ref: ① B. Simons book Chapter 9.
② Abanov, Stoney Brook

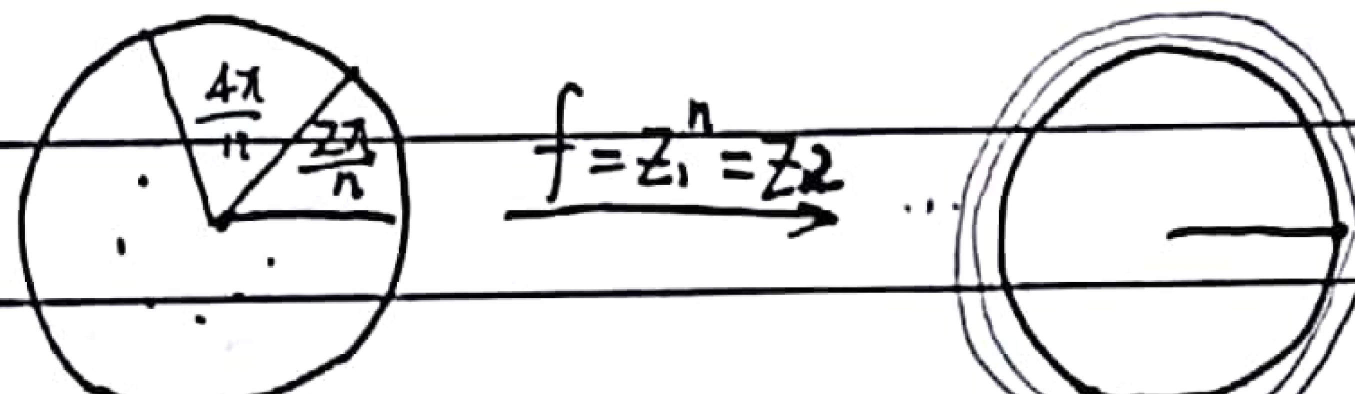
Date No.

2019.12.13 凝聚态场论及其在拓扑相变中的应用 裘明

Homotopy (同伦群) $f: X \to Y \Rightarrow \pi_n(M): S_n \to M = \begin{cases} S^m \\ U(N) \\ SU(N) \\ O(N) \\ \cdots \end{cases}$ Lie group

examples $\begin{cases} \pi_1(S^1) = Z \to \text{winding number} \\ \pi_2(S^2) = Z \to \text{monopole (real space)、Topo insulator/SC (momentum space)} \\ \pi_3(S^2) = Z \to \text{Hopf insulator / Hopf bundle} \end{cases}$

1. $\pi_1(S^1) = Z$



$$\boxed{\deg(f) \int \frac{dz_1}{2\pi i z_1} = \int \frac{dz_2}{2\pi i z_2}}$$

$\deg(f) = n$

$z_1 = e^{i2\pi/n} \Rightarrow z_2 = z_1^n = e^{i2\pi} = 1$

$\pi_1(S^1) \Rightarrow$ math : $\frac{1}{2\pi i}\int \frac{dz}{z}$

phys : SSH model

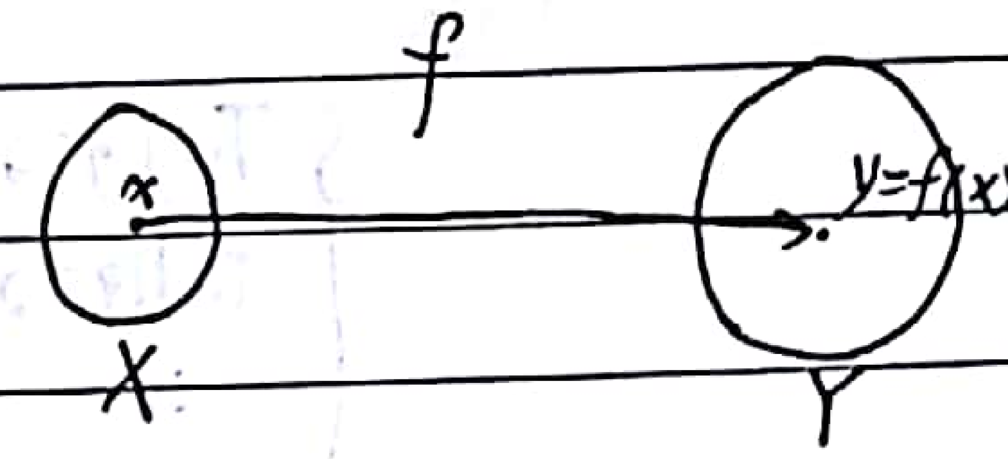
(base space) $S^1 \Rightarrow S^1$

In free space, k is a good quantum number.

$k \in [-\pi, \pi] = S^1$

Define : $H(k) = \begin{pmatrix} 0 & g(k) \\ g^*(k) & 0 \end{pmatrix}$

$f: x \to y \Rightarrow \begin{cases} f(x) = y \\ f(X) = Y \end{cases}$



$H: k \in S^1 \to \frac{g(k)}{|g(k)|} \in S^1$

$$\underset{\text{(物理上常用的是}k\text{)}}{\overset{\text{(base space)}}{S^1}} \xrightarrow{H(k)} \underset{\text{(物理上常用的是 Hamilton 空间)}}{\overset{\text{(target space)}}{S^1}}$$

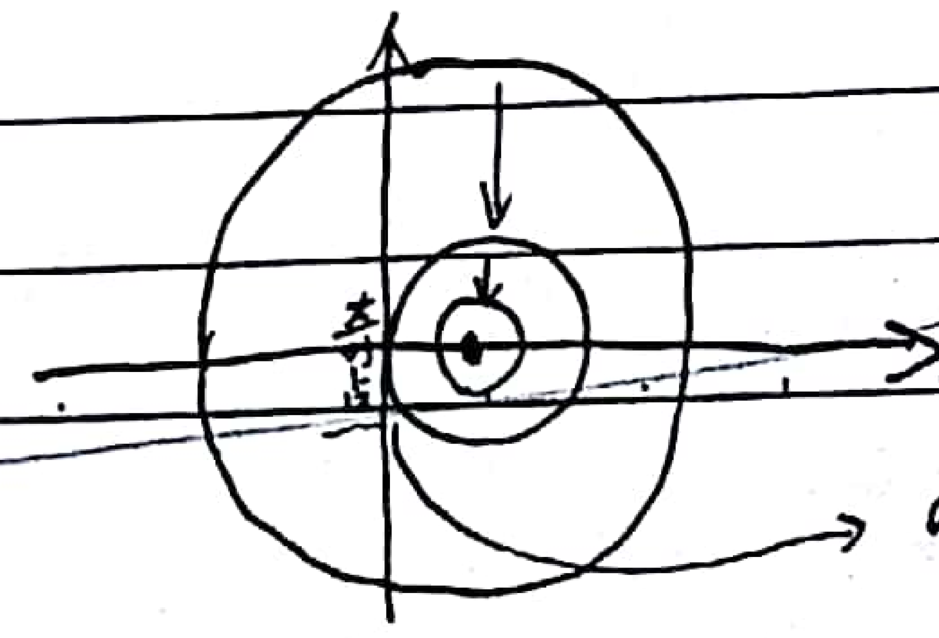
$$H = \begin{pmatrix} 0 & \underbrace{ia\sin k + a\cos k}_{ae^{ik}} - \mu \\ * & 0 \end{pmatrix}$$

