3

HOMOLOGY GROUPS

Among the topological invariants the Euler characteristic is a quantity readily computable by the 'polyhedronization' of space. The homology groups are *refinements*, so to speak, of the Euler characteristic. Moreover, we can easily read off the Euler characteristic from the homology groups. Let us look at figure 3.1. In figure 3.1(a), the interior is included but not in figure 3.1(b). How do we characterize this difference? An obvious observation is that the three edges of figure 3.1(a) form a boundary of the interior while the edges of figure 3.1(b) do not (the interior is *not* a part of figure 3.1(b)). Clearly the edges in both cases form a closed path (loop), having no boundary. In other words, the existence of a loop that is not a boundary of some area implies the existence of a hole within the loop. This is our guiding principle in classifying spaces here: *find a region without boundaries, which is not itself a boundary of some region*. This principle is mathematically elaborated into the theory of homology groups.

Our exposition follows Armstrong (1983), Croom (1978) and Nash and Sen (1983). An introduction to group theory is found in Fraleigh (1976).

3.1 Abelian groups

The mathematical structures underlying homology groups are *finitely generated Abelian groups*. Throughout this chapter, the group operation is denoted by $+$ since all the groups considered here are Abelian (commutative). The unit element is denoted by 0.

3.1.1 Elementary group theory

Let G_1 and G_2 be Abelian groups. A map $f : G_1 \to G_2$ is said to be a **homomorphism** if

$$f(x + y) = f(x) + f(y) \tag{3.1}$$

for any $x, y \in G_1$. 1f f is also a *bijection*, f is called an **isomorphism**. If there exists an isomorphism $f : G_1 \to G_2$, G_1 is said to be **isomorphic** to G_2, denoted by $G_1 \cong G_2$. For example, a map $f : \mathbb{Z} \to \mathbb{Z}_2 = \{0, 1\}$ defined by

$$f(2n) = 0 \qquad f(2n + 1) = 1$$

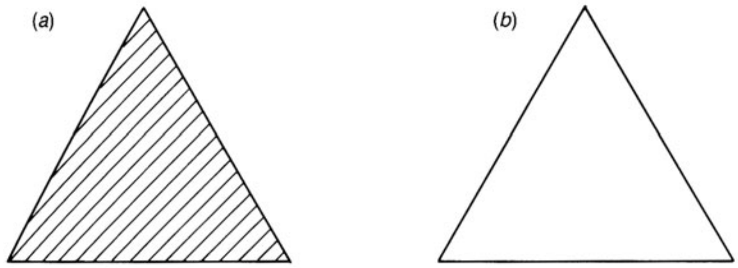

Figure 3.1. (*a*) is a solid triangle while (*b*) is the edges of a triangle without an interior.

is a homomorphism. Indeed

$$\begin{aligned} f(2m+2n) &= f(2(m+n)) = 0 = 0+0 = f(2m)+f(2n) \\ f(2m+1+2n+1) &= f(2(m+n+1)) = 0 = 1+1 \\ &= f(2m+1)+f(2n+1) \\ f(2m+1+2n) &= f(2(m+n)+1) = 1 = 1+0 \\ &= f(2m+1)+f(2n). \end{aligned}$$

A subset $H \subset G$ is a subgroup if it is a group with respect to the group operation of G. For example,

$$k\mathbb{Z} \equiv \{kn | n \in \mathbb{Z}\} \qquad k \in \mathbb{N}$$

is a subgroup of $\mathbb{Z}$, while $\mathbb{Z}_2 = \{0, 1\}$ is not.

Let H be a subgroup of G. We say $x, y \in G$ are equivalent if

$$x - y \in H \tag{3.2}$$

and write $x \sim y$. Clearly $\sim$ is an equivalence relation. The equivalence class to which x belongs is denoted by $[x]$. Let G/H be the quotient space. The group operation $+$ in G naturally induces the group operation $+$ in G/H by

$$[x] + [y] = [x + y]. \tag{3.3}$$

Note that $+$ on the LHS is an operation in G/H while $+$ on the RHS is that in G. The operation in G/H should be independent of the choice of representatives. In fact, if $[x'] = [x]$, $[y'] = [y]$, then $x - x' = h$, $y - y' = g$ for some $h, g \in H$ and we find that

$$x' + y' = x + y - (h + g) \in [x + y]$$

Furthermore, G/H becomes a group with this operation, since H is always a normal subgroup of G; see example 2.6. The unit element of G/H is $[0] = [h]$,

$h \in H$. If $H = G$, $0 - x \in G$ for any $x \in G$ and G/G has just one element $[0]$. If $H = \{0\}$, G/H is G itself since $x - y = 0$ if and only if $x = y$.

