6

DE RHAM COHOMOLOGY GROUPS

The homology groups of topological spaces have been defined in chapter 3. If a topological space M is a manifold, we may define the *dual* of the homology groups out of differential forms defined on M. The dual groups are called the de Rham cohomology groups. Besides physicists' familiarity with differential forms, cohomology groups have several advantages over homology groups.

We follow closely Nash and Sen (1983) and Flanders (1963). Bott and Tu (1982) contains more advanced topics.

6.1 Stokes' theorem

One of the main tools in the study of de Rham cohomology groups is Stokes' theorem with which most physicists are familiar from electromagnetism. Gauss' theorem and Stokes' theorem are treated in a unified manner here.

6.1.1 Preliminary consideration

Let us define an integration of an r-form over an r-simplex in a Euclidean space. To do this, we need first to define the **standard n-simplex** $\bar{\sigma}_r = (p_0 p_1 \ldots p_r)$ in $\mathbb{R}^r$ where

$$\begin{aligned} p_0 &= (0, 0, \ldots, 0) \\ p_1 &= (1, 0, \ldots, 0) \\ \ldots \\ p_r &= (0, 0, \ldots, 1) \end{aligned}$$

see figure 6.1. If $\{x^\mu\}$ is a coordinate of $\mathbb{R}^r$, $\bar{\sigma}_r$ is given by

$$\bar{\sigma}_r = \left\{ (x^1, \ldots, x^r) \in \mathbb{R}^r \middle| x^\mu \geq 0, \sum_{\mu=1}^{r} x^\mu \leq 1 \right\}. \tag{6.1}$$

An r-form ω (the volume element) in $\mathbb{R}^r$ is written as

$$\omega = a(x)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^r.$$

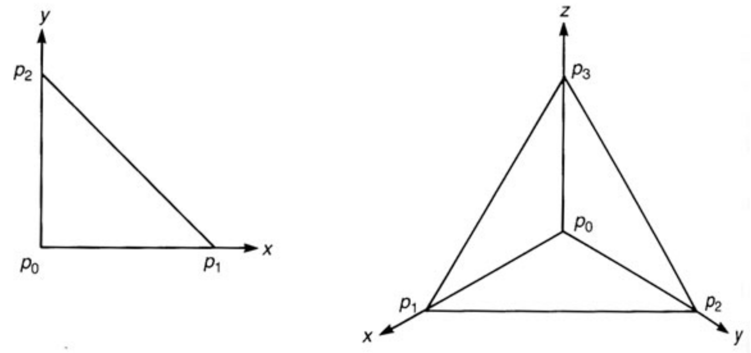

Figure 6.1. The standard 2-simplex $\bar{\sigma}_2 = (p_0 p_1 p_2)$ and the standard 3-simplex $\bar{\sigma}_3 = (p_0 p_1 p_2 p_3)$.

We define the integration of ω over $\bar{\sigma}_r$ by

$$\int_{\bar{\sigma}_r} \omega \equiv \int_{\bar{\sigma}_r} a(x)\, \mathrm{d}x^1\, \mathrm{d}x^2 \ldots \mathrm{d}x^r \tag{6.2}$$

where the RHS is the usual r-fold integration. For example, if $r = 2$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y$, we have

$$\int_{\bar{\sigma}_2} \omega = \int_{\bar{\sigma}_2} \mathrm{d}x\, \mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y = \tfrac{1}{2}.$$

Next we define an r-chain, an r-cycle and an r-boundary in an m-dimensional manifold M. Let σ_r be an r-simplex in $\mathbb{R}^r$ and let $f : \sigma_r \to M$ be a smooth map. [To avoid the subtlety associated with the differentiability of f at the boundary of σ_r, f may be defined over an open subset U of $\mathbb{R}^r$, which contains σ_r.] Here we assume f is not required to have an inverse. For example, im f may be a point in M. We denote the image of σ_r in M by s_r and call it a (**singular**) ***r*-simplex** in M. These simplexes are called singular since they do not provide a triangulation of M and, moreover, *geometrical independence* of points makes no sense in a manifold (see section 3.2). If $\{s_{r,i}\}$ is the set of r-simplexes in M, we define an ***r*-chain** in M by a formal sum of $\{s_{r,i}\}$ with $\mathbb{R}$-coefficients

$$c = \sum_i a_i s_{r,i} \qquad a_i \in \mathbb{R}. \tag{6.3}$$

In the following, we are concerned with $\mathbb{R}$-coefficients only and we omit the explicit quotation of $\mathbb{R}$. The r-chains in M form the **chain group** $C_r(M)$. Under $f : \sigma_r \to M$, the boundary $\partial\sigma_r$ is also mapped to a subset of M. Clearly, $\partial s_r \equiv f(\partial\sigma_r)$ is a set of $(r-1)$-simplexes in M and is called the **boundary** of

s_r. ∂s_r corresponds to the geometrical boundary of s_r with an induced orientation defined in section 3.3. We have a map

$$\partial : C_r(M) \to C_{r-1}(M). \tag{6.4}$$

The result of section 3.3 tells us that ∂ is nilpotent; $\partial^2 = 0$.

Cycles and boundaries are defined in exactly the same way as in section 3.3 (note, however, that $\mathbb{Z}$ is replaced by $\mathbb{R}$). If c_r is an ***r*-cycle**, $\partial c_r = 0$ while if c_r is an ***r*-boundary**, there exists an $(r+1)$-chain c_{r+1} such that $c_r = \partial c_{r+1}$. The **boundary group** $B_r(M)$ is the set of r-boundaries and the **cycle group** $Z_r(M)$ is the set of r-cycles. There are infinitely many singular simplexes which make up $C_r(M)$, $B_r(M)$ and $Z_r(M)$. It follows from $\partial^2 = 0$ that $Z_r(M) \supset B_r(M)$; cf theorem 3.3. The singular homology group is defined by

$$H_r(M) \equiv Z_r(M)/B_r(M). \tag{6.5}$$

With mild topological assumptions, the singular homology group is isomorphic to the corresponding simplicial homology group with $\mathbb{R}$-coefficients and we employ the same symbol to denote both of them.

