to a rotation by $\phi \in (2\mathbb{Z}+1)\pi$. (For $\phi = \pi$, see the center of the figure.)

Now, let us suppose that the topologically distinct processes differ by some phase $\kappa(\phi)$. If we iterate transmutation processes, $(x^1, x^2) \stackrel{t}{\to} (x^2, x^1) \stackrel{t'}{\to} (x^1, x^2)$, the winding angles add while quantum mechanics requires that the topological phases multiply, $\kappa(\phi + \phi') = \kappa(\phi)\kappa(\phi')$, implying that $\kappa(\phi) = \exp(i\psi\phi)$, where $\psi \in [0, 2]$ is some parameter.[95]

Specifically, a single exchange operation $(x^1, x^2) \to (x^2, x^1)$ corresponds to a phase $e^{i\pi\psi}$. Let us compare this with the formal exchange definition above. According to Eq. (9.78), a twofold exchange, $(x^1, x^2) \to (x^2, x^1) \to (x^1, x^2)$, leaves the wavefunction unaltered. In contrast, even the most elementary "physical" exchange procedure corresponds to a winding angle $\phi = \pi$ and, therefore, to a topological phase $\exp(2\pi i\psi)$. Only for the special choices $\psi = 0$ (bosons) or $\psi = 1$ (fermions) do we recover the result of the formal exchange. For a beautiful (yet non-path-integral-oriented) extension of the arguments above to a physically meaningful exchange prescription for general N-particle systems we refer to the seminal paper of Leinaas and Myrheim.[96] Presently, all we need to appreciate is that a constructive exchange operation appears to leave more room for non-trivial (i.e. $\neq 1, -1$) transmutation statistics. Particles with $\psi \neq 0, 1$ have been dubbed **anyons**, where the "any" stands for "any exchange statistics." Skeptical readers may justly object that the overwhelming majority of particles observed in physics are of either bosonic or fermionic type. So where, then, does anyonic exchange statistics play a role? The short answer to this question is that our discussion above was critically tied to the two-dimensional system. In three or more dimensions, the winding angle ϕ is defined only mod 2π. For example, in the three-dimensional world, you might use the dimension perpendicular to the paper plane to contract the process shown in the right hand of Fig. 9.18, $\phi = 2\pi$, to a $\phi = 0$ type process. This forces ψ to be an integer, i.e. $\psi = 0$ (bosons) and $\psi = 1$ (fermions) simply exhaust the list of possible options:[97] in $d \neq 2$ anyons do not exist.[98] Yet, in various two-dimensional applications, anyonic excitations do play an important role. For example, a theoretical approach to the FQHE competing with the CF approach discussed in the text is based on composite *bosons*.[99] Although these are not anyons in the strict sense, here, too, the exchange statistics has been externally modified (in this case to manufacture bosons from fermions). Indeed, for a while, attempts were made to link anyons to the subject

[95] Since ϕ is a multiple of π, ψ is defined only mod 2.

[96] J. M. Leinaas and J. Myrheim, On the theory of identical particles, *Il Nuovo Cimento B* **37** (1977), 1–23.

[97] In $d = 1$, particles cannot be exchanged anyway, which accounts for the interchangeability of bosonic and fermionic modelings.

[98] Another argument can be developed to the same effect: according to the spin-statistics theorem, the exchange statistics of particles is intimately tied to the quantization of angular momentum. The latter, in turn, is a direct consequence of the SU(2) commutation relations $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}J_k$: the structure of the right-hand side fixes the dimension of the irreducible representations of SU(2) and, thereby, the particle statistics. Yet, when restricted to a two-dimensional world, the algebra of angular momentum becomes one-dimensional, i.e. abelian. The absence of angular momentum quantization then implies that the spin statistics theorem loses its meaning.

[99] S. C. Zhang, T. H. Hansson, and S. Kivelson, Effective field theory model for the fractional quantum Hall effect, *Phys. Rev. Lett.* **62** (1989), 82–5.

of high-temperature superconductivity.[100] Further, there is compelling evidence that the quasi-particle excitations in quasi-one-dimensional FQHE systems show fractional statistics, etc. Finally, field theories of anyons have been considered as prototypical model systems of matter fields coupled to massive gauge field excitations. These applications are motivation enough to briefly discuss the quantitative theoretical framework of anyon dynamics in two-dimensional systems.

Let us start out from the following functional setup:

$$\mathcal{Z} = \int Da\ D\phi\ \exp\left[iS_0[\phi] + i\int d^3x\, j_\mu^\uparrow a_\downarrow^\mu + \frac{i\theta}{4} S_{\rm CS}[a]\right],$$

where either $\int D\phi = \int D(\bar\psi,\psi)$ may stand for a coherent state field integral, or $\int D\phi = \int \prod_{i=1}^N D(\mathbf{x}^i, \dot{\mathbf{x}}^i)$ for a multiple path integral over the configuration space trajectories $\{\mathbf{x}^i\}$ of N particles. In the latter case (see the exercise below), $j(x) = \sum_{i=1}^N (1, \dot{\mathbf{x}}^i)\delta(\mathbf{x} - \mathbf{x}^i(t))$ is the current density carried by the world lines of N particles.

EXERCISE Subject the first-quantized many-particle Hamiltonian $\hat H = \sum_{i=1}^N (\frac{\hat{\mathbf{p}}^{i2}}{2m} + V(\hat{\mathbf{x}}^i))$ to the singular gauge transformation (9.70). Then construct a path integral representation for the transition amplitude $\langle \mathbf{x}^1, \ldots, \mathbf{x}^N | \hat U(t) | \mathbf{y}^1, \ldots, \mathbf{y}^N\rangle$. Show that the decoupling of the nonlinear gauge field contribution acquired by the Hamiltonian exactly parallels the field integral scheme discussed above. As before, the action contains the Chern–Simons contribution Eq. (9.75) and a current vector potential coupling. The latter is given by the canonical expression

$$\int dt \sum_i (-\phi(\mathbf{x}^i) + \mathbf{x}_i \cdot \mathbf{a}_i(\mathbf{x}^i)) \quad \equiv \quad \int dt \sum_i a_\mu(\mathbf{x}^i) j^\mu(\mathbf{x}^i, \mathbf{x}^i) \equiv \int d^3x\, a_\mu(x) j^\mu(x).$$

In the first line, $j(\mathbf{x}) \equiv (1, \mathbf{x})$ denotes the $(2+1)$-dimensional current carried by a point particle while, in the second line, $j(x) \equiv \sum_i (1, \mathbf{x}^i)\delta(\mathbf{x} - \mathbf{x}^i)$ represents the corresponding current density. (Upon space integration, $\int d^2x\, j^\mu(x) = \sum_i j^\mu(\mathbf{x}^i)$.)

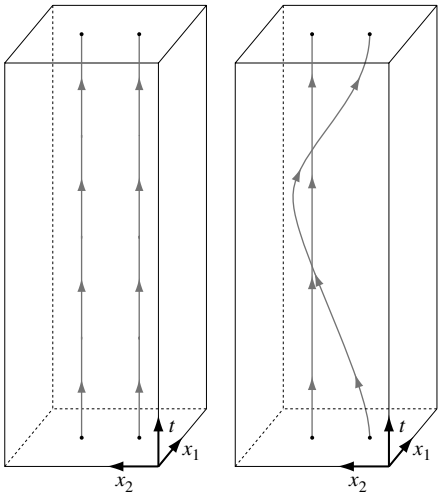
We next employ the path integral variant to show that the statistical angle θ controls the transmutation statistics of the planar particle system. Let us consider a system of $N = 2$ particles initially ($t = t_{\rm i}$) prepared to occupy the coordinates (y^1, y^2). We want to analyze the transition amplitude $\langle y^1, y^2 | \hat U(t_{\rm f}, t_{\rm i}) | y^1, y^2\rangle$ to reoccupy the same state at a time $t_{\rm f}$. As in our qualitative construction above, the path integral describing this amplitude receives contributions from topologically distinct pairs of trajectories. (The world lines of two trajectories with winding angle $\phi = 0$ (left) and $\phi = 2\pi$ (right) are shown in the figure.)

To explore the impact of the statistical vector potential on the transition amplitude it is convenient to switch to an imaginary time formalism. Upon analytic continuation $t \to -i\tau$, $a_0 \to i a_0$, the path integral for the two transition amplitudes assumes the form

$$\mathcal{Z} = \int D(\mathbf{x}^1, \mathbf{x}^2)\ \exp\left[-S_0[\phi] + \int d^3x\, j_\mu a_\mu - \frac{i\theta}{4} S_{\rm CS}\right],$$

[100] R. B. Laughlin, The relationship between high-temperature superconductivity and the fractional quantum Hall effect, *Science* **242** (1988), 525–33.

where $j_\mu = i\frac{\partial x_\mu(\tau)}{\partial\tau}\delta(\mathbf{x}-\mathbf{x}(\tau))$ is the $(2+1)$-dimensional imaginary time current. The most important consequence of the transition $t \to -i\tau \in -i[0,\beta]$ is that the initial and final points of our two space-time trajectories now have to be identified (the usual temporally periodic boundary conditions of the imaginary time formulation[101]). A glance at the figure shows that this leads to a pair of two closed world line curves which, for a winding angle $\phi = 2\pi n$, are n-fold intertwined.[102] Focusing on the two most elementary variants $\phi = 0$ and $\phi = 2\pi$, we next evaluate the Chern–Simons action on these trajectory pairs.



We begin by integrating out the statistical vector potential. For now, it will be convenient to perform this integration in the radiation gauge, $\partial_\mu a_\mu = 0$. (Remember that, to integrate over a gauge field, a specific gauge has to be chosen.) This gauge can be selected by adding a **gauge fixing** contribution $\alpha\int d^3x\,(\partial_\mu a_\mu)^2$ to the action. In the limit $\alpha\to\infty$, configurations with $\partial_\mu a_\mu$ no longer contribute to the integration and the gauge is effectively fixed. Equivalently, one may limit the integration from the outset so that only the two components of $a(q)$ perpendicular to q are integrated over. Either way, one finds (exercise) $\langle a_\mu(q)a_\nu(q')\rangle_a = \frac{2}{\theta}\frac{\epsilon_{\mu\nu\sigma}q_\sigma}{q^2}\delta_{q+q'}$, where $\langle\cdots\rangle_a$ denotes functional averaging over the (imaginary time) Chern–Simons action. Using this result, we obtain

$$\begin{aligned}
\mathcal{Z} &= \int D(\mathbf{x}^1,\mathbf{x}^2)\,\exp\left[-S_0[\phi]+\frac{1}{2}\int d^3x\;d^3x'\,j_\mu(x)j_\nu(x')\,\langle a_\mu(x)a_\nu(x')\rangle_a\right]\\
&= \int D(\mathbf{x}^1,\mathbf{x}^2)\,\exp\left[-S_0[\phi]+\frac{1}{2}\sum_{qq'}\,j_\mu(q)j_\nu(q')\,\langle a_\mu(q)a_\nu(q')\rangle_a\right]\\
&= \int D(\mathbf{x}^1,\mathbf{x}^2)\,\exp\left[-S_0[\phi]+\frac{\epsilon_{\mu\nu\sigma}}{\theta}\sum_q\,j_\mu(q)\frac{q_\sigma}{q^2}j_\nu(-q)\right].
\end{aligned}$$

Let us now try to understand the meaning of the last term in the action, the remnant of the Chern–Simons integration. In fact, this term will turn out to be a topological invariant which tells us about the degree of knotting of the two integration paths. To see this, let us define

$$\tilde{j}_\mu \equiv -ij_\mu = \partial_\tau\sum_{a=1,2}x^a_\mu(\tau)\delta(x_1-x^a_1(\tau))\delta(x_2-x^a_2(\tau)),$$

where $x=(\tau,x_1,x_2)$. Temporarily forgetting about the time-like origin of the first component, we may think of $\tilde{j}$ as the current vector field created by two loops in three-dimensional

[101] For the present, however, the parameter β does not carry any physical significance.
[102] For the moment, we do not consider odd multiples $\phi=(2n+1)\pi$ (corresponding to a particle *exchange*) since for $y^1\neq y^2$ these cases do not have a meaningful imaginary time extension.