$g = ae^{ik} - \mu$

$\begin{cases} k = 0 \Rightarrow a - \mu \\ k = \pi \Rightarrow -a - \mu \end{cases}$

$W = \frac{1}{2\pi i} \oint \frac{dg}{g}$

$= \frac{1}{2\pi i} \int_0^{2\pi} \left(\frac{a i e^{ik} dk}{ae^{ik} - \mu} \right)$



$\det(g) = 0 \Rightarrow \boxed{a = \pm\mu}$

可以将 $g(k)$ 推广为矩阵 $H(k)=\left(\begin{array}{c|c} 0 & g(k)I \\ \hline g^*(k)I & 0 \end{array}\right)$

$$W=\frac{1}{2\pi i}\oint \mathrm{Tr}(g^{-1}dg)$$

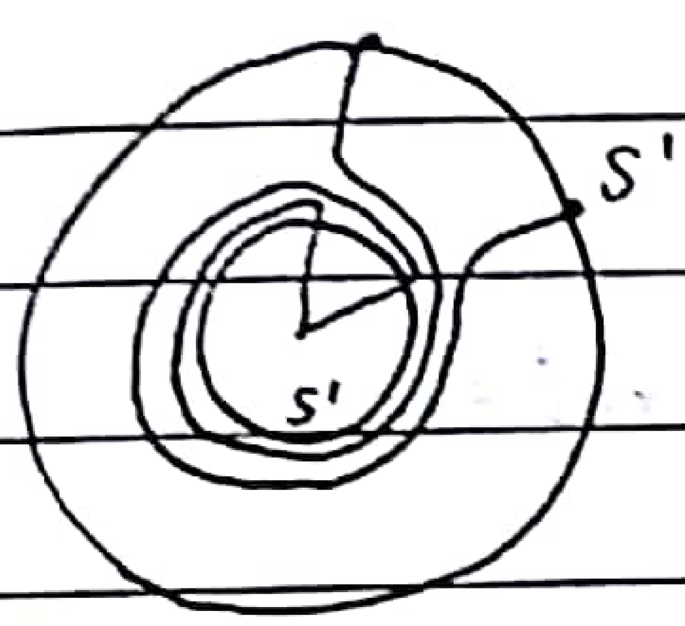
$$=\int C_{2n+1}\,\mathrm{Tr}\left[(g^{-1}dg)^{2n+1}\right]$$

注意：二维空间本来应该是 T^2

但实际只有在有限面积上才有贡献，到无穷远处快速衰减

故将 $T^2 \longrightarrow S^2$
(较弱) (较强)

$\pi_1(S^1)=Z$



2. $\pi_2(S^2)$

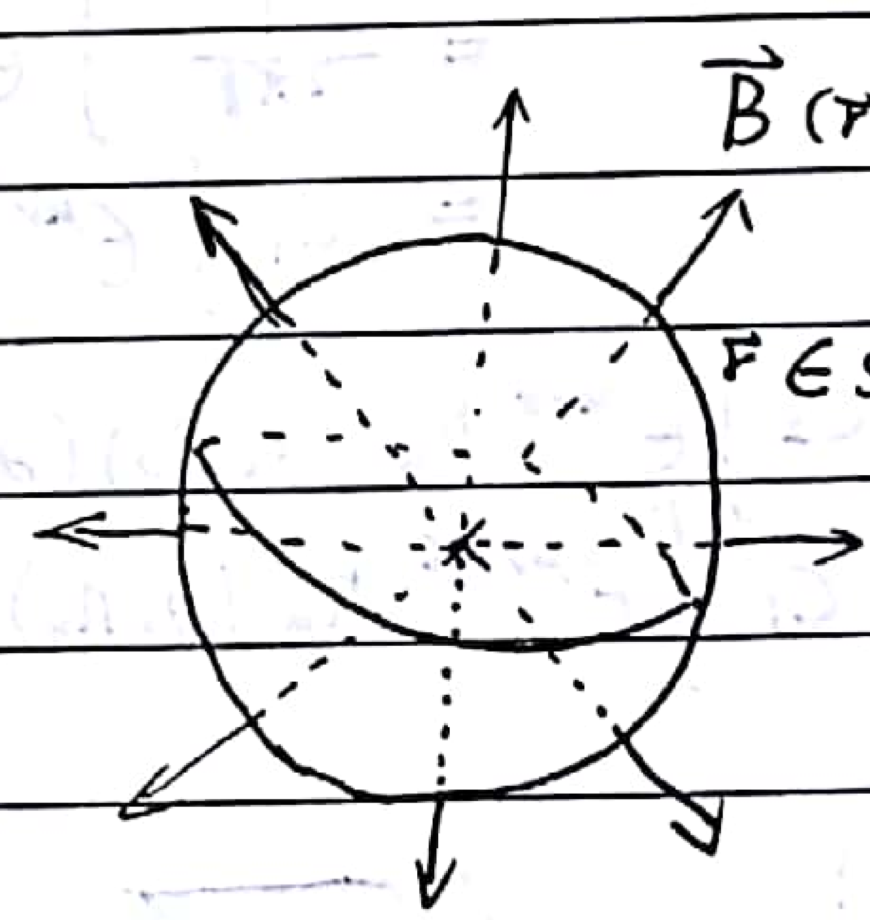
Real space: $\boxed{S^2 \longrightarrow S^2}$

$\vec{r}\in S^2=\{x^2+y^2+z^2=1 \mid x,y,z\in R\}$

Monopole $\boxed{\vec{B}=g\frac{\vec{r}}{4\pi r^3}}$

$$\oint \vec{B}\cdot d\vec{S}=g\oint \frac{\vec{r}}{4\pi r^3}\, r^4 d\Omega\, \partial\hat{n}$$

$$=\boxed{g}$$



$$S^2 \longrightarrow S^2$$

$$\underset{S^2}{\vec{B}: r} \xrightarrow{\vec{B}(r)} \underset{S^2}{\vec{B}(r)}$$