Now we are ready to define an integration of an r-form ω over an r-chain in M. We first define an integration of ω on an r-simplex s_r of M by

$$\int_{s_r} \omega = \int_{\bar{\sigma}_r} f^*\omega \tag{6.6}$$

where $f : \bar{\sigma}_r \to M$ is a smooth map such that $s_r = f(\bar{\sigma}_r)$. Since $f^*\omega$ is an r-form in $\mathbb{R}^r$, the RHS is the usual r-fold integral. For a general r-chain $c = \sum_i a_i s_{r,i} \in C_r(M)$, we define

$$\int_c \omega = \sum_i a_i \int_{s_{r,i}} \omega. \tag{6.7}$$

6.1.2 Stokes' theorem

Theorem 6.1. (**Stokes' theorem**) Let $\omega \in \Omega^{r-1}(M)$ and $c \in C_r(M)$. Then

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega. \tag{6.8}$$

Proof. Since c is a linear combination of r-simplexes, it suffices to prove (6.8) for an r-simplex s_r in M. Let $f : \bar{\sigma}_r \to M$ be a map such that $f(\bar{\sigma}_r) = s_r$. Then

$$\int_{s_r} \mathrm{d}\omega = \int_{\bar{\sigma}_r} f^*(\mathrm{d}\omega) = \int_{\bar{\sigma}_r} \mathrm{d}(f^*\omega)$$

where (5.75) has been used. We also have

$$\int_{\partial s_r} \omega = \int_{\partial\bar{\sigma}_r} f^*\omega.$$

Note that $f^*\omega$ is an $(r-1)$-form in $\mathbb{R}^r$. Thus, to prove Stokes' theorem

$$\int_{s_r} \mathrm{d}\omega = \int_{\partial s_r} \omega \tag{6.9a}$$

it suffices to prove an alternative formula

$$\int_{\bar{\sigma}_r} \mathrm{d}\psi = \int_{\partial\bar{\sigma}_r} \psi \tag{6.9b}$$

for an $(r-1)$-form ψ in $\mathbb{R}^r$. The most general form of ψ is

$$\psi = \sum a_\mu(x)\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{\mu-1} \wedge \mathrm{d}x^{\mu+1} \wedge \ldots \wedge \mathrm{d}x^r.$$

Since an integration is distributive, it suffices to prove (6.9b) for $\psi = a(x)\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{r-1}$. We note that

$$\mathrm{d}\psi = \frac{\partial a}{\partial x^r}\, \mathrm{d}x^r \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{r-1} = (-1)^{r-1} \frac{\partial a}{\partial x^r}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{r-1} \wedge \mathrm{d}x^r.$$

By direct computation, we find, from (6.2), that

$$\begin{aligned}
\int_{\bar{\sigma}_r} \mathrm{d}\psi &= (-1)^{r-1} \int_{\bar{\sigma}_r} \frac{\partial a}{\partial x^r}\, \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1}\, \mathrm{d}x^r \\
&= (-1)^{r-1} \int_{x^\mu \geq 0, \sum_{\mu=1}^{r-1} x^\mu \leq 1} \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1} \int_0^{1-\sum_{\mu=1}^{r-1} x^\mu} \frac{\partial a}{\partial x^r}\, \mathrm{d}x^r \\
&= (-1)^{r-1} \int \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1} \\
&\quad \times \left[a\left(x^1, \ldots, x^{r-1}, 1 - \sum_{\mu=1}^{r-1} x^\mu\right) - a\left(x^1, \ldots, x^{r-1}, 0\right) \right].
\end{aligned}$$

For the boundary of $\bar{\sigma}_r$, we have

$$\begin{aligned}
\partial\bar{\sigma}_r = (p_1, p_2, \ldots, p_r) &- (p_0, p_2, \ldots, p_r) \\
&+ \cdots + (-1)^r (p_0, p_1, \ldots, p_{r-1}).
\end{aligned}$$

Note that $\psi = a(x)\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{r-1}$ vanishes when one of $x^1, \ldots, x^{r-1}$ is constant. Then it follows that

$$\int_{(p_0, p_2, \ldots, p_r)} \psi = 0$$

since $x^1 \equiv 0$ on $(p_0, p_2, \ldots, p_r)$. In fact, most of the faces of $\partial\bar{\sigma}_r$ do not contribute to the RHS of (6.9b) and we are left with

$$\int_{\partial\bar{\sigma}_r} \psi = \int_{(p_1, p_2, \ldots, p_r)} \psi + (-1)^r \int_{(p_0, p_1, \ldots, p_{r-1})} \psi.$$

Since $(p_0, p_1, \ldots, p_{r-1})$ is the standard $(r-1)$-simplex ($x^\mu \geq 0$, $\sum_{\mu=1}^{r-1} x^\mu \leq 1$), on which $x^r = 0$, the second term is

$$(-1)^r \int_{(p_0, p_1, \ldots, p_{r-1})} \psi = (-1)^r \int_{\bar{\sigma}_{r-1}} a(x^1, \ldots, x^{r-1}, 0)\, \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1}.$$

The first term is

$$\begin{aligned}\int_{(p_1, p_2, \ldots, p_r)} \psi &= \int_{(p_1, \ldots, p_{r-1}, p_0)} a\left(x^1, \ldots, x^{r-1}, 1 - \sum_{\mu=1}^{r-1} x^\mu\right) \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1} \\ &= (-1)^{r-1} \int_{\bar{\sigma}_{r-1}} a\left(x^1, \ldots, x^{r-1}, 1 - \sum_{\mu=1}^{r-1} x^\mu\right) \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1}\end{aligned}$$

where the integral domain $(p_1, \ldots, p_r)$ has been projected along x^r to the $(p_1, \ldots, p_{r-1}, p_0)$-plane, preserving the orientation. Collecting these results, we have proved (6.9b). [The reader is advised to verify this proof for $m = 3$ using figure 6.1.] □