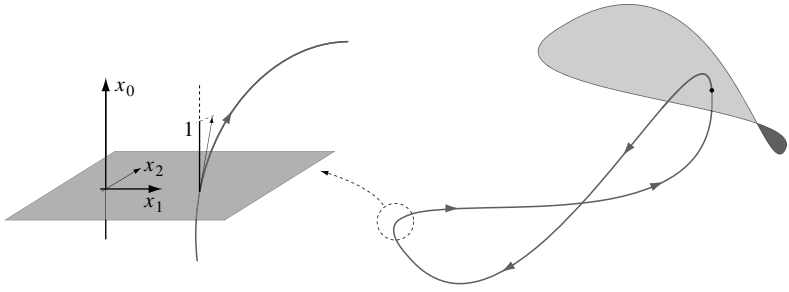


Figure 9.19 Two intertwined current loops in (2+1)-dimensional Euclidean space–time.

space, each carrying a current $I = 1$ of unit[103] strength (see Fig. 9.19). The key point now is that, according to **Ampère's law**, $b_\mu = -i\epsilon_{\mu\nu\sigma}\frac{q_\nu}{q^2}\tilde{j}_\sigma$ is the magnetic field created by a static[104] current distribution $\tilde{j}$. We may thus write the topological contribution to the action as

$$\begin{aligned} S_{\rm top}[x^1, x^2] &= -\frac{i}{\theta}\sum_{aa'}\int d^3x\, \tilde{j}^a_\mu(x) b^{a'}(x) = -\frac{i}{\theta}\sum_{aa'}\int d\tau\ \frac{dx^a(\tau)}{d\tau}\cdot b^{a'}(x) \\ &= -\frac{i}{\theta}\sum_{aa'}\oint_{\gamma^a} ds\cdot b^{a'} = -\frac{i}{\theta}\sum_{aa'}\int_{S^a} dS\cdot \mathrm{curl}\, b^{a'} = -\frac{i}{\theta}\sum_{aa'}\int_{S^a} dS\, \tilde{j}^{a'} \\ &= -\frac{2i}{\theta} I(\gamma^a, \gamma^{a'}). \end{aligned}$$

Here, γ^a is a shorthand for the curve $\{\tau, x^a_1, x^a_2\}$, and S^a a surface spanned by γ^a. The crucial last line states that the topological action is proportional to the current $I(\gamma^a, \gamma^{a'})$ flowing through the area spanned by loop a' due to the presence of loop a (see Fig. 9.19). Obviously, $I(\gamma^a, \gamma^a) = 0$. For $a \neq a'$, $I(\gamma^a, \gamma^{a'}) \in \mathbb{Z}$ is equal to the number of times γ^a pierces $S^{a'}$ or, in other words, the degree to which the two loops are intertwined.

Specifically, amplitudes where one of the particles encircles the other n times acquire a phase $\kappa(2\pi n) = e^{2\pi i n\psi}$, where $\psi = 1/(\pi\theta)$. We have thus found that the Chern–Simons action does the book-keeping of the anyonic exchange phases discussed qualitatively above. In fact, it is relatively straightforward to extend our present two-particle analysis to N particles, or to the fully-fledged formalism of the coherent state field integral. In the specific case of the Chern–Simons field theory of the FQHE, $\theta = (2\pi s)^{-1}$, i.e. fermions get transformed to (composite) fermions. In general, however, θ may be tuned so as to generate any form of exchange statistics.

[103] To compute the strength of the current, one may integrate $\tilde{j}$ over a space-like surface $x_0 = \tau = \text{const.}$ intersecting the loop (see Fig. 9.19). The integral gives $I = \int dx_1\, dx_2\, \tilde{j}_0(x) = \tilde{j}_0(\tau, x_1(\tau), x_2(\tau)) = 1$.

[104] Keep in mind that, in our present picture, the 0-direction of space no longer carries the significance of time.

9.5.3 Chern–Simons field theory II: analysis

Equation (9.76) defines an exact reformulation of the FQHE field integral in terms of a Chern–Simons action. Trusting that the CS gauge field degree of freedom is sufficiently benign, we now proceed to our familiar program "mean-field + fluctuations." We begin by subjecting the Coulomb interaction $S_{\rm int}$ to a Hubbard–Stratonovich transformation,

$$e^{iS_{\rm int}[\bar\psi,\psi]} = \int D\sigma\, e^{\frac{i}{2}\int d^3x\, d^3x'\, \sigma(x)[V^{-1}(\mathbf{x}-\mathbf{x}')\delta(x_0-x_0')]\sigma(x')+i\int d^3x\,(\hat\rho(x)-\rho_0)\sigma(x)},$$

where V^{-1} denotes the inverse of the interaction kernel. Integration over the – now Gaussian – CF degrees of freedom ψ then brings us to the partition function, $\mathcal{Z} = \mathcal{N}\int Da\; D\sigma\, e^{iS[a,\sigma]}$, where

$$S[a,\sigma] = -i\,\mathrm{tr}\ln\left[i\partial_0+\mu-\phi-\sigma+\frac{1}{2m}(-i\nabla+\mathbf{A})^2-V\right] - \rho_0\int d^3x\,\sigma(x)$$
$$+\frac{1}{2}\int d^3x\, d^3x'\,\sigma(x)V^{-1}(\mathbf{x}-\mathbf{x}')\delta(x_0-x_0')\sigma(x') + \frac{\theta}{4}S_{\rm CS}[a]. \qquad (9.79)$$

Starting from this representation, we now subject the theory to a mean-field analysis.

Mean-field equations

Let us seek for solutions of the equations

$$\left.\frac{\delta S[a,\sigma]}{\delta a_\mu(x)}\right|_{\bar\sigma,\bar a} = \left.\frac{\delta S[a,\sigma]}{\delta\sigma(x)}\right|_{\bar\sigma,\bar a} = 0.$$

Explicitly performing the differentiation with respect to $a_0=\phi$, one may see that the first of these equations translates to the – by now familiar – form

$$\rho[\bar a,\bar\sigma] = \frac{1}{4\pi s}\bar b, \qquad (9.80)$$

where $\rho[a,\sigma](x) = i(i\partial_0+\mu-\phi-\sigma+\frac{1}{2m}(-i\nabla+\mathbf{A})^2-V)^{-1}(x,x)$ denotes the local density of CFs, and the notation emphasizes the functional dependence of ρ on a and the Hubbard–Stratonovich potential σ. The differentiation with respect to the space-like components $\mathbf{a}$ does not yield independent new information; all it gives us is two relations expressing the compatibility of Eq. (9.80) with the continuity equation. Finally, differentiation with respect to σ gives

$$\sigma(x) = -\int d^2x'\, V(\mathbf{x}-\mathbf{x}')\;(\rho[\bar a,\bar\sigma](x')-\rho_0)|_{x_0'=x_0}\,. \qquad (9.81)$$

This equation also affords a transparent interpretation: on the mean-field level the potential $\sim\int V\,[\rho-\rho_0]$ created by local density fluctuations compensates the interaction potential. In the absence of fluctuations of the external potential $V(\mathbf{x})$, Eq. (9.80) and (9.81) possess the obvious homogeneous solution

$$\rho[\bar a,0] = \rho_0 = \text{const.},\quad \sigma=\phi=0,\quad \bar b = 4\pi s\rho_0 \qquad \mathbf{a} = 2s\nu\mathbf{A}_{\rm ext}.$$

The particle density is homogeneous, implying that no mean-field interaction σ is generated. The strength of the equally homogeneous CS mean-field $\bar b$ is set by the average density ρ of

composite fermions. As expected, the effective field seen by the CFs, $B = B_{\rm ext} - b$, turns out to be lower than that of the external magnetic field. Specifically, for a half-filled band, $\nu = 1/2$, $\rho_0 = B_{\rm ext}/4\pi$. In this case, CFs with two flux quanta attached, $s = 1$, experience a mean-field B of vanishing strength.

Now let us investigate the stability of the mean-field with respect to fluctuations. In many respects the N CFs behave like ordinary fermions subject to a perpendicular field of strength B. In particular, they will undergo a Landau quantization where, typically, the highest Landau level will be only partially occupied. Foreseeably, these generic configurations will be highly susceptible to all sorts of fluctuation effects (i.e. poor candidates for mean-field schemes): the massive degeneracy of the Landau levels leads to "small-energy denominators," so that even a slight perturbation/fluctuation may cause dramatic effects. We thus expect our present mean-field scheme to work only in the vicinity of filling fractions ν where an integer number p of CF Landau levels are fully occupied. These filling fractions are determined by the equation $\nu_{\rm eff} = p$ or $\Phi_{\rm eff} = 2\pi N/p$, where $\Phi_{\rm eff} = BL^2$ denotes the total effective flux piercing the system. Using the relation $B = B_{\rm ext} - b$ where $b = 4\pi s N L^{-2}$, it is straightforward to solve these equations to obtain

$$\nu = \frac{2\pi N}{B_{\rm ext} L^2} = \frac{p}{2sp+1},$$

in agreement with the experimental observation. Summarizing, the mean-field analysis of the Chern–Simons action confirms the basic expectation formulated on page 571 that

> the fractional QHE can be interpreted as an integer QHE of composite fermions.

INFO There is one more aspect of the theory that can be explored on the level of plain mean-field theory, namely the **charge of the composite fermions.** To this end, let us consider the functional expectation value of the operator

$$\mathcal{O}(x,x') \equiv \langle \bar\psi(x)\psi(x')\rangle\,.$$

The correlation function $\mathcal{O}$ describes the amplitude for creation of a CF at a space-time point x and its annihilation at x'. Let us regard this amplitude as the coherent sum over all paths γ connecting x and x'. Due to the presence of the external magnetic field, each such contribution acquires an Aharonov–Bohm phase $\phi_\gamma \equiv -q_{\rm eff} \int_\gamma dx_\mu\, A^\mu_{\rm ext}$, where the coefficient $q_{\rm eff}$ defines the effective charge of the particle. (As we shall see, the presence of the statistical field makes $q_{\rm eff}$ different from the bare electron value $q = 1$.) Performing the Gaussian integral over ψ, we represent $\mathcal{O}$ as

$$\begin{aligned}
\mathcal{O}(x,x') &= \left\langle \langle x|(i\partial_0 + \mu - \phi - \sigma + \frac{1}{2m}(-i\nabla + \mathbf{A})^2 - V)^{-1}|x'\rangle \right\rangle_{a,\sigma} \\
&= \langle\langle \mathbf{x}|U(t,t')|\mathbf{x}'\rangle\rangle_{a,\sigma} \\
&= \left\langle \int_{\substack{x(t)=x\\ x(t')=x'}} Dx\, \exp\left[iS_0[x,\sigma] - i\int dt\ (\phi - \mathbf{A}\cdot\mathbf{x})\right]\right\rangle_{a,\sigma} \\
&\overset{\rm MF}{\approx} \text{const.} \times \int\limits_{\substack{x(t)=x\\ x(t')=x'}} Dx \exp\left[iS_0[x,0] - iq_{\rm eff}\int d\mathbf{x}\cdot\mathbf{A}_{\rm ext}\right].
\end{aligned}$$

In the first line, the angular brackets denote the functional averaging over the Chern–Simons action (a) and the interaction kernel (σ). Further, we have made use of the fact that the (retarded) real time Green function is equal to the time evolution operator U. The latter is represented as a *path* (as opposed to a "field") integral. The action of the path integral contains a field-independent contribution S_0, the coupling to the scalar field component ϕ, and the canonical coupling to the vector potential $\mathbf{A}$. Until now, all manipulations have been exact. In the crucial last line, we then evaluate the (a,σ)-integration in a mean-field approximation. Assuming that fluctuations (lumped into the "const." in front of the integral) will be small, this amounts to a substitution $(a,\sigma) \to (\bar{a},0)$ in the action. Using the fact that $\mathbf{a}_{\rm ext} = 2s\nu\mathbf{A}_{\rm ext}$, we arrive at the last line, where the coupling constant

$$q_{\rm eff} = 1 - 2s\nu = \frac{1}{1+2sp}, \tag{9.82}$$

is identified as the effective charge of the CF. The line of reasoning above tells us that CFs effectively carry **fractional charge**. The partial "screening" of the bare electron charge, $0 < q_{\rm eff} < 1$, is explained by the tendency of the phase vortices to expel electronic charge. In the specific case, $\nu = 1/2 \rightsquigarrow q_{\rm eff} = 0$, the electron charge becomes completely screened by $2s = 2$ vortices, a result that led some authors[105] to interpret the $\nu = 1/2$ FQHE as a dynamic phenomenon of **charge dipoles** – each comprising one electron ($q = 1$) and two vortices ("$q = -2 \times 1/2 = -1$").