k-space $\boxed{T^2 \rightarrow S^2}$

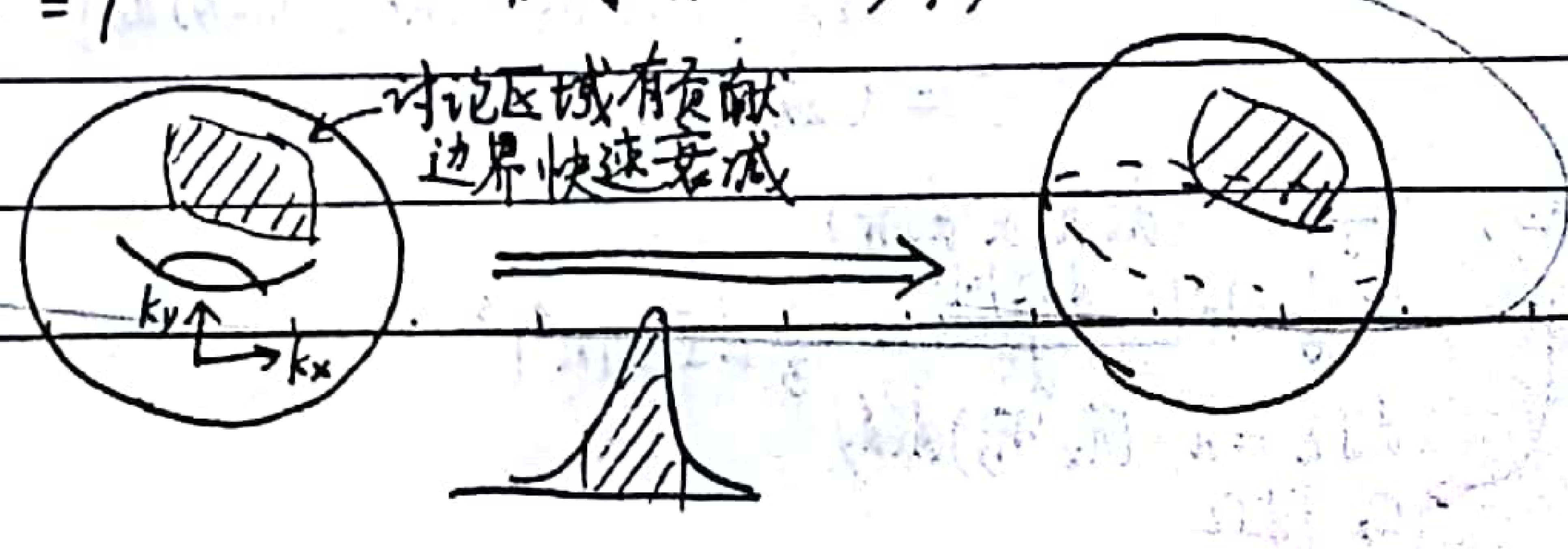
p-wave SC

p-wave pairing: $\Delta C_k^\dagger C_{-k}^\dagger(\sin k_x + i\sin k_y)$

$$(C_k^\dagger, C_{-k})\left(\begin{array}{c|c} \varepsilon_k & \Delta(\sin k_x + i\sin k_y) \\ \hline \Delta(\sin k_x - i\sin k_y) & -\varepsilon_k \end{array}\right)\begin{pmatrix} C_k \\ C_{-k}^\dagger \end{pmatrix}$$

$\varepsilon_k=(\cos k_x+\cos k_y-\mu)$

$\left.\begin{cases} k_x\in[-\pi,\pi]=S^1 \\ k_y\in[-\pi,\pi]=S^1 \end{cases}\right\} S^1\times S^1=T^2$



$\boxed{(T^2)S^2 \longrightarrow S^2}$

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$$(C_k^\dagger, C_{-k}) \begin{pmatrix} \varepsilon_k & \Delta(\sin k_x + i\sin k_y) \\ \Delta(\sin k_x - i\sin k_y) & -\varepsilon_k \end{pmatrix} \begin{pmatrix} C_k \\ C_{-k}^\dagger \end{pmatrix}$$

$$= n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$$

where $\begin{cases} n_1 = \Delta \sin k_x \\ n_2 = \Delta \sin k_y \\ \cancel{n_3 = \cos k_x + \cos k_y - \mu} \end{cases} \in S^2 = \{x^2+y^2+z^2=1\}$

$\cancel{\lambda(n_1^2+n_2^2+n_3^2)=1}$

$$\begin{array}{ccc} S^2 & \longrightarrow & S^2 \\ \downarrow & & \cup\!\!\!\!\mid \\ \vec{k} & \xrightarrow{n(\vec{k})} & n(\vec{k}) \\ \cap\!\!\!\!\mid & & \cap\!\!\!\!\mid \\ S^2 & & S^2 \end{array}$$

具体计算:

Calculation of winding number:

$\pi_1(S^1)$: $\frac{1}{2\pi i}\oint\left(\frac{dz}{z}\right) \Rightarrow z = (x+iy) = n_1 + i n_2$ if $n_1^2+n_2^2=1$

$= \cancel{(\ldots)(实数时)} \frac{1}{2\pi} \int \cancel{(n_1 dn_2 - n_2 dn_1)}$

$= \frac{1}{2\pi} \int \epsilon^{\mu\nu} n_\mu (\partial_x n_\nu) dx$

$= C_1 \int \epsilon^{\mu\nu} (n_\mu \partial_x n_\nu) dx$

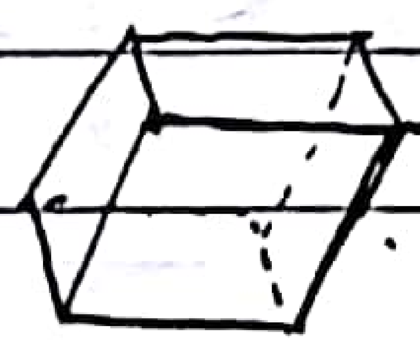
$\pi_2(S^2)$: $C_2 \int \epsilon^{\mu\nu\sigma} n_\mu (\partial_x n_\nu)(\partial_y n_\sigma) dx dy$ (W-Z term)

$\pi_3(S^3)$: $C_3 \int \epsilon^{\mu\nu\sigma\tau} n_\mu (\partial_x n_\nu)(\partial_y n_\sigma)(\partial_z n_\tau) dx dy dz$

$\vec{a}\cdot(\vec{b}\times\vec{c})$

$=$ volume

$= \epsilon^{ijk} a_i b_j C_k$



$\boxed{C_2 \int \vec{n}\cdot(\vec{n}_x \times \vec{n}_y) dx dy}$

$\updownarrow$ WZ term

$\begin{cases} \text{Berry phase} \\ \text{Haldane model} \end{cases}$

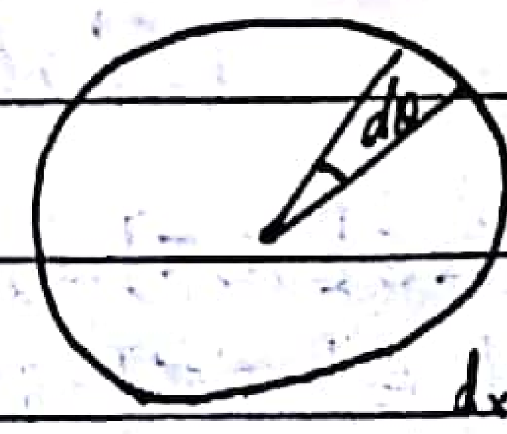
physical meaning of winding number.