Exercise 6.1. Let $M = \mathbb{R}^3$ and $\omega = a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}z$. Show that Stokes' theorem is written as

$$\int_S \operatorname{curl} \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S} = \oint_C \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{S} \qquad \text{(Stokes' theorem)} \qquad (6.10)$$

where $\boldsymbol{\omega} = (a, b, c)$ and C is the boundary of a surface S. Similarly, for $\psi = \frac{1}{2}\psi_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, show that

$$\int_V \operatorname{div} \boldsymbol{\psi}\, \mathrm{d}V = \oint_S \boldsymbol{\psi} \cdot \mathrm{d}\mathbf{S} \qquad \text{(Gauss' theorem)}$$

where $\psi^\lambda = \varepsilon^{\lambda\mu\nu}\psi_{\mu\nu}$ and S is the boundary of a volume V.

6.2 de Rham cohomology groups

6.2.1 Definitions

Definition 6.1. Let M be an m-dimensional differentiable manifold. The set of closed r-forms is called the rth **cocycle group**, denoted $Z^r(M)$. The set of exact r-forms is called the rth **coboundary group**, denoted $B^r(M)$. These are vector spaces with $\mathbb{R}$-coefficients. It follows from $\mathrm{d}^2 = 0$ that $Z^r(M) \supset B^r(M)$.

Exercise 6.2. Show that

(a) if $\omega \in Z^r(M)$ and $\psi \in Z^s(M)$, then $\omega \wedge \psi \in Z^{r+s}(M)$;
(b) if $\omega \in Z^r(M)$ and $\psi \in B^s(M)$, then $\omega \wedge \psi \in B^{r+s}(M)$; and

(c) if $\omega \in B^r(M)$ and $\psi \in B^s(M)$, then $\omega \wedge \psi \in B^{r+s}(M)$.

Definition 6.2. The rth **de Rham cohomology group** is defined by

$$H^r(M; \mathbb{R}) \equiv Z^r(M)/B^r(M). \tag{6.11}$$

If $r \leq -1$ or $r \geq m+1$, $H^r(M; \mathbb{R})$ may be defined to be trivial. In the following, we omit the explicit quotation of $\mathbb{R}$-coefficients.

Let $\omega \in Z^r(M)$. Then $[\omega] \in H^r(M)$ is the equivalence class $\{\omega' \in Z^r(M)|\omega' = \omega + \mathrm{d}\psi, \psi \in \Omega^{r-1}(M)\}$. Two forms which differ by an exact form are called **cohomologous**. We will see later that $H^r(M)$ is isomorphic to $H_r(M)$. The following examples will clarify the idea of de Rham cohomology groups.

Example 6.1. When $r = 0$, $B^0(M)$ has no meaning since there is no (-1)-form. We define $\Omega^{-1}(M)$ to be empty, hence $B^0(M) = 0$. Then $H^0(M) = Z^0(M) = \{f \in \Omega^0(M) = \mathcal{F}(M)|\mathrm{d}f = 0\}$. If M is connected, the condition $\mathrm{d}f = 0$ is satisfied if and only if f is constant over M. Hence, $H^0(M)$ is isomorphic to the vector space $\mathbb{R}$,

$$H^0(M) \cong \mathbb{R}. \tag{6.12}$$

If M has n connected components, $\mathrm{d}f = 0$ is satisfied if and only if f is constant on each connected component, hence it is specified by n real numbers,

$$H^0(M) \cong \underbrace{\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{n}. \tag{6.13}$$

Example 6.2. Let $M = \mathbb{R}$. From example 6.1, we have $H^0(\mathbb{R}) = \mathbb{R}$. Let us find $H^1(\mathbb{R})$ next. Let x be a coordinate of $\mathbb{R}$. Since $\dim \mathbb{R} = 1$, any one-form $\omega \in \Omega^1(\mathbb{R})$ is closed, $\mathrm{d}\omega = 0$. Let $\omega = f\mathrm{d}x$, where $f \in \mathcal{F}(\mathbb{R})$. Define a function $F(x)$ by

$$F(x) = \int_0^x f(s)\,\mathrm{d}s \in \mathcal{F}(\mathbb{R}) = \Omega^0(\mathbb{R}).$$

Since $\mathrm{d}F(x)/\mathrm{d}x = f(x)$, ω is an exact form,

$$\omega = f\,\mathrm{d}x = \frac{\mathrm{d}F(x)}{\mathrm{d}x}\,\mathrm{d}x = \mathrm{d}F.$$

Thus, any one-form is closed as well as exact. We have established

$$H^1(\mathbb{R}) = \{0\}. \tag{6.14}$$

Example 6.3. Let $S^1 = \{\mathrm{e}^{\mathrm{i}\theta}|0 \leq \theta < 2\pi\}$. Since S^1 is connected, we have $H^0(S^1) = \mathbb{R}$. We compute $H^1(S^1)$ next. Let $\omega = f(\theta)\,\mathrm{d}\theta \in \Omega^1(S^1)$. Is it possible to write $\omega = \mathrm{d}F$ for some $F \in \mathcal{F}(S^1)$? Let us repeat the analysis of the previous example. If $\omega = \mathrm{d}F$, then $F \in \mathcal{F}(S^1)$ must be given by

$$F(\theta) = \int_0^\theta f(\theta')\,\mathrm{d}\theta'.$$

For F to be defined uniquely on S^1, F must satisfy the periodicity $F(2\pi) = F(0)$ (=0). Namely F must satisfy

$$F(2\pi) = \int_0^{2\pi} f(\theta')\,\mathrm{d}\theta' = 0.$$

If we define a map $\lambda : \Omega^1(S^1) \to \mathbb{R}$ by

$$\lambda : \omega = f\,\mathrm{d}\theta \mapsto \int_0^{2\pi} f(\theta')\,\mathrm{d}\theta' \tag{6.15}$$

then $B^1(S^1)$ is identified with $\ker\lambda$. Now we have (theorem 3.1)

$$H^1(S^1) = \Omega^1(S^1)/\ker\lambda = \operatorname{im}\lambda = \mathbb{R}. \tag{6.16}$$