Fluctuations

We now proceed to explore the role of quadratic fluctuations around the homogeneous mean-field. In particular:

- ▷ What does the field theory have to say about the electromagnetic response of the system?
- ▷ What else (beyond the fractional transmutation statistics) does it tell us about the microscopic features of the composite fermions?

To answer these questions, we do not have to go into much quantitative detail. Rather the ubiquitous condition of gauge invariance and the presence of an excitation gap (an integer number of CF Landau levels are fully occupied!) suffice to fix the structure of the quadratic action.

Shifting $a \to \bar{a} + a$ and $\sigma \to \bar{\sigma} + \sigma$, our goal is to expand the action to second order in the deviations (a,σ). Let us begin by considering the interaction field σ. A shift $a_0 = \phi \to \phi + \sigma$ removes σ from the "tr ln" component of the action and makes it reappear in the Chern–Simons action. More specifically, the Chern–Simons acquires a linear contribution[106] $S_{\rm CS}[a] \to S_{\rm CS}[a] - 2\int d^3x\, \sigma b$ so that the total σ-expansion of the action now reads

$$S[a,\sigma] = S[a] - \frac{\theta}{2}\int d^3x\, \sigma(x) b(x) + \frac{1}{2}\int d^3x\; d^3x'\, \sigma(x) V^{-1}(\mathbf{x}-\mathbf{x}')\delta(x_0 - x_0')\sigma(x').$$

[105] A. Stern, B. I. Halperin, F. von Oppen, and S. H. Simon, Half-filled Landau level as a Fermi liquid of dipolar quasiparticles, *Phys. Rev. B* **59** (1999), 12547–67.

[106] Here $b = \epsilon_{ij}\partial_i a_j$ describes the *uctuations* of the statistical magnetic field around its mean value $\bar{b}$.

After the straightforward Gaussian integration over σ we then obtain

$$S[a,\sigma] \xrightarrow{\int D\sigma} S[a] + \frac{1}{2}\int d^3x\; d^3x'\, b(x)V(\mathbf{x}-\mathbf{x}')\delta(x_0-x_0')b(x').$$

The physical interpretation of the induced term is obvious: in the CS theory, fluctuations of the particle density are tied to fluctuations of the statistical magnetic field. Accordingly, spatial inhomogeneities of the statistical magnetic field get penalized by an interaction contribution, as described by the second term above. However, as far as the two basic questions above are concerned, we may temporarily forget about this contribution (see, however, the concluding remarks below) and focus attention on the action $S[a]$.

Let us, then, consider the action $S[a,A'] \equiv S_{\rm CF}[\bar{a}+a+A'] + \frac{\theta}{4}S_{\rm CS}[\bar{a}+a]$, where

$$S_{\rm CF}[a] = -i\,\mathrm{tr}\ln\left(i\partial_0 + \mu - a_0 - \frac{1}{2m}(-i\nabla - \mathbf{a})^2 - V\right),$$

denotes the CF contribution to the action to which we have coupled a source potential A'. As usual, a two-fold differentiation with respect to elements of A' will later tell us about the relevant transport coefficients of the system. Next, we develop the formal expansion $S[a,A'] \equiv \sum_n S^{(n)}[a,A']$, where $S^{(n)}[a,A']$ is of total order n in a and A'. The zeroth-order term $S^{(0)}$ describes the CF system on the mean-field level and will not be of further interest to us. The first-order term $S^{(1)}$ does not contain a as we are expanding around a stationary configuration. Moreover, its A dependence $S^{(1)}[A] = iA_\mu\bar{j}^\mu$ is inessential because the mean-field CF density $\bar{j}^0$ is structureless and a mean-field current $\bar{\mathbf{j}} = 0$ does not flow. However, at the second-order level $S^{(2)}$, things start becoming more interesting. Formally, the second-order contribution can be represented as

$$S^{(2)}[a,A'] = \frac{1}{2}\int d^3x\; d^3x'\,(a+A')_\mu(x)K_{\mu\nu}(x,x')(a+A')_\nu(x') + \frac{\theta}{4}S_{\rm CS}[a], \tag{9.83}$$

where $K_{\mu\nu}(x,x') = \frac{\delta^2 S_{\rm CF}[a]}{\delta a_\mu(x)\delta a_\nu(x')}\Big|_{a=\bar{a}}$. By construction, K is but the linear response kernel discussed in Section 7.2. For the moment, all we need to recall about this object is that it (a) is gauge invariant, $\partial_\mu K^{\mu\nu} = K^{\mu\nu}\;\partial_\nu = 0$, (b) is generally short-range ($K(q)$ can be expanded in powers of q), and (c) contains information about both the polarizability of the medium and its conductivity.

Building on property (b), one can expand the second-order action in derivatives: $S^{(2)}[a,A'] = \sum_{l=0}^{\infty} S^{(2,l)}[a,A']$, where $S^{(2,l)}[a,A']$ is of lth order in derivatives (∂_0,∂_x) and, as usual, one may focus attention on the contribution with the least number of derivatives. As discussed before, no gauge-invariant zero derivative term $S^{(2,0)}$ can be constructed. However, with the contribution linear in the number of derivatives, $S^{(2,1)}$, the situation is more tricky. While in general there are no gauge invariant contributions of first order in $q \leftrightarrow -i\nabla$, terms nominally scaling as $|q|$ for $q \to 0$ exist. (For example, in systems with a non-vanishing longitudinal conductivity, the action takes the form $S^{(2,1)}[A'] \sim A'_\mu(q)\frac{q^\mu q^\nu - q^2}{Dq^2 + i\omega}A'_\nu(-q)$, where D denotes the diffusion constant.) However, the system at hand is an insulator, so that no such term is present.

Yet, in two-dimensional systems, there exists one more gauge-invariant first-order derivative term, namely the notorious Chern–Simons term! Thus, we are led to the preliminary

identification $S^{(2,1)}[a,A'] = c\,S_{\rm CS}[a+A'] + \frac{\theta}{4}S_{\rm CS}[a]$, where the first contribution is obtained by a first-order gradient expansion of the CF action while the second contribution has been present from the outset. Of course, the coupling constant c remains to be determined. This can be done by explicit – and quite laborious – calculation or by the following more elegant construction: remembering that $S_{\rm CS}[a] = -\int d^3x\,\epsilon_{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma = -i\sum_q \epsilon_{\mu\nu\sigma}a_\mu(q)q_\nu a_\sigma(-q)$, and comparing with Eq. (9.83), our findings so far translate to the relation

$$K_{\mu\nu}(q) = -i2c\,\epsilon_{\mu\sigma\nu}q_\sigma + \mathcal{O}(q^2). \tag{9.84}$$

INFO To fully describe the **electromagnetism of the statistical gauge field**, we should also include the second-order derivative contribution $S^{(2,2)}$ in our analysis: gauge-invariant contributions of second order are obtained as bilinears formed from elements of the field tensor $F_\mu = \partial_\mu a' - \partial a'_\mu$. These terms are then combined into an action $S^{(2,2)} = \frac{1}{2}\int d^3x\,(\epsilon\mathbf{e}'\cdot\mathbf{e}' + \chi b^2)$, where $e'_i = \partial_0 a'_i - \partial_i a'_0$ is the electric field derived from the gauge field a', and ϵ and χ are the electric and magnetic permeabilities, respectively. Physically, the action $S^{(2,2)}$ describes the electric and magnetic susceptibility of the (mean-field) CF medium to the presence of gauge field fluctuations. However, as far as our present objectives are concerned, these effects turn out to be of lesser relevance.

On the other hand, we know the conductivity of the system relates to the linear response kernel through the relation $\sigma^0_{12} = -i\lim_{q\to 0}\omega^{-1}K_{12}(\omega,\mathbf{q})$. For the present, we have to interpret $\sigma^0_{12} = \frac{p}{2\pi}$ as the quantized Hall conductivity carried by the CF system at the mean-field level. Comparison with Eq. (9.84) then leads to the identification $c = \sigma_{12}/2$, or

$$S^{(2,1)}[a,A'] = \frac{\sigma^0_{12}}{2}\,S_{\rm CS}[a+A'] + \frac{\theta}{4}S_{\rm CS}[a].$$

Of course, σ^0_{12} does not coincide with the actual Hall conductance σ_{12} carried by the system – within our present level of approximation, the latter is obtained by two-fold differentiation with respect to A' *after* the statistical gauge field has been integrated out. Now, there is a simple general formula telling us what happens to two *quadratic* actions (such as the Chern–Simons actions) upon integration over one half of the fields:

$$c_1 S[a+b] + c_2 S[a] \overset{\int Da}{\longrightarrow} \left(c_1^{-1} + c_2^{-1}\right)^{-1} S[b]. \tag{9.85}$$

EXERCISE Of course, Eq. (9.85) can be proven by straightforward Gaussian integration over a. A more elegant procedure is based on the fact that, for any *quadratic* action, $S[a,b] \overset{\int Da}{\longrightarrow} S[b,\bar{a}[b]]$, where $\bar{a}[b]$ is the solution of the mean-field equation $\delta_a S[a,b]\big|_{a=\bar{a}} = 0$. To make use of this identity, let us formally write $S[a] = a^T K a$, where K is a non-degenerate[107] operator kernel. Find the solution of the mean-field equations corresponding to the left-hand side of Eq. (9.85) and show that, upon substitution back into the action, we obtain the right-hand side. If you are critical, you may object that our CS actions are not, in fact, non-degenerate: when evaluated on

[107] Otherwise the field integration would create a headache!

a pure gauge configuration $a_\mu = \partial_\mu f$, they vanish. Nevertheless, you may convince yourself that Eq. (9.85) remains valid if the integration over a is performed in some fixed reference gauge.