$d=1$, $n_1^2+n_2^2=1 \Rightarrow n_1 = \cos\theta$, $n_2 = \sin\theta$

$\frac{1}{2\pi}(n_1 \partial n_2 - n_2 \partial n_1)$

$= \frac{1}{2\pi}[\cos^2\theta\, d\theta - \sin\theta(-\sin\theta) d\theta]$

$= (\frac{1}{2\pi} d\theta)$

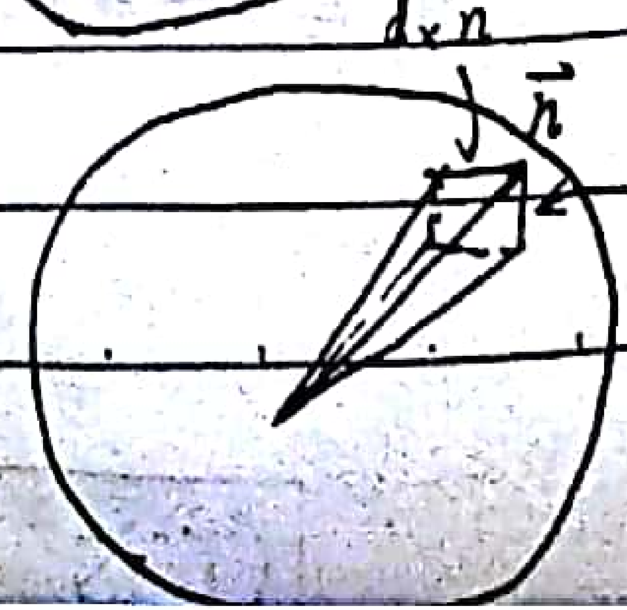


$d=2$, $\frac{1}{3}\vec{n}\cdot(d_x\vec{n} \times d_y\vec{n})$

$= \frac{1}{3}|\vec{n}| \frac{d\Omega |n|^2}{ds} = \frac{1}{3} d\Omega |n|^3$

$d\Omega = \vec{n}\cdot(\vec{n}_x \cdot \vec{n}_y) dx dy$

$\rightarrow C_2 \int d\Omega$



$\left\{ d=3 : \right.$

check

HW1: ① $|\vec{n}| \neq 1$ 时，$\overset{出来}{\frac{1}{n^3}} \vec{n}\cdot(\vec{n}_x \times \vec{n}_y)$

② $\begin{cases} n_x = \Delta \sin k_x \\ n_y = \Delta \sin k_y \\ n_z = \cos k_x + \cos k_y - \mu \end{cases}$ calculate the phase diag scanning Δ, μ.

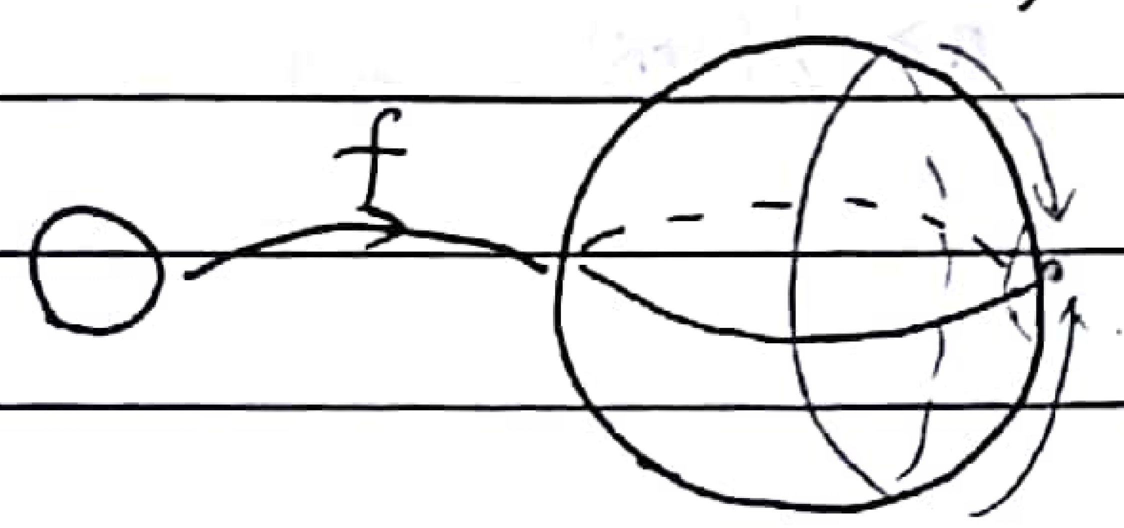
3. $\pi_3(S^2) = Z$

先看 $\pi_2(S^1)$: $\underset{\underset{x^2+y^2+z^2=1}{S^2}}{\underset{\cap}{(x,y,z)}} \xrightarrow{f} \underset{\underset{x'^2+y'^2=1}{S^1}}{\underset{\cap}{(x',y')}}$

let $x'^2 = \frac{x^2}{2} + \frac{y^2}{2} + xy$

$y'^2 = z^2 + \frac{x^2}{2} + \frac{y^2}{2} - xy$

$\pi_1(S^2)$



$\pi_1(S^2) = 0$

$\pi_3(S^2)$: $\underset{\underset{S^3}{\cap}}{(x^2+y^2+z^2+w^2)} \xrightarrow{f} \underset{\underset{S^2}{\cap}}{x'^2+y'^2+z'^2=1}$

HW2: L.M. Duan group. PRB, 2015

Hopf insulator.

$S^n \to S^m$

$S^n \to$ Lie group $M = SU(N), O(N), SO(N) \dots$

$g_k \in$ matrix

winding number: $\frac{1}{2\pi i}\oint\left(\frac{dg}{g}\right)$ complex scalar $(S^1 \to S^1)$

$\Downarrow$ 推广

Matrix-form winding number $\frac{1}{2\pi i}\oint \mathrm{Tr}(g^{-1}\overset{\text{Matrix}}{dg})$ $(S^1 \to M)$

if $g = U^{+}\lambda U$

$= \frac{1}{2\pi i}\int \sum_i \left(\frac{d\lambda_i}{\lambda_i}\right) \in Z$

其中 $g = \begin{cases} z\, I \\ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} \\ \text{more general matrix} \end{cases}$

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$\Rightarrow \quad C_n \int \mathrm{Tr}\left[(g^{-1}dg)^n\right]$. here $A^n \equiv \underbrace{A \wedge A \wedge A \cdots A}_{n}$

Proof $n=2k$, $W \equiv 0$ (偶数维恒为0)

例: $n=2$, $C_2 \int \mathrm{Tr}\left[(g^{-1}dg)^2\right] \equiv 0$

$$g^{-1}dg = F_i\, dx^i$$

$$2\mathrm{Tr}\left[(g^{-1}dg)\wedge(g^{-1}dg)\right]$$
$$= 2\mathrm{Tr}\left[(F_i\, dx^i)\wedge(F_j\, dx^j)\right]$$
$$= \mathrm{Tr}\left[(F_iF_j)\, dx^i \wedge dx^j\right] + \mathrm{Tr}\left[(\underset{F_iF_j}{\overset{\parallel}{F_jF_i}})\, dx^j \wedge dx^i\right]$$
$$= 0$$

$$\pi_k(U(n)) = \begin{cases} 0, & k \text{ is even} \\ \mathbb{Z}, & k \text{ is odd} \end{cases} \qquad n \geq \frac{1}{2}(1+k)$$

找好 S^n、M

注意: $S^n \longrightarrow M$

(物理一般 k-space) $\left\{ \begin{array}{l} \left.\begin{array}{l} \vec{B}(\vec{x}) \\ \vec{n}(\vec{x}) \end{array}\right\} S^m \\ H(\vec{x}) \in \text{Lie group} \end{array} \right.$