This is also obtained from the following consideration. Let ω and ω' be closed forms that are not exact. Although $\omega - \omega'$ is not exact in general, we can show that there exists a number $a \in \mathbb{R}$ such that $\omega' - a\omega$ is exact. In fact, if we put

$$a = \int_0^{2\pi} \omega' \Big/ \int_0^{2\pi} \omega$$

we have

$$\int_0^{2\pi} (\omega' - a\omega) = 0.$$

This shows that, given a closed form ω which is not exact, any closed form ω' is cohomologous to $a\omega$ for some $a \in \mathbb{R}$. Thus, each cohomology class is specified by a real number a, hence $H^1(S^1) = \mathbb{R}$.

Exercise 6.3. Let $M = \mathbb{R}^2 - \{0\}$. Define a one-form ω by

$$\omega = \frac{-y}{x^2 + y^2}\,\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y. \tag{6.17}$$

(a) Show that ω is closed.
(b) Define a ‘function’ $F(x, y) = \tan^{-1}(y/x)$. Show that $\omega = \mathrm{d}F$. Is ω exact?

6.2.2 Duality of $H_r(M)$ and $H^r(M)$; de Rham's theorem

As the name itself suggests, the *co*homology group is a dual space of the homology group. The duality is provided by Stokes' theorem. We first define the inner product of an r-form and an r-chain in M. Let M be an m-dimensional manifold and let $C_r(M)$ be the chain group of M. Take $c \in C_r(M)$ and $\omega \in \Omega^r(M)$ where $1 \le r \le m$. Define an inner product $(\quad,\quad) : C_r(M) \times \Omega^r(M) \to \mathbb{R}$ by

$$c, \omega \mapsto (c, \omega) \equiv \int_c \omega. \tag{6.18}$$

Clearly, (c, ω) is linear in both c and ω and $(\quad, \omega)$ may be regarded as a linear map acting on c and *vice versa*,

$$(c_1 + c_2, \omega) = \int_{c_1+c_2} \omega = \int_{c_1} \omega + \int_{c_2} \omega \tag{6.19a}$$

$$(c, \omega_1 + \omega_2) = \int_c (\omega_1 + \omega_2) = \int_c \omega_1 + \int_c \omega_2. \tag{6.19b}$$

Now Stokes' theorem takes a compact form:

$$(c, \mathrm{d}\omega) = (\partial c, \omega). \tag{6.20}$$

In this sense, the exterior derivative operator d is the adjoint of the boundary operator ∂ and *vice versa*.

Exercise 6.4. Let (i) $c \in B_r(M)$, $\omega \in Z^r(M)$ or (ii) $c \in Z_r(M)$, $\omega \in B^r(M)$. Show, in both cases, that $(c, \omega) = 0$.

The inner product $(\quad,\quad)$ naturally induces an inner product λ between the elements of $H_r(M)$ and $H^r(M)$. We now show that $H_r(M)$ is the dual of $H^r(M)$. Let $[c] \in H_r(M)$ and $[\omega] \in H^r(M)$ and define an inner product $\Lambda : H_r(M) \times H^r(M) \to \mathbb{R}$ by

$$\Lambda([c], [\omega]) \equiv (c, \omega) = \int_c \omega. \tag{6.21}$$

This is well defined since (6.21) is independent of the choice of the representatives. In fact, if we take $c + \partial c'$, $c' \in C_{r+1}(M)$, we have, from Stokes' theorem,

$$(c + \partial c', \omega) = (c, \omega) + (c', \mathrm{d}\omega) = (c, \omega)$$

where $\mathrm{d}\omega = 0$ has been used. Similarly, for $\omega + \mathrm{d}\psi$, $\psi \in \Omega^{r-1}(M)$,

$$(c, \omega + \mathrm{d}\psi) = (c, \omega) + (\partial c, \psi) = (c, \omega)$$

since $\partial c = 0$. Note that $\Lambda(\quad, [\omega])$ is a linear map $H_r(M) \to \mathbb{R}$, and $\Lambda([c], \quad)$ is a linear map $H^r(M) \to \mathbb{R}$. To prove the duality of $H_r(M)$ and $H^r(M)$, we have to show that $\Lambda(\quad, [\omega])$ has the maximal rank, that is, $\dim H_r(M) = \dim H^r(M)$. We accept the following theorem due to de Rham without the proof which is highly non-trivial.

Theorem 6.2. (**de Rham's theorem**) If M is a compact manifold, $H_r(M)$ and $H^r(M)$ are finite dimensional. Moreover the map

$$\Lambda : H_r(M) \times H^r(M) \to \mathbb{R}$$

is bilinear and non-degenerate. Thus, $H^r(M)$ is the dual vector space of $H_r(M)$.

A **period** of a closed r-form ω over a cycle c is defined by $(c, \omega) = \int_c \omega$. Exercise 6.4 shows that the period vanishes if ω is exact or if c is a boundary. The following corollary is easily derived from de Rham's theorem.

Corollary 6.1. Let M be a compact manifold and let k be the rth Betti number (see section 3.4). Let $c_1, c_2, \ldots, c_k$ be properly chosen elements of $Z_r(M)$ such that $[c_i] \neq [c_j]$.