When applied to our action $S^{(2,1)}$ above, the auxiliary identity Eq. (9.85) tells us that

$$S^{(2,1)}[a, A'] \stackrel{\int Da}{\longrightarrow} S_{\rm eff}[A'] \equiv \frac{\sigma_{12}}{2} S_{\rm CS}[A'],$$

where $\sigma_{12} \equiv (\frac{1}{\sigma^0_{12}} + \frac{2}{\theta})^{-1}$. The terminology "$\sigma_{12}$" is justified because, as seen above, the constant in front of a CS source action is but the **Hall conductance** of the system. Recalling that $\theta = 1/2\pi s$ and $\sigma^0_{12} = p/2\pi$, we find

$$\sigma_{12} = \frac{1}{2\pi} \frac{p}{1 + 2sp},$$

i.e. we confirm the expectation of fractional Hall quantization. Let us recall that this result is critically linked to the absence of a **longitudinal conductance** σ_{11}. In our analysis above, alluding to the striking analogies that exist between CFs and ordinary electrons in a magnetic field, we simply *postulated* $\sigma_{11} = 0$ at CF filling p. (Technically, this happened when we said that the CF "tr ln" does not support a longitudinal current–voltage relation.) However, one may note that the actual problem of **Anderson localization of CFs** has not yet been attacked on a truly microscopic level. What makes the problem so difficult is the massive inter-CF correlations induced by fluctuations of the statistical gauge field. In the absence of disorder, the full extent of this correlation mechanism does not yet become clear (at least not on the level of our simplistic "mean-field plus quadratic fluctuations" analysis). However, once external inhomogeneities are present, things instantly become more complicated. For example, a static impurity potential will be screened by an inhomogeneous CF distribution. This creates an accumulation of statistical flux that in turn acts as a scattering center of CFs, etc. Thus, we readily wind up with a full-blown problem "interaction + disorder + strong magnetic field" whose rigorous microscopic solution seems to be elusive. Nonetheless, all evidence suggests that eventually the CFs will be localized, so that the analogy (FQHE of fermions) $\leftrightarrow$ (IQHE of composite fermions) remains valid.

INFO As mentioned above, the field theory approach to the FQHE was preceded by a number of other developments. Perhaps most importantly, shortly after the experimental discovery of the effect, Laughlin proposed a trial wavefunction – nowadays generally referred to as **Laughlin s wavefunction**[108] – which did a fantastic job at explaining much of the phenomenology of the FQHE.

With the benefit of hindsight (!) it is not difficult to motivate the structure of the Laughlin wavefunctions from a few simple considerations. Consider a clean FQHE system at a filling fraction $\nu = 1/(2s+1)$ belonging to the principal series.[109] Any ground state many-body eigenfunction should (a) be built by superposition of single-particle states belonging to the lowest

[108] R. B. Laughlin, Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations, *Phys. Rev. Lett.* **50** (1983), 1395–8.

[109] For an extension to more complex fractions, see F. D. M. Haldane, Fractional quantization of the Hall effect: a hierarchy of incompressible quantum fluid states, *Phys. Rev. Lett.* **51** (1983), 605–8, and B. I. Halperin, Statistics of quasiparticles and the hierarchy of fractional quantized Hall states, *Phys. Rev. Lett.* **52** (1984), 1583–6.

Landau level, (b) optimally account for the effects of Coulomb repulsion (recall that the strongest player in the Hamiltonian, the kinetic energy, is completely degenerate), and (c) obey Fermi statistics.

As to (a), we have seen in Section 9.3.4 that any function of the form $\psi(z) = f(z)\exp(-|z|^2/(4l_0^2))$ automatically belongs to the lowest Landau level, provided that $f(z)$ is analytic.[110] So let us seek a many-body wavefunction of the form

$$= F(\{z_i - z_j\})e^{-\frac{1}{4l_0^2}\sum_i |z_i|^2},$$

where F must depend only on differences of coordinates (the translational invariance of the clean system). The antisymmetry of the wavefunction (c) requires that F is skew-symmetric under any exchange $z_i \quad z_j$. Also, (b) F should vanish whenever two coordinates approach each other. Taken together, (a)–(c) motivate the *ansatz*



Robert B. Laughlin 1950–
Theoretical condensed matter physicist. Laughlin was awarded one third of the 1998 Nobel Prsize for his groundbreaking contributions to the explanation of the FQHE. The complementary share went to the experimentalists **Horst L. Störmer** (1949–) and **Dan C. Tsui** (1939–) for the experimental discovery of the effect.

$$(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = \mathcal{N} \prod_{i<j} (z_i - z_j)^m e^{-\frac{1}{4l_0^2}\sum_i |z_i|^2} \tag{9.86}$$

with some *a priori* undetermined integer coefficient m. Equation (9.86) defines Laughlin's wavefunction. Notice its high degree of universality (except for the integer m, the trial wavefunction does not contain a single adjustable parameter!) and the simplicity of its structure. Nonetheless, the *ansatz* Eq. (9.86) offers a straightforward explanation of many observable features of the FQHE. Specifically, it can be shown that:

The integer m relates to the filling factor as $m = \nu^{-1} = 2s + 1$.
For certain types of short-range interaction, ψ is an exact ground state of the Hamiltonian. Numerical analyses have shown that, even for the long-range Coulomb interaction, ψ has a close to perfect overlap with the exact ground state.
Single-particle excitations superimposed on ψ are gapped and fractionally charged.
In fact, one may readily rewrite ψ in a way that suggests an interpretation in terms of composite fermions; simply factor out a power $(z_i - z_j)^{m-1}$ to obtain

$$\psi = \prod_{i<j} (z_i - z_j)^{2s} \psi\big|_{m=1}.$$

Here, $\psi_{m=1}$ is the wavefunction at an integer filling factor while the prefactor adds $2s$ winding phases to each particle coordinate. In other words, the prefactor converts fermions to composite fermions, so that ψ can be interpreted as a CF wavefunction at integer filling. Indeed, this anticipates the field theoretical picture constructed above.

[110] To be precise, in Section 9.3.4, we considered wavefunctions $\psi(z) = f(\bar{z})\exp(-|z|^2/(4l_0^2))$ with an anti-holomorphic prefactor. One can switch from one form to the other by inversion of the external field $B \to -B$. For notational simplicity, we shall use the analytic form throughout.

9.6 Summary and outlook

This concludes our introduction to topological quantum field theory.[111] We have seen the way "large-scale" geometric structures of quantum fields may lead to intriguing physical phenomena – phenomena, in fact, whose understanding required the application of the entire spectrum of field integral techniques developed earlier in this text. Once again, this chapter could provide no more than a very preliminary impression of the diversity of topology-related *quantum* phenomena in condensed matter physics. Given the rapid development of the field, we encourage readers motivated to deepen their knowledge in topological condensed matter field theory to turn to the original literature. (For an excellent particle-physics-oriented introduction we refer to Ryder.[13] A comprehensive discussion of topological textures in classical physics can be found in Efetov.[112])

9.7 Problems

9.7.1 Persistent current of a disordered ring

In the main text we have argued that, contrary to a long-standing belief, the presence of disorder does not conflict with the formation of persistent currents in normal metal rings. Here we support this assertion by a microscopic calculation.

Consider a quasi-one-dimensional ring of circumference $L \gg \ell$ and transverse extension $\lambda_F \ll L_\perp < \ell$, where ℓ is the mean free path. We assume that the ring is pierced by a magnetic flux . Using the fact that $L_\perp/L \ll 1$, variations in transverse direction of the vector potential $\mathbf{A}$ may be neglected, i.e. $\mathbf{A} = (\Phi/L)\mathbf{e}_\parallel$, where $\mathbf{e}_\parallel$ is the unit vector in the longitudinal direction and units $e = \hbar = 1$ are used. We are interested in the typical value $I_{\rm typ} \equiv \sqrt{\langle I^2\rangle}$ of the persistent current $I = -\partial_\Phi F$, where F is the flux-dependent free energy.[113] Assuming that the ring is metallic in the sense that its dimensionless conductance $g \gg 1$, we will be content with computing the first contribution to $I_{\rm typ}$ in an expansion in powers of g^{-1}. For notational simplicity, we assume the flux Φ to be measured in units of Φ_0 throughout.

(a) Represent $\langle I(\Phi_1)I(\Phi_2)\rangle$ as a correlation function of two non-interacting Green functions. Without going into quantitative detail, convince yourself that the dominant contribution to the correlation function is given by the sum of diagrams shown in Fig. 9.20, where the dots denote the operator $\left(-i\partial_\parallel - (\Phi/L)\right)/m \equiv -\hat{v}_\parallel$ and $\partial_\parallel$ is the derivative along the ring. Estimate the relative contributions of the four diagrams. (Hint: Recall the derivation of low-momentum diffusion modes in Section 6.5. Owing to the thinness of the ring, fluctuations of the diffuson/cooperon modes in the radial direction may safely be neglected.)

[111] Strictly speaking, the terminology "topological quantum field theory" is reserved for field theories whose behavior is *solely* determined by topological terms. However, we use it here in a more liberal sense to denote field theories where topological aspects play a significant role.

[112] K. B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, 1996).

[113] In the presence of static disorder, the sign of I sensitively depends on the impurity configuration, i.e. $\langle I\rangle \ll I_{\rm typ}$ is much smaller than the typical value of the current for a fixed impurity configuration.

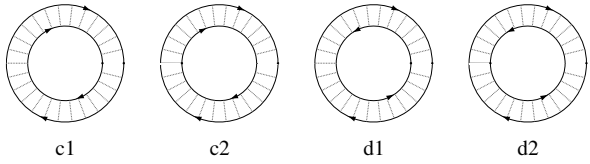


Figure 9.20 Four diagrams contributing at leading order to the correlation function $\langle I(\Phi_1)I(\Phi_2)\rangle$ of the persistent current in a disordered ring.

(b) To quantitatively compute the correlation function, we employ the formalism of the field integral. Using the fact that $F = -T\lim_{R\to 0}\frac{1}{R}(\mathcal{Z}^R - 1)$, construct a generalization of the nonlinear σ-model action $S[Q]$ suitable to compute the current correlation function. (Hint: Introduce a two-component index $t = 1, 2$ discriminating between the two currents, and derive the form of the vector potential by minimal substitution, cf. Eq. (6.64), i.e. by recalling that the flux dependence of the Hamiltonian may be formally removed by a gauge transformation.)

(c) As in Problem 8.8.4, introduce the parameterization $Q = e^{iW}\Lambda e^{-iW} = \Lambda(1 - 2iW - 2W^2 + \cdots)$, where the generators W are given in Eq. (8.53) (block structure in Matsubara space). Notice that $B = \{B^{nat\sigma,n'a't'\sigma'}\}$ are matrices in Matsubara (n), replica (a), current (t), and time-reversal (σ) spaces. Expand the action to second order in B. It is convenient to split $B = B_d + B_c$ into contributions diagonal and off-diagonal in time reversal space. Explore the symmetries of the constituent matrices $B_{d,c}$ and perform the Gaussian integrals. Show that

$$\langle I(\Phi_1)I(\Phi_2)\rangle \simeq \partial^2_{\Phi_1,\Phi_2}T^2 \sum_{q,\omega_n>0,\omega_{n'}<0} \Gamma_{q-(\Phi_1-\Phi_2)/L,nn'} + (\Phi_1 - \Phi_2 \to \Phi_1 + \Phi_2), \quad (9.87)$$

where $\Gamma_{qnn'} \equiv \frac{1}{2\pi\nu\tau}\left[Dq^2 + |\omega_n - \omega_{n'}|\right]^{-1}$ is the diffusion mode.

(d) The final step in the evaluation of the correlation function, the summation over $q, \omega_n, \omega_{n'}$, is a good exercise in executing tricky integrals (here defined as integrals which have to be simplified by physically motivated approximations).

EXERCISE Differentiate with respect to the flux before doing the frequency/momentum summations; identify four contributions corresponding to the diagrams above in Fig. 9.20.

