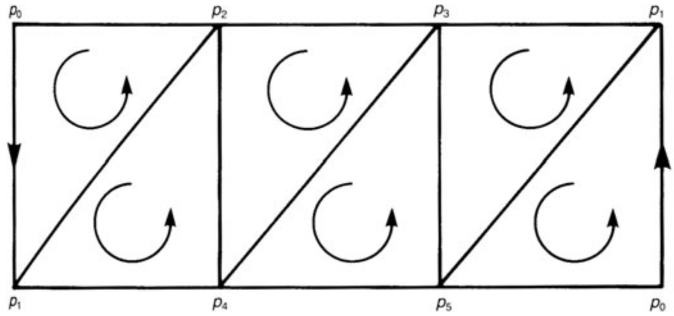

Figure 3.8. A triangulation of the Möbius strip.

Now we have proved for a connected complex K that $z = \sum n_i p_i \in B_0(K)$ if and only if $\sum n_i = 0$.

Define a surjective homomorphism $f : Z_0(K) \to \mathbb{Z}$ by

$$f(n_1 p_1 + \cdots + n_{I_0} p_{I_0}) = \sum_{i=1}^{I_0} n_i.$$

We then have $\ker f = f^{-1}(0) = B_0(K)$. It follows from theorem 3.1 that $H_0(K) = Z_0(K)/B_0(K) = Z_0(K)/\ker f \cong \operatorname{im} f = \mathbb{Z}$. □

3.3.5 More homology computations

Example 3.10. This and the next example deal with homology groups of non-orientable spaces. Figure 3.8 is a triangulation of the Möbius strip. Clearly $B_2(K) = 0$. Let us take a cycle $z \in Z_2(K)$,

$$\begin{aligned} z = {} & i(p_0 p_1 p_2) + j(p_2 p_1 p_4) + k(p_2 p_4 p_3) \\ & + l(p_3 p_4 p_5) + m(p_3 p_5 p_1) + n(p_1 p_5 p_0). \end{aligned}$$

z satisfies

$$\begin{aligned} \partial_2 z = {} & i\{(p_1 p_2) - (p_0 p_2) + (p_0 p_1)\} \\ & + j\{(p_1 p_4) - (p_2 p_4) + (p_2 p_1)\} \\ & + k\{(p_4 p_3) - (p_2 p_3) + (p_2 p_4)\} \\ & + l\{(p_4 p_5) - (p_3 p_5) + (p_3 p_4)\} \\ & + m\{(p_5 p_1) - (p_3 p_1) + (p_3 p_5)\} \\ & + n\{(p_5 p_0) - (p_1 p_0) + (p_1 p_5)\} = 0. \end{aligned}$$

Since each of $(p_0 p_2)$, $(p_1 p_4)$, $(p_2 p_3)$, $(p_4 p_5)$, $(p_3 p_1)$ and $(p_5 p_0)$ appears once and only once in $\partial_2 z$, all the coefficients must vanish, $i = j = k = l = m = n =$

0. Thus, $Z_2(K) = \{0\}$ and

$$H_2(K) = Z_2(K)/B_2(K) \cong \{0\}. \tag{3.44}$$

To find $H_1(K)$, we use our intuition rather than doing tedious computations. Let us find the loops which make complete circuits. One such loop is

$$z = (p_0 p_1) + (p_1 p_4) + (p_4 p_5) + (p_5 p_0).$$

Then all the other complete circuits are homologous to multiples of z. For example, let us take

$$z' = (p_1 p_2) + (p_2 p_3) + (p_3 p_5) + (p_5 p_1).$$

We find that $z \sim z'$ since

$$z - z' = \partial_2\{(p_2 p_1 p_4) + (p_2 p_4 p_3) + (p_3 p_4 p_5) + (p_1 p_5 p_0)\}.$$

If, however, we take

$$z'' = (p_1 p_4) + (p_4 p_5) + (p_5 p_0) + (p_0 p_2) + (p_2 p_3) + (p_3 p_1)$$

we find that $z'' \sim 2z$ since

$$\begin{aligned} 2z - z'' &= 2(p_0 p_1) + (p_1 p_4) + (p_4 p_5) + (p_5 p_0) - (p_0 p_2) \\ &\quad - (p_2 p_3) - (p_3 p_1) \\ &= \partial_2\{(p_0 p_1 p_2) + (p_1 p_4 p_2) + (p_2 p_4 p_3) + (p_3 p_4 p_5) \\ &\quad + (p_3 p_5 p_1) + (p_0 p_1 p_5)\}. \end{aligned}$$

We easily verify that all the closed circuits are homologous to nz, $n \in \mathbb{Z}$. $H_1(K)$ is generated by just one element $[z]$,

$$H_1(K) = \{i[z] | i \in \mathbb{Z}\} \cong \mathbb{Z}. \tag{3.45}$$

Since K is connected, it follows from theorem 3.5 that $H_0(K) = \{i[p_a] | i \in \mathbb{Z}\} \cong \mathbb{Z}$, p_a being any 0-simplex of K.