(a) A closed r-form ψ is exact if and only if

$$\int_{c_i} \psi = 0 \qquad (1 \le i \le k). \tag{6.22}$$

(b) For any set of real numbers $b_1, b_2, \ldots, b_k$ there exists a closed r-form ω such that

$$\int_{c_i} \omega = b_i \qquad (1 \le i \le k). \tag{6.23}$$

Proof. (a) de Rham's theorem states that the bilinear form $\Lambda([c], [\omega])$ is non-degenerate. Hence, if $\Lambda([c_i], \quad)$ is regarded as a linear map acting on $H^r(M)$, the kernel consists of the trivial element, the cohomology class of exact forms. Accordingly, ψ is an exact form.

(b) de Rham's theorem ensures that corresponding to the homology basis $\{[c_i]\}$, we may choose the dual basis $\{[\omega_i]\}$ of $H^r(M)$ such that

$$\Lambda([c_i], [\omega_j]) = \int_{c_i} \omega_j = \delta_{ij}. \tag{6.24}$$

If we define $\omega \equiv \sum_{i=1}^k b_i \omega_i$, the closed r-form ω satisfies

$$\int_{c_i} \omega = b_i$$

as claimed. □

For example. we observe the duality of the following groups.

(a) $H^0(M) \cong H_0(M) \cong \underbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{n}$ if M has n connected components.

(b) $H^1(S^1) \cong H_1(S^1) \cong \mathbb{R}$.

Since $H^r(M)$ is isomorphic to $H_r(M)$, we find that

$$b^r(M) \equiv \dim H^r(M) = \dim H_r(M) = b_r(M) \tag{6.25}$$

where $b_r(M)$ is the Betti number of M. The Euler characteristic is now written as

$$\chi(M) = \sum_{r=1}^{m} (-1)^r b^r(M). \tag{6.26}$$

This is quite an interesting formula; the LHS is purely *topological* while the RHS is given by an *analytic* condition (note that $d\omega = 0$ is a set of partial differential equations). We will frequently encounter this interplay between topology and analysis.

In summary, we have the chain complex $C(M)$ and the de Rham complex $\Omega^*(M)$,

$$\begin{aligned} &\longleftarrow C_{r-1}(M) \xleftarrow{\partial_r} C_r(M) \xleftarrow{\partial_{r+1}} C_{r+1}(M) \longleftarrow \\ &\longrightarrow \Omega^{r-1}(M) \xrightarrow{d_r} \Omega^r(M) \xrightarrow{d_{r+1}} \Omega^{r+1}(M) \longleftarrow \end{aligned} \tag{6.27}$$

for which the rth homology group is defined by

$$H_r(M) = Z_r(M)/B_r(M) = \ker \partial_r / \operatorname{im} \partial_{r+1}$$

and the rth de Rham cohomology group is defined by

$$H^r(M) = Z^r(M)/B^r(M) = \ker d_{r+1} / \operatorname{im} d_r.$$

6.3 Poincaré's lemma

An exact form is always closed but the converse is not necessarily true. However, the following theorem provides the situation in which the converse is also true.

Theorem 6.3. (**Poincaré's lemma**) If a coordinate neighbourhood U of a manifold M is contractible to a point $p_0 \in M$, any closed r-form on U is also exact.

Proof. We assume U is smoothly contractible to p_0, that is, there exists a smooth map $F : U \times I \to U$ such that

$$F(x, 0) = x, \qquad F(x, 1) = p_0 \qquad \text{for } x \in U.$$

Let us consider an r-form $\eta \in \Omega^r(U \times I)$,

$$\begin{aligned}\eta &= a_{i_1\ldots i_r}(x,t)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_r}\\&\quad + b_{j_1\ldots j_{r-1}}(x,t)\,\mathrm{d}t\wedge\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_{r-1}}\end{aligned}\tag{6.28}$$

where x is the coordinate of U and t of I. Define a map $P : \Omega^r(U \times I) \to \Omega^{r-1}(U)$ by

$$P\eta \equiv \left(\int_0^1 \mathrm{d}s\, b_{j_1\ldots j_{r-1}}(x,s)\right)\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_{r-1}}. \tag{6.29}$$

Next, define a map $f_t : U \to U \times I$ by $f_t(x) = (x,t)$. The pullback of the first term of (6.28) by f_t^* is an element of $\Omega^r(U)$,

$$f_t^*\eta = a_{i_1\ldots i_r}(x,t)\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_r} \in \Omega^r(U). \tag{6.30}$$

We now prove the following identity,

$$\mathrm{d}(P\eta) + P(\mathrm{d}\eta) = f_1{}^*\eta - f_0{}^*\eta. \tag{6.31}$$

Each term of the LHS is calculated to be

$$\begin{aligned}\mathrm{d}P\eta &= \mathrm{d}\left(\int_0^1 \mathrm{d}s\, b_{j_1\ldots j_{r-1}}\right)\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_{r-1}}\\&= \int_0^1 \mathrm{d}s\left(\frac{\partial b_{j_1\ldots j_{r-1}}}{\partial x^{j_r}}\right)\mathrm{d}x^{j_r}\wedge\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_{r-1}}\\P\,\mathrm{d}\eta &= P\left[\left(\frac{\partial a_{i_1\ldots i_r}}{\partial x^{i_{r+1}}}\right)\mathrm{d}x^{i_{r+1}}\wedge\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_r}\right.\\&\quad+\left(\frac{\partial a_{i_1\ldots i_r}}{\partial t}\right)\mathrm{d}t\wedge\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_r}\\&\quad\left.+\left(\frac{\partial b_{j_1\ldots j_{r-1}}}{\partial x^{j_r}}\right)\mathrm{d}x^{j_r}\wedge\mathrm{d}t\wedge\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_{r-1}}\right]\\&= \left[\int_0^1 \mathrm{d}s\left(\frac{\partial a_{i_1\ldots i_r}}{\partial s}\right)\right]\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_r}\\&\quad-\left[\int_0^1 \mathrm{d}s\left(\frac{\partial b_{j_1\ldots j_{r-1}}}{\partial x^{j_r}}\right)\right]\mathrm{d}x^{j_r}\wedge\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_{r-1}}.\end{aligned}$$