What prevents the expression above from vanishing is the discreteness of the momentum sum; were we able to approximate the sum by an integral, a shift $q \to q + \Phi/L$ would remove the flux dependence of the integrand. The validity of an integral approximation in turn depends on the value of $|\omega|$. For $|\omega| > (2\pi)^2 D/L^2 \equiv E_c$ much larger than the magnitude of the lowest non-vanishing quantized q-mode (physically, the Thouless energy, i.e. the inverse time of diffusion around the ring), many modes contribute to the integral and a continuum approximation is valid. Thus, the dominant contribution to

the correlation function comes from frequencies $|\omega| < E_c$, for which the discreteness of the sum really matters.[114]

Carry out the sum over momenta by the methods otherwise employed in performing frequency summations. Approximate the subsequent sum over Matsubara frequencies $|\omega| < E_c$ by an integral. Show that, for small values of the flux, $I_{typ} \sim E_c \Phi(-\ln(\Phi))^{1/2}$. (Hint: Do not try to do the frequency integral rigorously; keeping the relevant information on board, simplify the integrand to obtain a manageable integral. As to the q-summation, use the freedom to do the flux derivatives before or after the summation to obtain a contour integral that is as simple as possible.)

The most important feature of this result is that it depends only algebraically on the disorder concentration, i.e. through $E_C \sim D \sim \tau$. In contrast, the long-standing yet erroneous expectation that impurity scattering destroys the phase coherence required to maintain a stable persistent current inevitably leads to the prediction $I_{typ} \sim \exp(-L/\ell)$. For experimentally "realistic" values $L/\ell = \mathcal{O}(10^2)$, the difference between the two results is dramatic. Irritatingly, the actually observed value of the current turns out to be roughly two orders of magnitude larger than our result above. Although the origin of this discrepancy is unknown, it is clear that it cannot be resolved within the framework of a non-interacting theory.

Answer:

(a) Using the fact that the free energy of a non-interacting Fermion system is given by $F = -T\sum_n \mathrm{tr}\ln(i\omega_n - \hat{H})$, where $\hat{H} = \frac{1}{2m}(\hat{\mathbf{p}} - \mathbf{A})^2 + \hat{V}$ and $\hat{V}$ represents the single-particle potential of the problem (the sum of disorder and confining potentials), we obtain $I(\Phi) = -T\sum_n \mathrm{tr}(\hat{G}_n \partial_\Phi \hat{H}) = \frac{T}{L}\sum_n \mathrm{tr}(\hat{G}_n \hat{v}_\parallel)$. Expanding the Green function in the impurity operator and constructing ladder diagrams at one-loop order, we identify the two cooperon (c1 and c2) and two diffuson (d1 and d2) diagrams in Fig. 9.20. Focusing on the cooperon sector, we note that the ladder diagram depends only on the sum $q-(\Phi_1+\Phi_2)/L$ of the fast momenta $\mathbf{p}-(\Phi_i/L)\mathbf{e}_\parallel$ carried by the Green functions.[115] More specifically, an individual cooperon ladder contributes a factor (see Section 6.5) $\sim \Gamma_{q-(\Phi_1+\Phi_2)/L,nn'} \equiv \frac{1}{2\pi\nu\tau}\left[D(q-(\Phi_1+\Phi_2)/L)^2 + |\omega_n - \omega_{n'}|\right]^{-1}$. Up to constants, the contribution of diagram c1 is thus given by $\Gamma_{q-(\Phi_1+\Phi_2)/L,nn'}$. Turning to diagram c2, we need to take into account the fact that the vector vertices $\sim v_\parallel$ indicated by the dots are now integrated independently over the fast momenta (because they are separated by impurity lines; recall the momentum structure of a ladder diagram). In the limit of zero momentum difference, $\mathbf{q} - (\Phi_1 + \Phi_2)/L \to 0$, the two Green function insertions carrying a dot vanish (think about it). This means that diagram c2 will be proportional

[114] This statement conforms with the expectation that the minimal time $t \sim \omega^{-1}$ required to sense the flux through a disordered ring is the diffusion time, i.e. the time for a quantum particle to traverse the ring at least once.

[115] Due to the thinness of the ring, fluctuations of the cooperon in the transverse direction are negligible, i.e. the momentum q is a scalar quantity measuring fluctuations in the longitudinal direction.

to $\sim (q-(\Phi_1+\Phi_2)/L)^2\Gamma^2_{q-(\Phi_1+\Phi_2)/L,nn'}$, where the factor Γ^2 accounts for the two ladders. The diffuson diagrams d1, d2 are obtained by replacing $\Phi_1+\Phi_2\to\Phi_1-\Phi_2$.

(b) Defining $\mathcal{Z}=\int D(\bar\psi,\psi)\exp(-S[\bar\psi,\psi])$ where

$$S[\bar\psi,\psi]=\int d^dr\ d\tau\ \bar\psi^{at}(\partial_\tau+\hat H(\mathbf{A}_t)-\mu)\psi^{at},$$

and $\mathbf{A}_t=(\Phi_t/L)\mathbf{e}_\parallel$, we have the representation $\langle I(\Phi_1)I(\Phi_2)\rangle=\lim_{R\to0}\frac{T^2}{R^2}\partial^2_{\Phi_1,\Phi_2}\mathcal{Z}$. All we need to do to deduce the structure of the low-energy action is to notice that the composite index (a,t) can be identified with a "replica index" of doubled dimension $2R$. Noting that the flux dependence of the Hamiltonian may be formally (i.e. at the price of changing boundary conditions) removed by the transformation $\psi\to e^{i\Phi_t r_\parallel}$, where $r_\parallel$ is the coordinate along the ring, and recalling the discussion above Eq. (6.64), we conclude that the action is given by $S[Q]=\frac{\pi\nu}{2}\int dr_\parallel\,\mathrm{tr}\left[\frac{D}{4}\left((\partial-i[A,])Q\right)^2-\hat\omega Q\right]$, where the vector potential $\mathbf{A}=\{\delta^{at,a't'}(\Phi_t/L)\}\otimes\sigma_3^{\rm tr}$.

(c) Using the symmetry relation $W=-\sigma_2^{\rm tr}W\sigma_2^{\rm tr}\Rightarrow B^T=-\sigma_2^{\rm tr}B^\dagger\sigma_2^{\rm tr}$, we verify that

$$B_{\rm d}=E^{\rm tr}_{11}b_{\rm d}-E^{\rm tr}_{22}b^*_{\rm d},\qquad B_{\rm c}=E^{\rm tr}_{12}b_{\rm c}+E^{\rm tr}_{21}b^*_{\rm c},$$

where $b_{\rm d,c}=\{b^{nat,n'a't'}_{\rm d,c}\}$ are complex matrices. Substituting this representation into the action and expanding to quadratic order, it is a straightforward matter to obtain

$$\begin{aligned}S^{(2)}[B,B^\dagger] &= \frac{L}{T\tau}\sum_{q,n>0,n'<0}\mathrm{tr}\left((b_{\rm d})^{tt'}_{nn'}\Gamma_{q-(\Phi_t-\Phi_{t'})/L,nn'}(b^\dagger_{\rm d})^{t't}_{n'n}\right)\\ &\quad+(b_{\rm d}\to b_{\rm c},\Phi_t-\Phi_{t'}\to\Phi_t+\Phi_{t'}).\end{aligned}$$

We finally integrate over the fields $b_{\rm d,c}$ and arrive at

$$\mathcal{Z}=\prod_{q,n>0,n'<0}\left(\Gamma_{q-(\Phi_1-\Phi_2)/L,nn'}\right)^{R^2}+(\Phi_1-\Phi_2\to\Phi_1+\Phi_2),$$

where the factor of R^2 counts the independent replica channels, constant factors $C^{R^2}\overset{R\to0}{\to}1$ have been omitted, and we also omitted those contributions to the integral that depend exclusively on Φ_1 or on Φ_2 (as they do not contribute to the two-fold derivative $\lim_{R\to0}R^{-2}\partial^2_{\Phi_1\Phi_2}$ [check!]). Differentiating with respect to the flux and sending $R\to0$, we obtain Eq. (9.87).

(d) Our strategy will be to do the momentum sum by contour integral methods. As it is not convenient to integrate over a function containing branch cuts (such as our logarithm), we do one of the flux derivatives, ∂_{Φ_1}, say, first. Focussing on the diffuson contribution, and introducing the shorthand notation $\Phi=\Phi_1-\Phi_2$, $|\omega|=|\omega_n-\omega'_n|$, we have

$$\begin{aligned}\partial_{\Phi_2}L^{-1}\sum_q\frac{D(q-\Phi/L)}{D(q-\Phi/L)^2+|\omega|} &= \partial_{\Phi_2}\frac{1}{4\pi i}\oint dz\coth(zL/2)\frac{D(-iz-\Phi/L)}{D(-iz-\Phi/L)^2+|\omega|}\\ &= -\frac{1}{2}\partial_{\Phi_2}\,\mathrm{Im}\coth([(|\omega|/E_{\rm c})^{1/2}+i\Phi]/2),\end{aligned}$$

where the integration contour is a circle at infinity avoiding the two poles of the integrand.[116] Notice that our result is 2π-periodic in the flux Φ, as it should be.

Turning to the frequency summation, we use the fact that $|\omega|/E_c < 1$ to approximate $\coth([(|\omega|/E_c)^{1/2} + i\Phi]/2) \simeq 2[(|\omega|/E_c)^{1/2} + i(\Phi \bmod 2\pi)]^{-1}$. This leads to the expression

$$\partial^2_{\Phi_1\Phi_2} T^2 \sum_{q,n,n'} \Theta(n)\Theta(-n')\Gamma_{q-\Phi/L,nn'} \simeq -\partial_{\Phi_2}(2\pi)^2 \operatorname{Im} \int_0^{E_c} d\omega \frac{\omega}{(|\omega|/E_c)^{1/2} + i\Phi}$$
$$= \partial_{\Phi_2}(2\pi E_c)^2 \Phi \int_0^1 dx \; \frac{x}{x+\Phi^2} = (2\pi E_c)^2 \partial_{\Phi_2}\left(\Phi - \Phi^3 \ln\left(\frac{\Phi^2+1}{\Phi^2}\right)\right).$$

For small values of the flux, the logarithm can be approximated as $-\ln(\Phi^2)$ and $\partial_{\Phi_2}\Phi^3 \ln(\Phi^2) \simeq 3\Phi^2 \ln(\Phi^2)$. Using this approximation and adding the cooperon contribution, we obtain

$$\langle I(\Phi_1) I(\Phi_2)\rangle \simeq -(2\pi E_c)^2 \left(3(\Phi_1-\Phi_2)^2 \ln((\Phi_1-\Phi_2)^2) - 3(\Phi_1+\Phi_2)^2 \ln((\Phi_1+\Phi_2)^2)\right).$$

Setting $\Phi_1 = \Phi_2$ we arrive at the required result.

9.7.2 Working with the $SU(N)$ Wess–Zumino term

In this problem, we develop some of the amazing mathematical features of the SU(N) WZ term Eq. (9.58). The problem also illustrates the superiority of the coordinate-free calculus of differential forms in topological quantum field theory. (Expressed in the standard languages of calculus, all formulae below become intolerably messy – not to mention the fact that the underlying structures are much more di cult to understand!)

(a) WZ terms generally originate from a closed (but only locally exact) differential form on the target manifold ω. Show that, on SU(N), the form $\omega = \mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)$ is closed. (Why is it only locally exact? This question will be addressed in part (c) below.)