Winding number

	scalar	Matrix
d=1		
d=2		
d=3		

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Topo field Theory

先写出 action $S = \int \mathcal{L}\, dx\, dt$

$$= \underbrace{S_0}_{\text{(trivial term)}} + S_{topo} \quad (\theta\text{-term})$$

kinetic energy, $V(\pi)$, $V(\phi)$, $(\nabla\pi)^2$, $(\nabla\phi)^2$

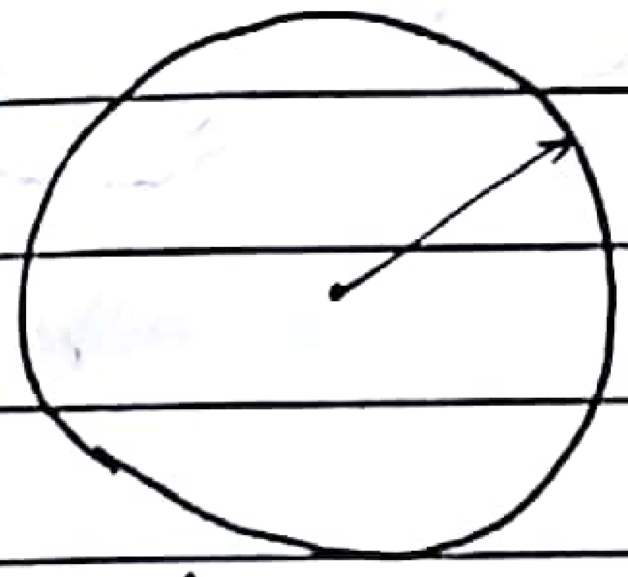
$$= i\theta W$$

(winding number/topo number)

例: W-Z term

$\boxed{\text{Model.}}$ Particle in a ring (P153, problem 3.5; P497, 9.1)

1. (P153, problem 3.5)



$S^1 \neq \mathbb{R}^1$

$$H = -\frac{1}{2I}\frac{\partial^2}{\partial\theta^2} \qquad (I \propto mr^2)$$

$$\varepsilon_n = \frac{n^2}{2I}$$

$$\psi_n = \frac{1}{\sqrt{2\pi}} e^{inx}$$

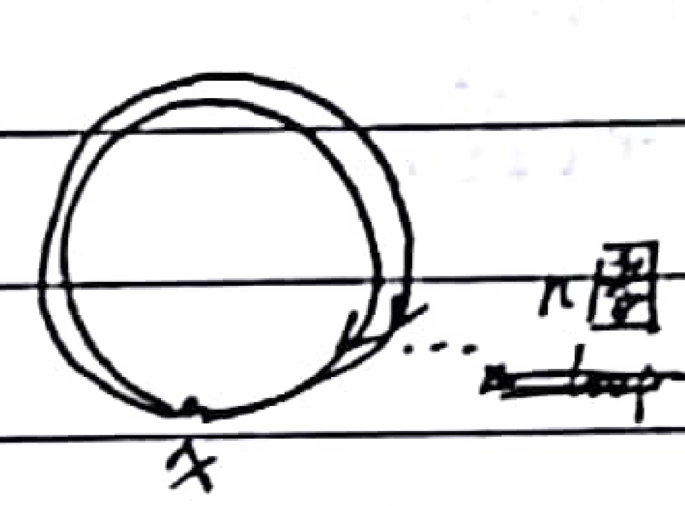
$$Z = \sum_n e^{-\beta\varepsilon_n} = \sum_n e^{-\beta n^2/2I}$$

Path Integral:

$$Z = \mathrm{Tr}(e^{-\beta H})$$

$$= \int \langle x| e^{-\beta H} |x\rangle\, dx$$

$$= \sum_n \int \langle x| e^{-\beta H} |x\rangle_n\, dx$$



duality

$$\sum_n f(n) = \sum_n \tilde{f}(n)$$

$\tilde{f}(n) = FT(f(n))$

$$\mathcal{L} = \frac{I}{2}\dot{\theta}^2$$

$$\pi = \frac{\partial \mathcal{L}}{\partial\dot{\theta}} = I\dot{\theta}$$

$$H = \pi\dot{\theta} - \mathcal{L} = \frac{\pi^2}{I} - \frac{\pi^2}{2I} = \frac{\pi^2}{2I}$$

eq of motion: $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{\partial\mathcal{L}}{\partial\theta} \Longleftrightarrow \boxed{I\ddot{\theta} = 0}$

Solution: $\theta(t) = \theta_0 + \alpha t$

在这里, 要求 $x \longrightarrow x$ (能回到原点) 故 $\alpha\beta = 2\pi n$.

$$\theta = \underbrace{\theta_0 + \frac{2\pi n}{\beta} t}_{\text{classical}} + \varphi$$

fluctuation

$$\theta = \theta_0 + \frac{2\pi n}{\beta} t + \varphi_n$$

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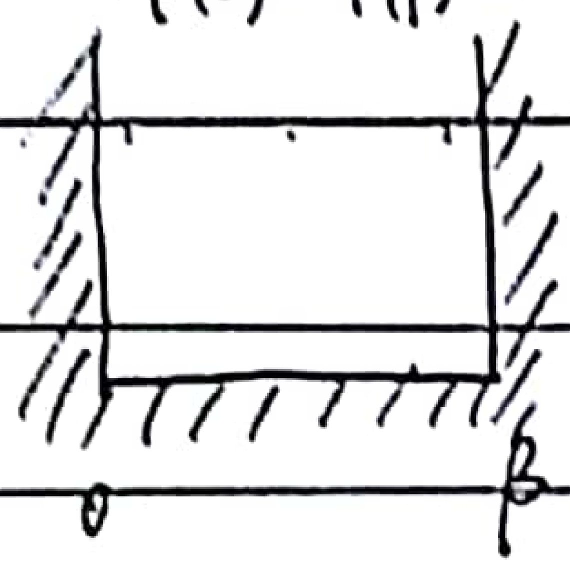
$$\dot\theta = \frac{2\pi n}{\beta} + \dot\varphi_n$$

$\varphi(0)=\varphi(\beta)=0$



Simons book P110 $\int_0^\beta \mathcal{L}\, dt$

$$\int_0^\beta \dot\theta^2 dt$$

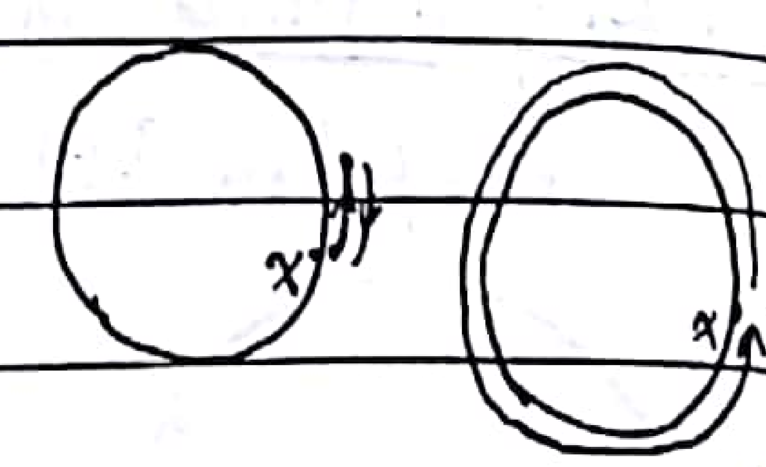
$$= \int_0^\beta \Big[\Big(\frac{2\pi n}{\beta}\Big)^2 + \cancel{\frac{4\pi n}{\beta}\dot\varphi} + \dot\varphi^2\Big] dt$$