Example 3.11. The projective plane $\mathbb{R}P^2$ has been defined in example 2.5(c) as the sphere S^2 whose antipodal points are identified. As a coset space, we may take the hemisphere (or the disc D^2) whose opposite points on the boundary S^1 are identified, see figure 2.5(*b*). Figure 3.9 is a triangulation of the projective plane. Clearly $B_2(K) = \{0\}$. Take a cycle $z \in Z_2(K)$,

$$\begin{aligned} z = &\, m_1(p_0 p_1 p_2) + m_2(p_0 p_4 p_1) + m_3(p_0 p_5 p_4) \\ &+ m_4(p_0 p_3 p_5) + m_5(p_0 p_2 p_3) + m_6(p_2 p_4 p_3) \\ &+ m_7(p_2 p_5 p_4) + m_8(p_2 p_1 p_5) + m_9(p_1 p_3 p_5) + m_{10}(p_1 p_4 p_3). \end{aligned}$$

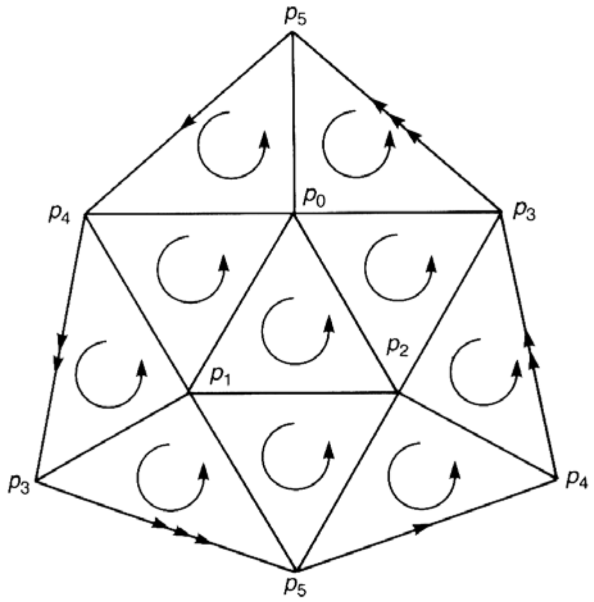

Figure 3.9. A triangulation of the projective plane.

The boundary of z is

$$\begin{aligned}\partial_2 z = m_1\{&(p_1p_2) - (p_0p_2) + (p_0p_1)\}\\ + m_2\{&(p_4p_1) - (p_0p_1) + (p_0p_4)\}\\ + m_3\{&(p_5p_4) - (p_0p_4) + (p_0p_5)\}\\ + m_4\{&(p_3p_5) - (p_0p_5) + (p_0p_3)\}\\ + m_5\{&(p_2p_3) - (p_0p_3) + (p_0p_2)\}\\ + m_6\{&(p_4p_3) - (p_2p_3) + (p_2p_4)\}\\ + m_7\{&(p_5p_4) - (p_2p_4) + (p_2p_5)\}\\ + m_8\{&(p_1p_5) - (p_2p_5) + (p_2p_1)\}\\ + m_9\{&(p_3p_5) - (p_1p_5) + (p_1p_3)\}\\ + m_{10}\{&(p_4p_3) - (p_1p_3) + (p_1p_4)\} = 0.\end{aligned}$$

Let us look at the coefficient of each 1-simplex. For example, we have $(m_1 - m_2)(p_0p_1)$, hence $m_1 - m_2 = 0$. Similarly,

$$\begin{aligned}&-m_1 + m_5 = 0, m_4 - m_5 = 0, m_2 - m_3 = 0, m_1 - m_8 = 0,\\ &m_9 - m_{10} = 0, -m_2 + m_{10} = 0, m_5 - m_6 = 0, m_6 - m_7 = 0,\\ &m_6 + m_{10} = 0.\end{aligned}$$