(b) Verify Eq. (9.61). Let us try to understand the **normalization of the WZW action**. To start with, let us recall a statement from group theory stated in the Info block on page 549: as far as topology is concerned, the subgroup SU(2) $\subset$ SU(N) preempts the structure of SU(N), i.e. without loss of generality, we may restrict our discussion to the SU(2) WZ action.

We next turn to the normalization of the WZW action. To start with, let us consider S^2 as the boundary of the northern hemisphere S^{3+} of the 3-sphere. We define the WZW term by integration over S^{3+}.[117] To parameterize S^{3+}, we introduce a third coordinate x_3 (in addition to the two coordinates (x_1, x_2) parameterizing S^2), such that $x_3 = 0$ defines the north pole and $x_3 = \pi/2$, the equatorial plane, S^2. For any field $g(x_1, x_2) \in$ SU(2), the WZW action then becomes $\Gamma^+[g] = i\mathcal{N} \int_0^{\pi/2} dx_3 \; \Gamma[g, x_3]$, where

[116] In spite of the weak convergence $\sim z^{-1}$ of the integrand at large values of z, it is permissible to do the integral in this manner: the second flux derivative implicitly levels the decay up to $\sim z^{-2}$.

[117] Topologically, this domain is equivalent to the three-dimensional unit ball B^3 used in the text (just as the northern hemisphere of the 2-sphere is topologically equivalent to the unit disk B^2).

$\Gamma[g,x_3] \equiv \int dx_1\ dx_2\ \epsilon^{\mu\nu\sigma}\,\mathrm{tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\sigma g)$, $\mu,\nu,\sigma = 1,2,3$, and $g(x_1,x_2,x_3)$ is a smooth extension of $g(x_1,x_2)$ from S^2 to S^{3+}. But, of course, it would have been just as good to take the southern hemisphere S^{3-} as our reference domain. To explore the consequences of this ambiguity, we extend the definition of the third coordinate in such a way that $x_3 = \pi$ corresponds to the south pole of S^3. In this case, the WZW action would assume the form $\Gamma^-[g] = -i\mathcal{N}\int_{\pi/2}^{\pi} dx_3, \Gamma\,[g,x_3]$, where $\Gamma[g,x_3]$ is defined as above (albeit for an extension $g(x_1,x_2,x_3<0)$.

For any reference field $g(x_1,x_2)$, the ambiguity,

$$\Gamma^+ - \Gamma^- = i\mathcal{N}\int_0^\pi dx^3 \int dx_1\ dx_2\ \epsilon^{\mu\nu\sigma}\mathrm{tr}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\sigma g) \stackrel{!}{=} 2\pi n,$$

must be an integer multiple of 2π.[118] Consider, thus, the field $g(x_1,x_2) \equiv i\mathbf{n}(x_1,x_2)\cdot\sigma$, where $\mathbf{n}(x_1,x_2)$ is a unit vector. As its extension into the northern/southern hemisphere, we choose $g(x_1,x_2,x_3) = \cos(x_3) + i\sin(x_3)g(x_1,x_2) = \exp(ix_3\mathbf{n}(x_1,x_2)\cdot\boldsymbol{\sigma}) \in \mathrm{SU}(2)$. This is an SU(2)**-instanton**, i.e. a mapping $S^3 \to \mathrm{SU}(2)$ that cannot be continuously deformed to the unit mapping.

(c) Show that $\Gamma^+[g] - \Gamma^-[g] = i\mathcal{N}24\pi^2 \times$ (integer), which enforces $\mathcal{N} = 1/12\pi$ as the normalization of the WZW action. As a corollary, we note that the form ω on which all of our discussion is based cannot be globally exact. (If it had been, Stokes' theorem would imply a vanishing of the integral over the boundary-less manifold S^3.)

(d) Show that, for the particular reference configuration considered in part (c), $g = i\mathbf{n}\cdot\boldsymbol{\sigma} \in \mathrm{SU}(2)$, the WZW action reduces to the θ-term for the unit-modular field $\mathbf{n} \in S^2$: $\Gamma[i\mathbf{n}\cdot\boldsymbol{\sigma}] = \pi S_{\mathrm{top}}[\mathbf{n}]$, where $S_{\mathrm{top}}[\mathbf{n}] = \frac{1}{4\pi}\int_{S^2} d^2x\ \mathbf{n}\cdot(\partial_1\mathbf{n}\times\partial_2\mathbf{n})$.

Answer:

(a) Using the fact that $g^{-1}dg\,g^{-1} = -dg^{-1}$ (why?) the form ω can be rewritten as $\omega = \mathrm{tr}(dg^{-1}\wedge dg\wedge dg^{-1}\,g)$, i.e. $d\omega = -\mathrm{tr}(dg^{-1}\wedge dg\wedge dg^{-1}\wedge dg)$. To show that this expression vanishes, one may make use of the fact that, for arbitrary matrix-valued forms, $\mathrm{tr}(\omega_1\wedge\omega_2) = (-)^{\deg(\omega_1)\deg(\omega_2)}\mathrm{tr}(\omega_2\wedge\omega_1)$.[119] Applied to $d\omega$, this yields $d\omega = +\,\mathrm{tr}(dg\wedge dg^{-1}\wedge dg\wedge dg^{-1}) = \mathrm{tr}(dg^{-1}\wedge dg\wedge dg^{-1}\wedge dg) = -d\omega$, where in the last step we applied the formula $g^{-1}dg = -dg^{-1}\,g$ to all derivatives.

[118] Geometrically, this is the integral of the differential 3-form $g^*\omega$ (i.e. the pullback of the SU(2) form ω by the field g to a form on S^3) over S^3.

[119] Proof: $\mathrm{tr}(\omega\wedge\eta) = \sum_{ij}\omega_{ij}\wedge\eta_{ji} = (-)^{\deg(\omega)\deg(\ )}\sum_{ij}\eta_{ji}\wedge\omega_{ij} = (-)^{\deg(\omega)\deg(\ )}\mathrm{tr}(\eta\wedge\omega)$.

(b) We want to explore what happens to the integral $\Gamma[g] = -\frac{i}{12\pi}\int_{B^3} g^*\omega$ upon variation $g \to e^W g \simeq (1+W)g$. Using the relation $g^{-1}dg \to g^{-1}dg + g^{-1}dW\, g$, we obtain

$$\begin{aligned}\Gamma[(1+W)g] - \Gamma[g] &\simeq -\frac{i}{4\pi}\int_{B^3} \mathrm{tr}(g^{-1}dW\, g \wedge g^{-1}dg \wedge g^{-1}dg) \\ &= \frac{i}{4\pi}\int_{B^3} \mathrm{tr}(dW \wedge dg \wedge dg^{-1}) = \frac{i}{4\pi}\int_{B^3} d\left[\mathrm{tr}(W \wedge dg \wedge dg^{-1})\right] \\ &= \frac{i}{4\pi}\int_{S^2} \mathrm{tr}(W \wedge dg \wedge dg^{-1}),\end{aligned}$$

where, in the last line, we have made use of Stokes' theorem.

(c) By an elementary rearrangement of fields,

$$\Gamma^+ - \Gamma^- = 3i\mathcal{N}\int_0^\pi dx_3 \int dx_1\, dx_2\, \epsilon^{ij}\mathrm{tr}(g\partial_3 g^{-1}\, \partial_i g\, \partial_j g^{-1}),$$

where the coordinates $i, j = 1, 2$. A straightforward calculation shows that $g\partial_3 g^{-1} = -i\mathbf{n}\cdot\boldsymbol{\sigma}$. Using the auxiliary identities $\partial_i g = i\sin(x_3)\partial_i \mathbf{n}\cdot\boldsymbol{\sigma}$ and $\mathrm{tr}(\mathbf{n}_1\cdot\boldsymbol{\sigma}\,\mathbf{n}_2\cdot\boldsymbol{\sigma}\,\mathbf{n}_3\cdot\boldsymbol{\sigma}) = 2i\mathbf{n}_1\cdot(\mathbf{n}_2\times\mathbf{n}_3)$, we thus obtain,

$$\begin{aligned}\Gamma^+ - \Gamma^- &= 6i\mathcal{N}\int_0^\pi dx_3 \sin^2(x_3)\int dx_1\, dx_2\, \epsilon^{ij}\mathbf{n}\cdot(\partial_i\mathbf{n}\times\partial_j\mathbf{n}) \\ &= 6i\pi\mathcal{N}\int dx_1 dx_2 \mathbf{n}\cdot(\partial_1\mathbf{n}\times\partial_2\mathbf{n}) = 24i\pi^2\times(\text{integer}),\end{aligned}$$

where the integer is the number of times the unit vector $\mathbf{n}$ wraps around the unit sphere.

(d) By analogy with the calculation performed in (c), we obtain $\Gamma^+[i\mathbf{n}\cdot\boldsymbol{\sigma}] = \Gamma^+ = 6i\pi\mathcal{N}\int dx_1\, dx_2\, \mathbf{n}\cdot(\partial_1\mathbf{n}\times\partial_2\mathbf{n}) = \pi S_{\mathrm{top}}[\mathbf{n}]$.

9.7.3 Renormalization group analysis of the $SU(N)$ Wess–Zumino model

Before embarking on this problem, it is helpful to recapitulate the RG analysis of the SU(N) nonlinear σ-model discussed in Section 8.5. We want to study the RG flow of the SU(N) WZW model, as specified by Eq. (9.65). As in our previous analyses of nonlinear σ-models, we can split a general field configuration $g = g_{\mathrm{s}}g_{\mathrm{f}}$ into a slow and a fast part, and expand the latter as $g_{\mathrm{f}} = 1 + W + W^2/2 + \cdots$, where $W \in$ su(N), i.e. W lives in the Lie algebra su(N) (the algebra of anti-Hermitian traceless matrices). To compute the one-loop RG equations, we need to expand the action to quadratic order in W and compute all contributions to the functional integral that contain (a) one fast momentum integration and (b) no more than two derivatives acting on a slow field.

(a) Show that the expansion of the action $S[g_{\mathrm{s}}g_{\mathrm{f}}]$ to second order in the generators takes the form $S[g_{\mathrm{f}}g_{\mathrm{s}}] = S[g_{\mathrm{s}}] + S[g_{\mathrm{s}}, W] + S^{(2)}[W]$, where

$$\begin{aligned}S^{(2)}[W] &= -\frac{1}{\lambda}\int_{S^2} d^2x\, \mathrm{tr}(\partial_\mu W \partial_\mu W), \\ S[g_{\mathrm{s}}, W] &= -\frac{1}{\lambda}\int_{S^2} d^2x\, \mathrm{tr}\left[\left(g_{\mathrm{s}}^{-1}\partial_\mu g_{\mathrm{s}} - \frac{i\lambda}{8\pi}\epsilon_{\mu\nu}g_{\mathrm{s}}^{-1}\partial_\nu g_{\mathrm{s}}\right)[\partial_\mu W, W]\right].\end{aligned} \tag{9.88}$$

(b) One-loop corrections to the action are obtained by expanding the functional to second order in $S^{(2)}[g_{\rm s}, W]$ and integrating over W: $S[g] \to S[g_{\rm s}] - \frac{1}{2}\langle S[g_{\rm s}, W]^2\rangle_W$. Use the results of Section 8.5 (in particular those derived in the exercises on page 463) to confirm that (i) only the gradient term of the action is renormalized and (ii) the RG equation for its coupling constant is given by Eq. (9.66).