边界项

$$Z = \sum_n \int D\theta\, e^{-\int_0^\beta \frac{I}{2}\dot\theta^2 dt}$$

$$= \sum_n e^{-\frac{I}{2}\left(\frac{2\pi n}{\beta}\right)^2\beta}\Big[\int D\varphi_n - \int_0^\beta \frac{I}{2}\dot\varphi_n^2 dt\Big] ?$$

与 n 无关，$\varphi_n \to \varphi$. 物理上来看 $e^{iS/\hbar}$, fluctuation 与圈数n无关



$$= \Big[\sum_n e^{-\frac{I}{2}\left(\frac{2\pi n}{\beta}\right)^2\beta}\Big]\cdot\Big[\int D\varphi - \int_0^\beta \frac{I}{2}\dot\varphi^2 dt\Big] ?$$

$$= \Big(\sum_n e^{-\frac{I}{2}\left(\frac{2\pi n}{\beta}\right)^2\beta}\Big)\cdot\sqrt{\frac{I}{2\pi\beta}}$$

fluctuation

如何实现

$$\sum_n e^{-\beta\left(\frac{n^2}{2I}\right)}$$

<u>Poisson Summation</u>

$$\boxed{\sum_{n=-\infty}^{+\infty} f(n) = \sum_n \tilde f(n)} \qquad \tilde f(n) = FT[f(n)]$$

Proof: $F(x) = \sum_n f(x+n) = \sum_m a_m e^{-i2\pi m x}$

$$a_m = \int_0^1 F(x)\, e^{2i\pi m x} dx$$

$$= \int_0^1 \sum_n f(x+n)\, e^{2i\pi m x} dx$$

$$= \sum_n \int_0^1 f(x+n)\, e^{2i\pi(x+n)} d(x+n) \quad \text{because } e^{i2\pi mn} = 1 \ (mn\text{为整数})$$

$$= \tilde f(m)$$

$$a_m = \tilde f(m)$$

$$\Leftrightarrow F(x) = \sum_n f(x+n) = \sum_n \tilde f(n)\, e^{2\pi i n x}$$

when $x=0$: $\boxed{\sum_n f(n) = \sum_n \tilde f(n)}$

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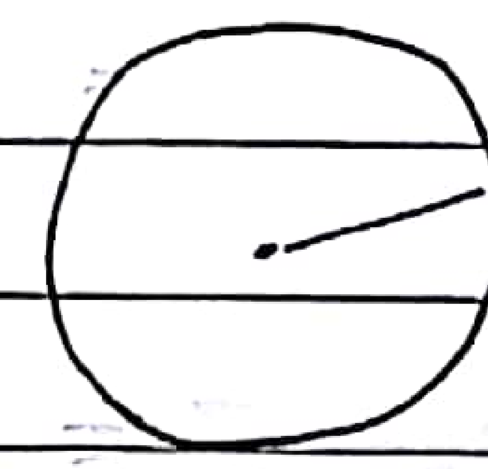
$$Z=\sqrt{\frac{2\pi I}{\beta}}\sum_m e^{-\frac{I}{2}\frac{(2\pi m)^2}{\beta^2}\beta}=\sum_m e^{-\frac{I}{2}\frac{(2\pi)^2 m^2}{\beta}}\cdot\sqrt{\frac{2\pi I}{\beta}}$$

$$=\boxed{\sqrt{\frac{2\pi I}{\beta}}}\underbrace{\sum_{n=-\infty}^{+\infty}\int d\phi\, e^{-\frac{(2\pi)^2 m^2 I}{2\beta}+2\pi i n\phi}}_{\text{classical}}$$

fluctuation

$$=\sum_n e^{-(\beta n^2/2I)}$$



fluctuation、classical 脱耦

目的：在拓扑空间做 Path Integral，分成不同部分来做.

2. (P497, 9.1) Particle in a ring with gauge potential.

$\phi=B\cdot S$

$$\boxed{\mathcal{L}=\frac{M}{2}\dot{\phi}^2+A\dot{\phi}}\qquad M\equiv I \ \text{(ref. Abanov)}$$

$$S=\int\mathcal{L}\,dt$$

☆ $A\dot{\phi}=\left(\frac{df}{dt}\right)$, $f=f(\phi)$

☆ $\mathcal{L}\to\mathcal{L}+\frac{df}{dt}=\mathcal{L}+f(q)\,\dot{q}$

$\updownarrow$ Topo term
- 在 S、Eq of motion 中 体现不出
- 但在 Topo effect 中 很重要. 如 $\begin{cases}\int_R df=0\\ \int_{S^1} df=2\pi n\end{cases}$

☆ Eq of motion $\Rightarrow$ $\boxed{M\ddot{\phi}=0}$ $\qquad \phi=\phi_0+\frac{2\pi n}{\beta}+\varphi$

注意：Topo term 不影响 S，不影响 Eq of motion

但对物理却有实在的影响！（energy spectrum、色散关系）

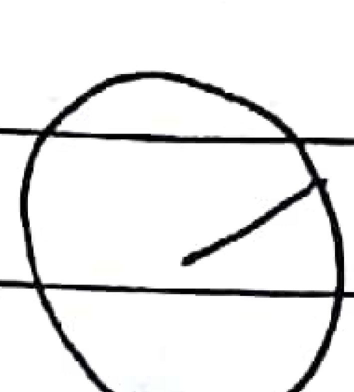
如：$\begin{cases}H=\frac{1}{2M}(p-A)^2\\ p=-i\partial_\phi\end{cases}$ Solution：$\left.\begin{array}{l}\varepsilon_n=\frac{1}{2M}(n-A)^2\\ \psi_n=\frac{1}{\sqrt{2\pi}}e^{in\phi}\end{array}\right\}$ energy spectrum 改变了!

$$\boxed{Z=\sum_n e^{-\beta\frac{1}{2M}(n-A)^2}}=e^{-\beta F(A)}$$

从另一个角度理解：

做么正变换：$U=e^{iAx}\ \to\ U^{+}HU=\frac{p^2}{2M}$

但边界条件改变了



$e^{in\phi}$ $\qquad \left.\begin{array}{l}e^{iA\cdot 0}\\ e^{iA\cdot 2\pi}\end{array}\right\}$ twist boundry condition

平时. $Z(i\beta) = \int D\phi \, e^{iS[\phi]/\hbar}$

$= \int D\phi \, e^{i\int_0^\beta \mathcal{L}(\phi,\dot{\phi})dt/\hbar}$

$\Downarrow$ Wick rotation $t \to i\tau$

but $Z \neq \int D\phi \, e^{-\int_0^\beta \mathcal{L}\, d\tau/\hbar}$ $\otimes$.

trivial term 中 $\quad i\int_0^t \mathcal{L}(\dot{\phi},\phi)dt$

$\left(\dot{\phi}^2 = \left(\frac{d\phi}{dt}\right)^2 \xrightarrow{t\to i\tau} \left(\frac{d\phi}{i\,d\tau}\right)^2 = -\dot{\phi}^2\right)$

$= i\int[\dot{\phi}^2 - V(\phi)]dt$

$\Rightarrow -\int_0^\beta [\dot{\phi}^2 + V(\phi)]dt$

topo term 中: $\quad i\int A\dot{\phi}\, dt = i\int A\frac{d\phi}{dt}dt$

$\Downarrow t\to i\tau$

$i\int A\frac{d\phi}{d(i\tau)}d(i\tau)$

$\Rightarrow i\int A\dot{\phi}\, dt$

☆ topo term $\xrightarrow{\text{Wick}}$ 实质不改变~~实质~~

形式也不改变！（还是一个纯虚数）.