Collecting these results, we have

$$\begin{aligned}\mathrm{d}(P\eta) + P(\mathrm{d}\eta) &= \left[\int_0^1 \mathrm{d}s\left(\frac{\partial a_{i_1\ldots i_r}}{\partial s}\right)\right]\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_r}\\&= [a_{i_1\ldots i_r}(x,1) - a_{i_1\ldots i_r}(x,0)]\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_r}\\&= f_1{}^*\eta - f_0{}^*\eta.\end{aligned}$$

Poincaré's lemma readily follows from (6.31). Let ω be a closed r-form on a contractible chart U. We will show that ω is written as an exact form,

$$\omega = \mathrm{d}(-PF^*\omega), \tag{6.32}$$

F being the smooth contraction map. In fact, if η in (6.31) is replaced by $F^*\omega \in \Omega^r(U \times I)$ we have

$$\begin{aligned} \mathrm{d}PF^*\omega + P\,\mathrm{d}F^*\omega &= f_1{}^* \circ F^*\omega - f_0{}^* \circ F^*\omega \\ &= (F \circ f_1)^*\omega - (F \circ f_0)^*\omega \end{aligned} \tag{6.33}$$

where use has been made of the relation $(f \circ g)^* = g^* \circ f^*$. Clearly $F \circ f_1 : U \to U$ is a constant map $x \mapsto p_0$, hence $(F \circ f_1)^* = 0$. However, $F \circ f_0 = \mathrm{id}_U$, hence $(F \circ f_0)^* : \Omega^r(U) \to \Omega^r(U)$ is the identity map. Thus, the RHS of (6.33) is simply $-\omega$. The second term of the LHS vanishes since ω is closed; $\mathrm{d}F^*\omega = F^*\,\mathrm{d}\omega = 0$, where use has been made of (5.75). Finally, (6.33) becomes $\omega = -\mathrm{d}P\,F^*\omega$, which proves the theorem. □

Any closed form is exact at least locally. The de Rham cohomology group is regarded as an obstruction to the *global* exactness of closed forms.

Example 6.4. Since $\mathbb{R}^n$ is contractible, we have

$$H^r(\mathbb{R}^n) = 0 \qquad 1 \le r \le n. \tag{6.34}$$

Note, however, that $H^0(\mathbb{R}^n) = \mathbb{R}$.

6.4 Structure of de Rham cohomology groups

de Rham cohomology groups exhibit quite an interesting structure that is very difficult or even impossible to appreciate with homology groups.

6.4.1 Poincaré duality

Let M be a compact m-dimensional manifold and let $\omega \in H^r(M)$ and $\eta \in H^{m-r}(M)$. Noting that $\omega \wedge \eta$ is a volume element, we define an inner product $\langle\ ,\ \rangle : H^r(M) \times H^{m-r}(M) \to \mathbb{R}$ by

$$\langle \omega, \eta \rangle \equiv \int_M \omega \wedge \eta. \tag{6.35}$$

The inner product is bilinear. Moreover, it is non-singular, that is, if $\omega \neq 0$ or $\eta \neq 0$, $\langle \omega, \eta \rangle$ cannot vanish identically. Thus, (6.35) defines the duality of $H^r(M)$ and $H^{m-r}(M)$,

$$H^r(M) \cong H^{m-r}(M) \tag{6.36}$$

called the **Poincaré duality**. Accordingly, the Betti numbers have a symmetry

$$b_r = b_{m-r}. \tag{6.37}$$

It follows from (6.37) that the Euler characteristic of an odd-dimensional space vanishes,

$$\begin{aligned}\chi(M) &= \sum(-1)^r b_r = \tfrac{1}{2}\left\{\sum(-1)^r b_r + \sum(-1)^{m-r} b_{m-r}\right\} \\ &= \tfrac{1}{2}\left\{\sum(-1)^r b_r - \sum(-1)^{-r} b_r\right\} = 0. \end{aligned}\tag{6.38}$$

6.4.2 Cohomology rings

Let $[\omega] \in H^q(M)$ and $[\eta] \in H^r(M)$. Define a product of $[\omega]$ and $[\eta]$ by

$$[\omega] \wedge [\eta] \equiv [\omega \wedge \eta]. \tag{6.39}$$

It follows from exercise 6.2 that $\omega \wedge \eta$ is closed, hence $[\omega \wedge \eta]$ is an element of $H^{q+r}(M)$. Moreover, $[\omega \wedge \eta]$ is independent of the choice of the representatives of $[\omega]$ and $[\eta]$. For example, if we take $\omega' = \omega + \mathrm{d}\psi$ instead of ω, we have

$$[\omega'] \wedge [\eta] \equiv [(\omega + \mathrm{d}\psi) \wedge \eta] = [\omega \wedge \eta + \mathrm{d}(\psi \wedge \eta)] = [\omega \wedge \eta].$$

Thus, the product $\wedge : H^q(M) \times H^r(M) \to H^{q+r}(M)$ is a well-defined map.

The **cohomology ring** $H^*(M)$ is defined by the direct sum,

$$H^*(M) \equiv \bigoplus_{r=1}^{m} H^r(M). \tag{6.40}$$

The product is provided by the exterior product defined earlier,

$$\wedge : H^*(M) \times H^*(M) \to H^*(M). \tag{6.41}$$

The addition is the formal sum of two elements of $H^*(M)$. One of the superiorities of cohomology groups over homology groups resides here. Products of chains are not well defined and homology groups cannot have a ring structure.