Answer:

(a) The first line of Eq. (9.88) and the first term in the second line are obtained by substitution of $g_{\rm s}g_{\rm f}$ into the gradient term of the action (9.57) and expanding to second order in the generators W. The second term in $S[g_{\rm s}, W]$, a descendant of the WZ action, is best derived in the language of differential forms: substitution of $(g_{\rm s}g_{\rm f})^{-1}d(g_{\rm s}g_{\rm f}) = g_{\rm f}^{-1}(g_{\rm s}^{-1}dg_{\rm s} + (dg_{\rm f})g_{\rm f}^{-1})g_{\rm f}$ into the pullback $(g_{\rm s}g_{\rm f})^*\omega = {\rm tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)|_{g=g_{\rm s}g_{\rm f}}$ gives

$$\begin{aligned}(g_{\rm s}g_{\rm f})^*\omega &= g_{\rm s}^*(\omega) + \frac{3}{2}{\rm tr}(dg_{\rm s}^{-1} \wedge dg_{\rm s} \wedge [W, dW]) + 3{\rm tr}(dW \wedge dW \wedge g_{\rm s}^{-1}dg_{\rm s}) + \mathcal{O}(W^3)\\ &= g_{\rm s}^*(\omega) + \frac{3}{2}d\left[{\rm tr}([W, dW] \wedge g_{\rm s}^{-1}dg_{\rm s})\right] + \mathcal{O}(W^3).\end{aligned}$$

Application of Stokes' theorem thus leads to

$$\begin{aligned}\Gamma[g_{\rm s}g_{\rm f}] &= -\frac{i}{12\pi}\int_{B^3}(g_{\rm s}g_{\rm f})^*\omega = \Gamma[g_{\rm s}] - \frac{i}{8\pi}\int_{B^3} d\left[{\rm tr}([W, dW] \wedge g_{\rm s}^{-1}dg_{\rm s})\right] + \mathcal{O}(W^3)\\ &= \Gamma[g_{\rm s}] - \frac{i}{8\pi}\int_{S^2}{\rm tr}([W, dW] \wedge g_{\rm s}^{-1}dg_{\rm s}) + \mathcal{O}(W^3)\\ &= \Gamma[g_{\rm s}] - \frac{i}{8\pi}\int_{S^2} d^2x\epsilon_{\mu\nu}{\rm tr}([W, \partial_\mu W] \wedge g_{\rm s}^{-1}\partial_\nu g_{\rm s}) + \mathcal{O}(W^3),\end{aligned}$$

where the $\mathcal{O}(W^2)$ term appears as the second contribution to the action $S[g_{\rm s}, W]$.

(b) Defining $\Phi_\mu \equiv g_{\rm s}^{-1}(\partial_\mu + \frac{i\lambda}{8\pi}\epsilon_{\mu\nu}\partial_\nu)g_{\rm s}$, the action $S[g_{\rm s}, W]$ assumes the form $S[g_{\rm s}, W] = \frac{2iL^d}{\lambda}\sum_{pq} p_\mu {\rm tr}(\Phi_\mu\Phi_\mu)$. But for the difference in the definition of the field Φ, this is equal to the fast–slow action of the standard SU(N) model. Using the results derived in the exercises on page 463, we thus obtain

$$\begin{aligned}-\frac{1}{2}\langle S[g_{\rm s}, W]^2\rangle_W &= -\frac{N\ln b}{8\pi}\int d^2x\,{\rm tr}(\Phi_\mu\Phi_\mu)\\ &= -\frac{N\ln b}{8\pi}\left(1 - \left(\frac{\lambda}{8\pi}\right)^2\right)\int d^2x\,{\rm tr}(\partial g_{\rm s}\partial g_{\rm s}^{-1}).\end{aligned}$$

This result confirms that only the gradient term in the action is renormalized. By proceeding in direct analogy to the discussion of Section 8.5, it is a straightforward matter to derive the corresponding RG equation. The result is given by Eq. (9.66)

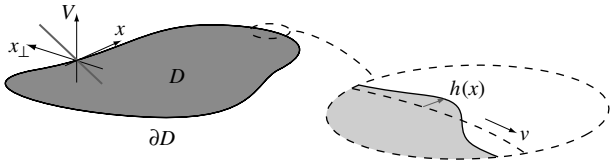


Figure 9.21 On the formation of surface wave excitations in an FQHE droplet. For an explanation, see the main text.

9.7.4 Fractional quantum Hall effect: physics at the edge

In Section 9.3.7 above, we have seen that the bulk physics of an integer quantum Hall system is intimately connected to that of its boundaries. These connections could be disclosed from both direct physical reasoning (the essence of the Laughlin–Halperin semi-phenomenological approach) and an analysis of the behavior of the microscopic theory under gauge transformations. Following the seminal work of Wen,[120] here we show that an equally close connection between bulk and boundaries exists for the FQHE. Again, we will be able to deduce the boundary physics by phenomenological, and by field-theory-oriented, reasoning.

Consider a finite FQHE system. For simplicity, we assume the system to be disc-shaped, although the details of the geometry will not be of relevance throughout. At the system boundaries, the two-dimensional electron gas is confined by a boundary potential V which we assume to be linear (see Fig. 9.21). In the first part of this problem, we want to describe the dynamics of edge excitations on phenomenological grounds. An important first observation is that the bulk system – presumed to sit at an FQHE plateau value – is in an "incompressible state." This follows from the fact that, for an integer number of composite fermion Landau levels, the system does not support gapless excitations: $\partial\mu/\partial_N \to \infty$. Now,[121] $\partial\mu/\partial_N \sim \kappa^{-1} \equiv -V(\partial P/\partial V)_N$ is proportional to the inverse compressibility, i.e. the lack of low-energy excitations implies an incompressible state. We should, therefore, think of our system as a rigid "liquid" rather than as a gas.

Given that the state is incompressible, the lowest-energy excitations of the system will be deformations of the boundary (similar to boundary distortions of a puddle of water). We may characterize these distortions by a surface density profile $\rho(x)$, where x parameterizes the boundary.[122]

(a) To derive the boundary action on phenomenological grounds, proceed in two steps. First derive the energy of a boundary distortion ρ. Second, argue why a distortion profile $\rho(x,t)$ propagates along the boundary at some constant velocity v. Derive an effective equation of motion for ρ and use it to determine the canonical momentum associated

[120] For a review, see X. G. Wen, Theory of the edge states in fractional quantum Hall effects, *Int. J. Mod. Phys.* **B6** (1992), 1711–62.

[121] G. Mahan, *Many Particle Physics* (Plenum Press, 1981).

[122] More precisely, the geometric shape of the system is described by some height function $h(x)$ (see Fig. 9.21). Using the fact that the constant charge density of the system (cf. Eq. (9.17)) is given by $\frac{\nu B}{2\pi} dx\, dx_\perp$, where $x_\perp$ is the coordinate perpendicular to the system boundary, $\rho(x)dx = \frac{\nu B}{2\pi}\int_0^{h(x)} dx_\perp\, dx = \frac{\nu B}{2\pi} h(x)dx$, i.e. the height profile is proportional to a density profile.

with the "displacement field" ϕ defined by $\partial_x \phi = 2\pi\rho$. Show that the Hamiltonian action of the displacement field is given by

$$S[\phi] = \frac{1}{4\pi\nu} \int dx\ d\tau \left(v(\partial_x \phi)^2 - i \partial_x \phi \partial_\tau \phi \right), \tag{9.89}$$

the action of a "**chiral Luttinger liquid**" – the terminology follows from the fact that Eq. (9.89) describes the uni-directional propagation of a density field. (The connection to the Luttinger liquid is made explicit by decomposing the coordinate–momentum pair describing one-dimensional charge density modes, $(\partial_x \phi, \theta)$, into left- and right-moving components, $\phi = \phi_{\rm L} + \phi_{\rm R}$, $\theta = \phi_{\rm L} - \phi_{\rm R}$. Substituting this representation into the action (4.49), the latter decomposes into a left- and a right-moving chiral action.) (Hint: A boundary distortion costs energy because of the presence of a voltage gradient. The presence of a constant drift follows, e.g. from the finiteness of the Hall conductivity. Also remember Eq. (9.17) for the density of a quantum Hall system.)

(b) We next derive the action from a complementary, bulk-oriented perspective. To this end, consider the CS action of the bulk system, Eq. (9.75). Throughout, it will be convenient (although not strictly necessary as one may translate all expressions back to the traditional representation) to use the language of differential forms. Interpreting a_μ as the coefficients of a differential 1-form, $a = a_\mu dx_\mu$, Eq. (9.75) assumes the compact form

$$S_{\rm CS}[a] = -\int_{D \times S^1} a \wedge da,$$

where the integral extends over the Cartesian product of the bulk of the system, D, and imaginary time S^1. Show that $S_{\rm CS}$ is gauge invariant *up to a boundary term*. To "cure" the gauge deficiency of the action, we adopt a gauge-fixing condition $a_0|_{\partial D} = 0$. Use this condition to integrate over the component a_0 in the entire system. Show that this integration leads to the global constraint $f_{ij} = 0$, where f_{ij} are the real-space components of the field strength tensor $f = da = \partial_\mu a_\nu dx_\mu \wedge dx_\nu$. This condition implies that the real-space component of the vector potential $\mathbf{a}$ can be represented as a pure gauge, $\mathbf{a} = \mathbf{d}\phi \equiv \partial_i \phi dx_i$, where we use the symbol $\mathbf{d}$ to denote the real space contribution to the exterior derivative. Plug this *ansatz* into the residual contribution to the action (after a_0 has been integrated out) to reduce the field strength tensor to the boundary action $S[\phi] = \int_{\partial B} dx \int d\tau\ \partial_x \phi \partial_\tau \phi$. Recalling that the CS action enters the theory as $\exp(\frac{i}{4\pi\nu} S_{\rm CS})$,[123] we conclude that the effective boundary action induced by the gauge non-invariance of $S_{\rm CS}$ reads

$$S_{\rm eff}[\phi] = -\frac{i}{4\pi\nu} \int_{\partial B} dx \int d\tau\ \partial_x \phi \partial_\tau \phi.$$

[123] To be precise, the coupling constant $1/4\pi\nu$ appeared after (a) the system had been coupled to an external vector potential, (b) the coupling between the statistical vector potential and the matter degrees of freedom had been taken into account, and (c) the statistical vector potential had been integrated out (see Section 9.5.3). Here we assume that all these steps are implied, i.e. we should think of a as an external electromagnetic field and $\frac{i}{4\pi\nu} S_{\rm CS}$ as the dominant (first-order derivative) contribution to its action.

(c) One may recognize $S_{\rm eff}[\phi]$ as the second contribution to the action (9.89). The obvious next question to ask is why we did not obtain the full action of the chiral Luttinger liquid. Indeed, it takes only a moment's thought to realize that a first principles derivation of the full boundary action from the bulk theory is out of the question. The point is that the missing contribution $\sim v(\partial_x\phi)^2$ depends on the boundary velocity, which in turn is determined by the steepness of the boundary potential. However, the bulk action does not know of the structure of the boundary. We can, however, employ a trick to infuse the required information on the system boundaries into the bulk theory. (Although this manipulation is fairly ad hoc, it shows, at least, that the bulk and the boundary theory are not inconsistent.) Indeed, the gauge-fixing condition employed in the construction above involves a lot of arbitrariness. For example, instead of $a_0 = 0$ the linear combination $a_0 + va_x = 0$ would have been just as good.

To explore the consequences of the new gauge-fixing condition, let us introduce a new set of coordinates, $\tilde{x} = x - vt$, $\tilde{t} = t$, and $\tilde{x}_\perp = x_\perp$. Recalling the transformation behavior of differential forms (see page 537) under coordinate changes, compute the components of the vector potential $a = \tilde{a}_\mu d\tilde{x}_\mu$. Show that the gauge fixing condition translates to $\tilde{a}_0 = 0$. We now benefit from the fact that we have expressed the CS term in a coordinate-invariant manner, i.e. it keeps the same form, no matter whether we express it in terms of the old or the new coordinates (a point on which to reflect!). However, in the new coordinates, the gauge-fixing condition assumes the same form $\tilde{a}_0$ as that considered in (b). This means that $S_{\rm CS}$ can be reduced to a boundary contribution, which, however, is expressed in new coordinates. Finally, translate back to the old coordinates to obtain the full chiral Luttinger action. The message to be taken home from this construction is that a boundary-gauge-fixing condition can be used to establish the equivalence between the boundary Luttinger and the bulk CS description of the system.