$Z(i\beta) = \int D\phi \, e^{iS[\phi]/\hbar}$

$\downarrow$ Wick

$Z = \int D\phi \, e^{-\int_0^\beta (\frac{M}{2}\dot{\phi}^2 - iA\dot{\phi})dt}$ $\quad \cdots$ (eq. 9.4)

$\phi = \phi_0 + \frac{2\pi n}{\beta}t + \varphi_n$

$= \sum_n \int D\varphi_n \, e^{-\int_0^\beta [\frac{M}{2}(\frac{2\pi n}{\beta}^2 + \dot{\varphi}_n^2) - iA(\frac{2\pi n}{\beta} + \dot{\varphi}_n)]dt}$

$= \sum_n e^{-\frac{M}{2}(\frac{2\pi n}{\beta})^2\beta - iA\frac{2\pi n}{\beta}\cdot\beta} \underbrace{\left(\int D\varphi \, e^{-\int_0^\beta \frac{M}{2}\dot{\varphi}^2 dt}\right)}_{\sqrt{\frac{M}{2\pi\beta}}}$

$\downarrow$ Poisson summation

$= \sum_n e^{-\beta\frac{(n-A)^2}{2M}}$

$A \to A+k$ 不变

~~Summary:~~

Simons book P500

§9.1

1. $\int D\phi$

$\Downarrow$

$\sum_n$ (sum in disjointed space)

~~要分类 $[\theta] = \{\theta + 2\pi n \mid n \in \mathbb{Z}\}$~~

2. $S_{topo} \equiv i\theta W = -iA \underbrace{\int_0^\beta d\phi}_{W}$ unchanged under Wick rotation.

3. $t \to i\tau$

$\frac{d\phi}{dt} dt \to \frac{d\phi}{d(\alpha t)} d(\alpha t)$ unchanged

~~4. S_{topo} is imaginary (纯虚数)~~

General idea (topo number & topo term).

Topo number : $\pi_n(S^m)$ $\quad \frac{1}{2\pi i}\oint \frac{dz}{z} \longrightarrow$ matrix

$\pi_n(M)$ $\quad \downarrow$ high dim

$\Updownarrow$

Topo term : $S \equiv \underbrace{S_0}_{trivial} + \underbrace{S_{topo}}_{topo}$ (QFT)

二者结合的一个例子.

WZ term: $\frac{1}{4\pi}\oint \vec{n}\cdot(\vec{n}_x \times \vec{n}_y)\, dx\, dy \Rightarrow$ skyrmion (P508 Fig)

(real space)

~~可以用在3个方向~~

$\Updownarrow$

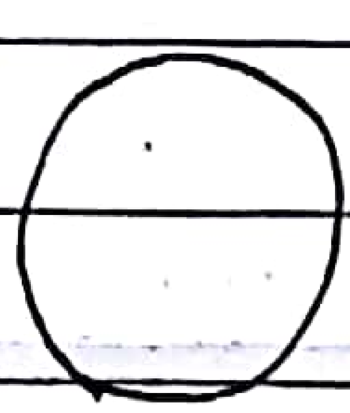
$\frac{1}{4\pi}\oint \vec{n}\cdot(\vec{n}_{k_x} \times \vec{n}_{k_y})\, dk_x\, dk_y \Rightarrow$ Topo SC/insulator

(k-space)

$\Updownarrow$

$\frac{1}{4\pi}\oint \vec{n}\cdot(\vec{n}_x \times \vec{n}_t)\, dx\, dt \Rightarrow$ WZ term in QFT

(Minkowski space)



~~$W = \frac{1}{2\pi i}\oint \frac{dz}{z}$.~~

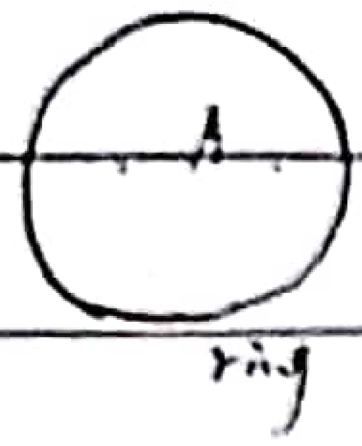
$i\theta W$

$\theta \propto A$.

Date. No.

~~gauge potential A~~

1d.



$S_{topo} = iAW$

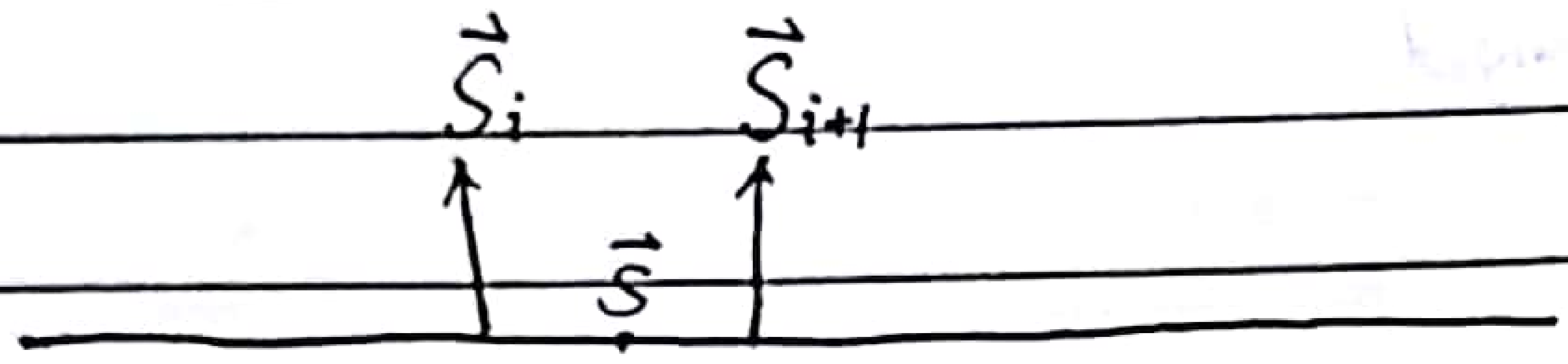
2d (1+1d)

$$H = J\sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

AFM (J > 0) $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\cdots$

令 $\underbrace{\vec{S}_i \to (-1)^i \vec{S}_i}$ $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$ (为了连续化)

注 这个变换也不应该改变 topo term 的性质.



$$\begin{cases} \vec{S}_{i+1} = \vec{S} - (\partial_x \vec{S})\cdot \frac{a}{2} \\ \vec{S}_i = \vec{S} + \frac{a}{2}(\partial_x \vec{S}) \end{cases}$$

$$\vec{S}_i\cdot\vec{S}_{i+1} = \vec{S}\cdot\vec{S} + \frac{a}{2}(\partial\vec{S})\cdot\vec{S} - \frac{a}{2}\vec{S}\cdot(\partial\vec{S}) - \frac{a^2}{4}(\partial S)^2$$