6.4.3 The Künneth formula

Let M be a product of two manifolds $M = M_1 \times M_2$. Let $\{\omega_i^p\}$ $(1 \le i \le b^p(M_1))$ be a basis of $H^p(M_1)$ and $\{\eta_i^p\}$ $(1 \le i \le b^p(M_2))$ be that of $H^p(M_2)$. Clearly $\omega_i^p \wedge \eta_j^{r-p}$ $(1 \le p \le r)$ is a closed r-form in M. We show that it is not exact. If it were exact, it would be written as

$$\omega_i^p \wedge \eta_j^{r-p} = \mathrm{d}(\alpha^{p-1} \wedge \beta^{r-p} + \gamma^p \wedge \delta^{r-p-1}) \tag{6.42}$$

for some $\alpha^{p-1} \in \Omega^{p-1}(M_1)$, $\beta^{r-p} \in \Omega^{r-p}(M_2)$, $\gamma^p \in \Omega^p(M_1)$ and $\delta^{r-p-1} \in \Omega^{r-p-1}(M_2)$. [If $p = 0$, we put $\alpha^{p-1} = 0$.] By executing the exterior derivative in (6.42), we have

$$\begin{aligned}\omega_i^p \wedge \eta_j^{r-p} &= \mathrm{d}\alpha^{p-1} \wedge \beta^{r-p} + (-1)^{p-1}\alpha^{p-1} \wedge \mathrm{d}\beta^{r-p} \\ &\quad + \mathrm{d}\gamma^p \wedge \delta^{r-p-1} + (-1)^p \gamma^p \wedge \mathrm{d}\delta^{r-p-1}. \qquad (6.43)\end{aligned}$$

By comparing the LHS with the RHS, we find $\alpha^{p-1} = \delta^{r-p-1} = 0$, hence $\omega_i^p \wedge \eta_j^{r-p} = 0$ in contradiction to our assumption. Thus, $\omega_i^p \wedge \eta_j^{r-p}$ is a non-trivial element of $H^r(M)$. Conversely, any element of $H^r(M)$ can be decomposed into a sum of a product of the elements of $H^p(M_1)$ and $H^{r-p}(M_2)$ for $0 \le p \le r$. Now we have obtained the **Künneth formula**

$$H^r(M) = \bigoplus_{p+q=r} [H^p(M_1) \otimes H^q(M_2)]. \qquad (6.44)$$

This is rewritten in terms of the Betti numbers as

$$b^r(M) = \sum_{p+q=r} b^p(M_1) b^q(M_2). \qquad (6.45)$$

The Künneth formula also gives a relation between the cohomology rings of the respective manifolds,

$$\begin{aligned}H^*(M) &= \sum_{r=1}^{m} H^r(M) = \sum_{r=1}^{m} \bigoplus_{p+q=r} H^p(M_1) \otimes H^q(M_2) \\ &= \sum_p H^p(M_1) \otimes \sum_q H^q(M_2) = H^*(M_1) \otimes H^*(M_2). \qquad (6.46)\end{aligned}$$

Exercise 6.5. Let $M = M_1 \times M_2$. Show that

$$\chi(M) = \chi(M_1) \cdot \chi(M_2). \qquad (6.47)$$

Example 6.5. Let $T^2 = S^1 \times S^1$ be the torus. Since $H^0(S^1) = \mathbb{R}$ and $H^1(S^1) = \mathbb{R}$, we have

$$H^0(T^2) = \mathbb{R} \otimes \mathbb{R} = \mathbb{R} \qquad (6.48a)$$

$$H^1(T^2) = (\mathbb{R} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes \mathbb{R}) = \mathbb{R} \oplus \mathbb{R} \qquad (6.48b)$$

$$H^2(T^2) = \mathbb{R} \otimes \mathbb{R} = \mathbb{R}. \qquad (6.48c)$$

Observe the Poincaré duality $H^0(T^2) = H^2(T^2)$. [*Remark*: $\mathbb{R} \otimes \mathbb{R}$ is the tensor product and should not be confused with the direct product. Clearly the product of two real numbers is a real number.] Let us parametrize the coordinate of T^2

as (θ_1, θ_2) where θ_i is the coordinate of S^1. The groups $H^r(T^2)$ are generated by the following forms:

$$\begin{aligned} r = 0: &\quad \omega_0 = c_0 \qquad c_0 \in \mathbb{R} \\ r = 1: &\quad \omega_1 = c_1\,\mathrm{d}\theta_1 + c_1'\,\mathrm{d}\theta_2 \qquad c_1, c_1' \in \mathbb{R} \\ r = 2: &\quad \omega_2 = c_2\,\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \qquad c_2 \in \mathbb{R}. \end{aligned} \tag{6.49a}$$

Although the one-form $\mathrm{d}\theta_i$ looks like an exact form, there is no *function* θ_i which is defined uniquely on S^1. Since $\chi(S^1) = 0$, we have $\chi(T^2) = 0$.

The de Rham cohomology groups of

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$$

are obtained similarly. $H^r(T^n)$ is generated by r-forms of the form

$$\mathrm{d}\theta^{i_1} \wedge \mathrm{d}\theta^{i_2} \wedge \ldots \wedge \mathrm{d}\theta^{i_r} \tag{6.50}$$

where $i_1 < i_2 < \cdots < i_r$ are chosen from $1, \ldots, n$. Clearly

$$b^r = \dim H^r(T^n) = \binom{n}{r}. \tag{6.51}$$

The Euler characteristic is directly obtained from (6.51) as

$$\chi(T^n) = \sum (-1)^r \binom{n}{r} = (1-1)^n = 0. \tag{6.52}$$

6.4.4 Pullback of de Rham cohomology groups

Let $f : M \to N$ be a smooth map. Equation (5.75) shows that the pullback f^* maps closed forms to closed forms and exact forms to exact forms. Accordingly, we may define a pullback of the cohomology groups $f^* : H^r(N) \to H^r(M)$ by

$$f^*[\omega] = [f^*\omega] \qquad [\omega] \in H^r(N). \tag{6.53}$$

The pullback f^* preserves the ring structure of $H^*(N)$. In fact, if $[\omega] \in H^p(N)$ and $[\eta] \in H^q(N)$, we find

$$\begin{aligned} f^*([\omega] \wedge [\eta]) &= f^*[\omega \wedge \eta] = [f^*(\omega \wedge \eta)] \\ &= [f^*\omega \wedge f^*\eta] = [f^*\omega] \wedge [f^*\eta]. \end{aligned} \tag{6.54}$$

6.4.5 Homotopy and $H^1(M)$

Let $f, g : M \to N$ be smooth maps. We assume f and g are homotopic to each other, that is, there exists a smooth map $F : M \times I \to N$ such that $F(p, 0) =$

$f(p)$ and $F(p, 1) = g(p)$. We now prove that $f^* : H^r(N) \to H^r(M)$ is equal to $g^* : H^r(N) \to H^r(M)$.