(d) Notwithstanding its innocuous appearance, the action (9.89) describes astonishingly rich physics. Referring for an in-depth discussion to the literature (see Wen's review article[120]), we here mention only a few of the characteristic features of the system described by Eq. (9.89). To establish contact with physically observable quantities, we first need to derive an expression for the electron operator $c^\dagger$ (the chiral analog of the fermion relation (4.46)). Arguing as in Section 4.3 above, we start from the observation that the creation of an electron goes along with the creation of a unit charge. Building on the connection between the displacement field and the charge operator, show that the chiral bosonic representation of the fermion operator will contain a piece

$$c^\dagger \sim e^{-i\nu^{-1}\phi}.$$

To complete the manufacturing of a fermion operator, we need to ensure that $c^\dagger$ obeys fermionic exchange statistics. Use the commutator relations of the field ϕ to show that

$$e^{i\nu^{-1}\phi(x)}e^{i\nu^{-1}\phi(x')} = e^{i\nu^{-1}\phi(x')}e^{i\nu^{-1}\phi(x)} \times e^{i\pi\nu^{-1}\mathrm{sgn}(x-x')}.$$

This means that $\exp(i\nu^{-1}\phi)$ describes a fermion provided that $\nu = 1/m$ belongs to the principal sequence.

INFO But what will happen for a general $\nu = n/m$? It is, in fact, not clear how to repair the *ansatz* above so as to generate **fermion statistics for general filling fractions**. Indeed, it turns out that the entire edge construction above is too narrow to describe the general case. We have seen in our discussion of the IQHE above that, in cases where $p > 1$ Landau levels are occupied, the edge hosts p edge channels. Similarly, an FQHE system with filling fraction $\nu = p/(2sp+1)$ (i.e. a system in which the composite fermions experience an effective $\nu = 1/p$ IQHE) will have p chiral edge channels circulating at its boundaries. However, the construction of this edge channel hierarchy is beyond the scope of the present text and we refer to the literature.

Notice that we may interpret $e^{i\nu^{-1}\phi} = (e^{i\phi})^{\nu^{-1}}$ as the ν^{-1}th power of the more elementary object $e^{i\phi}$. According to the construction above, $e^{i\phi}$ creates an entity of fractional charge $e \times \nu$. Also, the states created by $e^{i\phi}$ obey fractional exchange statistics. Comparing with our discussion in the main text, we identify $\exp(i\phi)$ as the creator of the boundary variant of the fractionally charged Laughlin quasi-particles.

Answer:

(a) Owing to the presence of an (approximately linear) confining potential $V = Ex_\perp$, a boundary distortion costs an energy (notation taken from footnote[122]) $H = \int dx \int_0^{h(x)} dx_\perp \sigma E x_\perp = \frac{\nu EB}{4\pi} \int dx\ h^2(x) = \frac{E}{4\pi\nu B} \int dx\ (\partial_x \phi)^2$.

To understand why the density profile propagates along the boundary, notice that the (confining) electric field E perpendicular to the boundary will generate a Hall current density $j \sim \sigma_{12} E = \sigma B^{-1} E$ tangential to the boundary. The total boundary current is obtained by integrating the current density from 0 to $h(x)$, i.e. $I \sim \rho E/B = \partial_x \phi E/2\pi B$. Integrating the continuity equation $\partial_t \rho = \partial_x j$ over x and substituting the identification of the current above, we obtain the equation of motion $\partial_t \phi = v \partial_x \phi$, where the velocity $v \sim E/B$ depends in a non-universal way on the boundary potential. This equation is solved by $\phi = \phi(x + vt)$, i.e. a uniformly propagating density distribution. Switching to momentum space, we may interpret $\dot{\phi}_k = \delta H/\delta \pi_k = vik\phi_k$ as an Hamiltonian equation of motion determining the momentum π conjugate to ϕ. Comparison[124] with $H = \frac{v}{2\pi\nu} \sum_k k^2 \phi_k \phi_{-k}$ leads to the identification $\pi_k = \frac{1}{2\pi\nu}(-ik)\phi_{-k}$, i.e. the strange looking prediction that the variable ϕ_k is canonically conjugate to its own derivative,

$$[\partial_x \phi(x), \phi(x')] = 2\pi i \nu \delta(x - x'). \tag{9.90}$$

Remembering that the Hamiltonian action of a system is given by $\int d\tau\ (H - i\pi \partial_\tau \phi)$, we arrive at the required result.

[124] Deviating from our standard conventions, we define the Fourier transform by $\phi_k = L^{-1/2} \int dx\ e^{ikx} \phi(x)$.

(b) Under a change of gauge, $a \to a + dg$, where g is a function, the CS action transforms as

$$\begin{aligned} S_{\rm CS}[a+dg] &= -\int_{D\times S^1} (a+dg)\wedge d(a+dg) = S_{\rm CS}[a] - \int_{D\times S^1} dg\wedge da \\ &= S_{\rm CS}[a] - \int_{\partial D\times S^1} dag, \end{aligned}$$

where we noticed that $dg \wedge da = d(gda)$ and applied Stokes, theorem. To remove the boundary gauge ambiguity, we need to fix a gauge at $\partial D \times S^1$, e.g. by setting $a_0|_{\partial D} = 0$. Decomposing $a = a_0 + \mathbf{a}$ and $d = \partial_0 d\tau + \partial_i dx_i \equiv d_0 + \mathbf{d}$ into temporal and real-space contributions, the action assumes the form

$$\begin{aligned} S_{\rm CS}[a_0,\mathbf{a}] &= -\int_{D\times S^1} (a_0+\mathbf{a})\wedge(d_0+\mathbf{d})(a_0+\mathbf{a}) \\ &= -\int_{D\times S^1} (a_0\wedge\mathbf{da} + \mathbf{a}\wedge d_0\mathbf{a} + \mathbf{a}\wedge\mathbf{d}a_0 + \mathbf{a}\wedge\mathbf{da}) \\ &= -\int_{D\times S^1} (2a_0\wedge\mathbf{da} + \mathbf{a}\wedge d_0\mathbf{a} + \mathbf{a}\wedge\mathbf{da}), \end{aligned}$$

where we have used the skew-symmetry of the exterior product and in the crucial second equality integrated by parts.[125] The boundary term corresponding to this integration vanishes due to the gauge-fixing condition $a_0|_{\partial D} = 0$. Linearly coupled to the action, the temporal component a_0 can be integrated out. As a result, we obtain a functional δ-distribution globally enforcing the constraint $da \equiv f = 0$, i.e. a vanishing of the spatial components of the field strength tensor. This in turn implies that $\mathbf{a}$ is a pure gauge, $\mathbf{a} = \mathbf{d}\phi$, where ϕ is some function. Substituting this representation into the action, we obtain $S[\phi] = -\int_{D\times S^1} \mathbf{d}\phi\wedge d_0\mathbf{d}\phi = -\int_{D\times S^1}\mathbf{d}(\mathbf{d}\phi\wedge d_0\phi) = -\int_{\partial D\times S^1}(\mathbf{d}\phi\wedge d_0\phi) = \int_{\partial D} dx\int d\tau\,\partial_x\phi\,\partial_\tau\phi$, where in the last step we switched back to conventional notation.

(c) Using the fact that $a = a_\mu dx_\mu = a_\mu(\partial x_\mu/\partial\tilde{x}_\nu)d\tilde{x}_\nu \equiv \tilde{a}_\nu d\tilde{x}_\nu$, and comparing coefficients, we obtain the identifications $\tilde{a}_x = a_x, \tilde{a}_{x_\perp} = a_{x_\perp}, \tilde{a}_0 = a_0 + v a_x$. "Form invariance" of the CS-action means that it does not matter whether we express it in terms of the old or the new coordinates:

$$S_{\rm CS}[a] = -\int (a_\mu dx_\mu)\wedge\left(dx_\nu\,\frac{\partial}{\partial x_\nu}\right)\wedge(a_\nu dx_\nu) = -\int(\tilde{a}_\mu d\tilde{x}_\mu)\wedge\left(d\tilde{x}_\nu\,\frac{\partial}{\partial\tilde{x}_\nu}\right)\wedge(\tilde{a}_\nu d\tilde{x}_\nu).$$

Focusing on the second representation (wherein $\tilde{a}_0|_{\partial D} = 0$), and repeating the analysis of (b), we obtain the real-time representation of the action $S_{\rm CS} = \int_{\partial D} d\tilde{x}\int d\tilde{t}\;\partial_{\tilde{x}}\phi\;\partial_{\tilde{t}}\phi$. Finally, using the fact that $\partial_x = \partial_{\tilde{x}}$ and $\partial_{\tilde{t}} = v\partial_x + \partial_t$, we arrive at $S_{\rm CS}[\phi] = \int_{\partial D} dx\int dt\,\left[v(\partial_x\phi)^2 + \partial_x\phi\;\partial_t\phi\right]$. Switching back to imaginary time and attaching the coupling constant, we obtain Eq. (9.89).

[125] That is, we have applied Stokes' theorem:

$$\int_{D\times S^1}\mathbf{a}\wedge\mathbf{d}a_0 = -\int_{D\times S^1}[\mathbf{d}(\mathbf{a}\wedge a_0) - \mathbf{da}\wedge a_0] = -\int_{D\times S^1}\mathbf{a}\wedge a_0 + \int_{D\times S^1}\mathbf{da}\wedge a_0 \overset{a_0|_{\partial D}=0}{=} \int_{D\times S^1}\mathbf{da}\wedge a_0.$$

(d) Since $\rho = \frac{1}{2\pi}\partial_x\phi$, the presence of a unit charge localized at x_0, $\rho(x) = \rho(x - x_0)$, amounts to the presence of a "kink" of height 2π in ϕ, i.e. $\phi(x) = 2\pi\theta(x - x_0)$. Now, according to Eq. (9.90), ϕ and $\partial_x\phi/2\pi\nu$ form a canonically conjugate pair. This implies that $U_d(x) \equiv \exp(-id\partial_x\phi/2\pi\nu)$ acts like a unit translation operator on ϕ: $U_d(x)\phi(x') = \phi(x') + d\delta(x - x')$. The unit-charge kink is generated by the action of $\int_{x_0}^{\infty} dx'\, U_{2\pi}(x') = \exp(-i\phi/\nu)$ on the field ϕ. Reformulating Eq. (9.90) as $[\phi(x), \phi(x')] = i\pi\nu\,\mathrm{sgn}(x - x')$, we find that

$$\begin{aligned} e^{i\nu^{-1}\phi(x)}e^{i\nu^{-1}\phi(x')} &= e^{i\nu^{-1}\phi(x')}\left(e^{-i\nu^{-1}\phi(x')}e^{i\nu^{-1}\phi(x)}e^{i\nu^{-1}\phi(x')}\right) \\ &= e^{i\nu^{-1}\phi(x')}\, e^{i\nu^{-1}\exp(-i\nu^{-1}[\phi(x'),\,])\phi(x)} \\ &= e^{i\nu^{-1}\phi(x')}e^{i\nu^{-1}\phi(x)} \times e^{i\pi\nu^{-1}\,\mathrm{sgn}(x-x')}. \end{aligned}$$