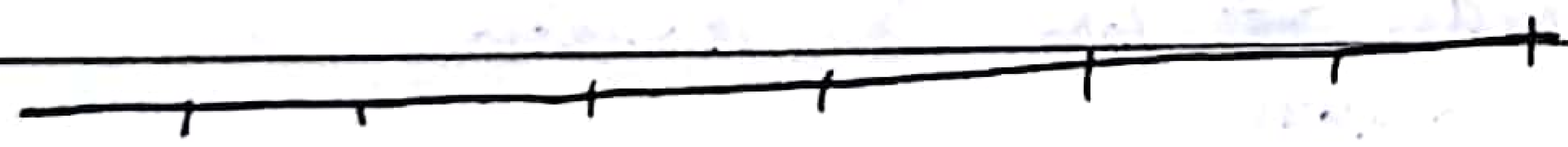
积分后只剩

$-(\partial\vec{S})^2 \Rightarrow NL\sigma M$

A single spin

$$\int_0^\beta (1-\cos\theta)\,d\phi = \int_0^\beta p\,dq \begin{cases} p \sim 1-\cos\theta \\ q \sim \phi \end{cases}$$

$$= \int_0^\beta \langle \vec{n}|\partial_t|\vec{n}\rangle$$

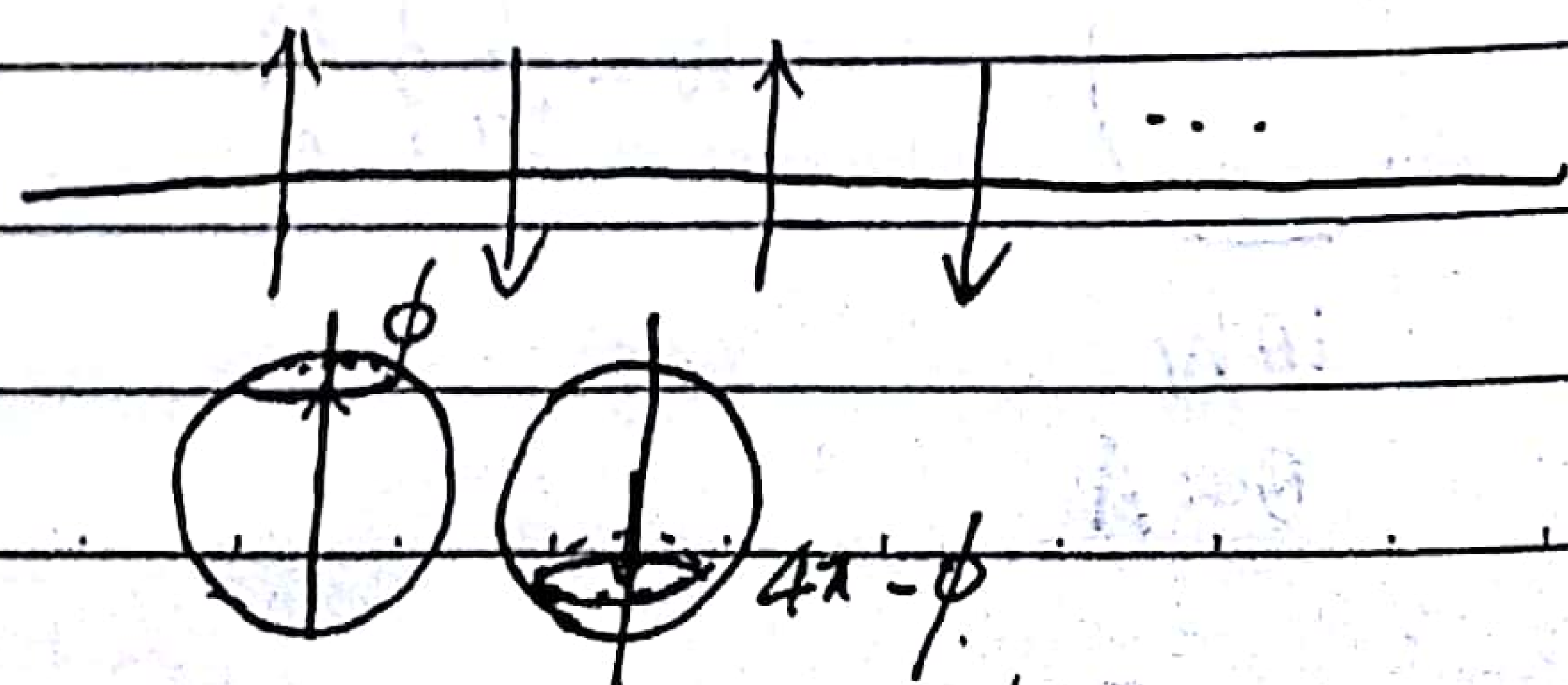


$$\int D\vec{n}_i\, e^{i\sum_i(\underbrace{\langle \vec{n}_i|\partial_t|\vec{n}_i\rangle}_{\text{Local}} - H_{i,i+1})dt}$$

前面的 $\sim \int (\partial S)^2 dx\,dt$.

In a lattice chain

$$\sum_i [\langle \vec{n}_i|\partial_t|\vec{n}_i\rangle]dt$$



奇数格点 = − 偶数格点.

$$\sum_{i=even} \left(\langle \vec{n}_i | \partial_t | \vec{n}_i \rangle + 4\pi - \langle \vec{n}_{i+1} | \partial_t | \vec{n}_{i+1} \rangle \right)$$

$$= -\sum_i \left(\langle n_{i+1} | \partial_t | \partial_{i+1} \rangle - \langle n_i | \partial_t | n_i \rangle \right) dt$$

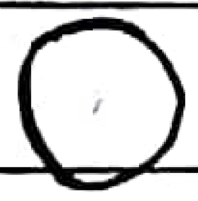
let $|n_{i+1}\rangle = |n_i\rangle + (\partial_x |n_i\rangle) dx$

$$= \int \left(\langle \partial_x \vec{n} | \partial_t \vec{n} \rangle - \langle \partial_t \vec{n} | \partial_x \vec{n} \rangle \right) dx dt$$

$$\Updownarrow$$

$$\int \vec{n} \cdot (\vec{n}_x \times \vec{n}_y) \, dx dt$$

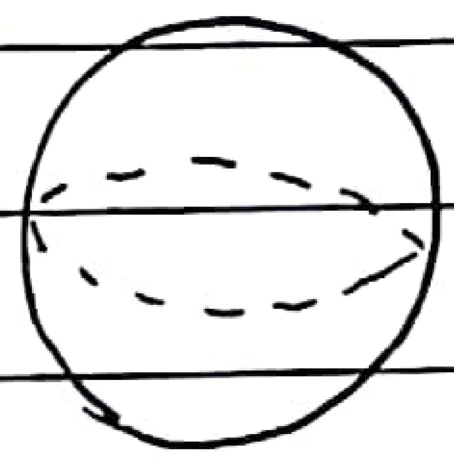
1d $\int d\phi$



2d $\frac{1}{4\pi} \int d\Omega$

$$= \frac{1}{4\pi} \int \vec{n} \cdot (\vec{n}_x \times \vec{n}_t) dx dt$$

WZ term



Anti-fermagnetism

(Haldane model)

$$\oint \quad S = S_{topo} + S_0$$

$$= i\theta W + S_0$$

核心: WZ term (不同用途, 不同名字)

$\Updownarrow$

Berry phase

下一次 SSB