Lemma 6.1. Let f^* and g^* be defined as before. If $\omega \in \Omega^r(N)$ is a closed form, the difference of the pullback images is exact,

$$f^*\omega - g^*\omega = \mathrm{d}\psi \qquad \psi \in \Omega^{r-1}(M). \tag{6.55}$$

Proof. We first note that

$$f = F \circ f_0, \qquad g = F \circ f_1$$

where $f_t : M \to M \times I \quad (p \mapsto (p, t))$ has been defined in theorem 6.3. The LHS of (6.55) is

$$\begin{aligned}(F \circ f_0)^*\omega - (F \circ f_1)^*\omega &= f_0^* \circ F^*\omega - f_1^* \circ F^*\omega \\ &= -[\mathrm{d}P(F^*\omega) + P\,\mathrm{d}(F^*\omega)] = -\mathrm{d}P\,F^*\omega\end{aligned}$$

where (6.33) has been used. This shows that $f^*\omega - g^*\omega = \mathrm{d}(-PF^*\omega)$. □

Now it is easy to see that $f^* = g^*$ as the pullback maps $H^r(N) \to H^r(M)$. In fact, from the previous lemma,

$$[f^*\omega - g^*\omega] = [f^*\omega] - [g^*\omega] = [\mathrm{d}\psi] = 0.$$

We have established the following theorem.

Theorem 6.4. Let $f, g : M \to N$ be maps which are homotopic to each other. Then the pullback maps f^* and g^* of the de Rham cohomology groups $H^r(N) \to H^r(M)$ are identical.

Let M be a simply connected manifold, namely $\pi_1(M) \cong \{0\}$. Since $H_1(M) = \pi_1(M)$ modulo the commutator subgroup (theorem 4.9), it follows that $H_1(M)$ is also trivial. In terms of the de Rham cohomology group this can be expressed as follows.

Theorem 6.5. Let M be a simply connected manifold. Then its first de Rham cohomology group is trivial.

Proof. Let ω be a closed one-form on M. It is clear that if $\omega = \mathrm{d}f$, then a function f must be of the form

$$f(p) = \int_{p_0}^{p} \omega \tag{6.56}$$

$p_0 \in M$ being a fixed point.

We first prove that an integral of a closed form along a loop vanishes. Let $\alpha : I \to M$ be a loop at $p \in M$ and let $c_p : I \to M$ $(t \mapsto p)$ be a constant loop. Since M is simply connected, there exists a homotopy $F(s,t)$ such that $F(s,0) = \alpha(s)$ and $F(s,1) = c_p(s)$. We assume $F : I \times I \to M$ is smooth. Define the integral of a one-form ω over $\alpha(I)$ by

$$\int_{\alpha(I)} \omega = \int_{S^1} \alpha^* \omega \tag{6.57}$$

where we have taken the integral domain in the RHS to be S^1 since $I = [0,1]$ in the LHS is compactified to S^1. From lemma 6.1, we have, for a closed one-form ω,

$$\alpha^*\omega - c_p^*\omega = \mathrm{d}g \tag{6.58}$$

where $g = -PF^*\omega$. The pullback $c_p\omega$ vanishes since c_p is a constant map. Then (6.57) vanishes since ∂S^1 is empty,

$$\int_{S^1} \alpha^*\omega = \int_{S^1} \mathrm{d}g = \int_{\partial S^1} g = 0. \tag{6.59}$$

Let β and γ be two paths connecting p_0 and p. According to (6.59), integrals of ω along β and along γ are identical,

$$\int_{\beta(I)} \omega = \int_{\gamma(I)} \omega.$$

This shows that (6.56) is indeed well defined, hence ω is exact. □

Example 6.6. The n-sphere S^n $(n \geq 2)$ is simply connected, hence

$$H^1(S^n) = 0 \qquad n \geq 2. \tag{6.60}$$

From the Poincaré duality, we find

$$H^0(S^n) \cong H^n(S^n) = \mathbb{R}. \tag{6.61}$$

It can be shown that

$$H^r(S^n) = 0 \qquad 1 \leq r \leq n-1. \tag{6.62}$$

$H^n(S^n)$ is generated by the volume element Ω. Since there are no $(n+1)$-forms on S^n, every n-form is closed. Ω cannot be exact since if $\Omega = \mathrm{d}\psi$, we would have

$$\int_{S^n} \Omega = \int_{S^n} \mathrm{d}\psi = \int_{\partial S^n} \psi = 0.$$

The Euler characteristic is

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 0 & n \text{ is odd}, \\ 2 & n \text{ is even}. \end{cases} \tag{6.63}$$

Example 6.7. Take S^2 embedded in $\mathbb{R}^3$ and define

$$\Omega = \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi \tag{6.64}$$

where (θ, ϕ) is the usual polar coordinate. Verify that Ω is closed. We may *formally* write Ω as

$$\Omega = -\mathrm{d}(\cos\theta) \wedge \mathrm{d}\phi = -\mathrm{d}(\cos\theta \, \mathrm{d}\phi).$$

Note, however, that Ω is not exact.