Example 3.1. Let us work out the quotient group $\mathbb{Z}/2\mathbb{Z}$. For even numbers we have $2n - 2m = 2(n - m) \in 2\mathbb{Z}$ and $[2m] = [2n]$. For odd numbers $(2n+1)-(2m+1) = 2(n-m) \in 2\mathbb{Z}$ and $[2m+1] = [2n+1]$. Even numbers and odd numbers never belong to the same equivalence class since $2n-(2m+1) \notin 2\mathbb{Z}$. Thus, it follows that

$$\mathbb{Z}/2\mathbb{Z} = \{[0], [1]\}. \tag{3.4}$$

If we define an isomorphism $\varphi : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}_2$ by $\varphi([0]) = 0$ and $\varphi([1]) = 1$, we find $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$. For general $k \in \mathbb{N}$, we have

$$\mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}_k. \tag{3.5}$$

Lemma 3.1. Let $f : G_1 \to G_2$ be a homomorphism. Then

(a) $\ker f = \{x | x \in G_1, f(x) = 0\}$ is a subgroup of G_1,

(b) $\mathrm{im}\, f = \{x | x \in f(G_1) \subset G_2\}$ is a subgroup of G_2.

Proof. (a) Let $x, y \in \ker f$. Then $x + y \in \ker f$ since $f(x + y) = f(x) + f(y) = 0+0 = 0$. Note that $0 \in \ker f$ for $f(0) = f(0)+f(0)$. We also have $-x \in \ker f$ since $f(0) = f(x - x) = f(x) + f(-x) = 0$.

(b) Let $y_1 = f(x_1), y_2 = f(x_2) \in \mathrm{im}\, f$ where $x_1, x_2 \in G_1$. Since f is a homomorphism we have $y_1 + y_2 = f(x_1) + f(x_2) = f(x_1 + x_2) \in \mathrm{im}\, f$. Clearly $0 \in \mathrm{im}\, f$ since $f(0) = 0$. If $y = f(x)$, $-y \in \mathrm{im}\, f$ since $0 = f(x - x) = f(x) + f(-x)$ implies $f(-x) = -y$. □

Theorem 3.1. (**Fundamental theorem of homomorphism**) Let $f : G_1 \to G_2$ be a homomorphism. Then

$$G_1/\ker f \cong \mathrm{im}\, f. \tag{3.6}$$

Proof. Both sides are groups according to lemma 3.1. Define a map $\varphi : G_1/\ker f \to \mathrm{im}\, f$ by $\varphi([x]) = f(x)$. This map is well defined since for $x' \in [x]$, there exists $h \in \ker f$ such that $x' = x + h$ and $f(x') = f(x + h) = f(x) + f(h) = f(x)$. Now we show that φ is an isomorphism. First, φ is a homomorphism,

$$\begin{aligned}\varphi([x] + [y]) &= \varphi([x + y]) = f(x + y) \\ &= f(x) + f(y) = \varphi([x]) + \varphi([y]).\end{aligned}$$

Second, φ is one to one: if $\varphi([x]) = \varphi([y])$, then $f(x) = f(y)$ or $f(x) - f(y) = f(x - y) = 0$. This shows that $x - y \in \ker f$ and $[x] = [y]$. Finally, φ is onto: if $y \in \mathrm{im}\, f$, there exists $x \in G_1$ such that $f(x) = y = \varphi([x])$. □

Example 3.2. Let $f : \mathbb{Z} \to \mathbb{Z}_2$ be defined by $f(2n) = 0$ and $f(2n+1) = 1$. Then $\ker f = 2\mathbb{Z}$ and $\mathrm{im}\, f = \mathbb{Z}_2$ are groups. Theorem 3.1 states that $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$, in agreement with example 3.1.

3.1.2 Finitely generated Abelian groups and free Abelian groups

Let x be an element of a group G. For $n \in \mathbb{Z}, nx$ denotes

$$\underbrace{x + \cdots + x}_{n} \qquad (\text{if } n > 0)$$

and

$$\underbrace{(-x) + \cdots + (-x)}_{|n|} \qquad (\text{if } n < 0).$$

If $n = 0$, we put $0x = 0$. Take r elements $x_1, \ldots, x_r$ of G. The elements of G of the form

$$n_1 x_1 + \cdots + n_r x_r \qquad (n_i \in \mathbb{Z}, 1 \le i \le r) \tag{3.7}$$

form a subgroup of G, which we denote H. H is called a subgroup of G **generated** by the **generators** $x_1, \ldots, x_r$. If G itself is generated by finite elements $x_1, \ldots, x_r$, G is said to be **finitely generated**. If $n_1 x_1 + \cdots + n_r x_r = 0$ is satisfied only when $n_1 = \cdots = n_r = 0$, $x_1, \ldots, x_r$ are said to be **linearly independent**.

Definition 3.1. If G is finitely generated by r *linearly independent* elements, G is called a **free Abelian group** of **rank** r.

Example 3.3. $\mathbb{Z}$ is a free Abelian group of rank 1 finitely generated by 1 (or -1). Let $\mathbb{Z} \oplus \mathbb{Z}$ be the set of pairs $\{(i, j) | i, j \in \mathbb{Z}\}$. It is a free Abelian group of rank 2 finitely generated by generators $(1, 0)$ and $(0, 1)$. More generally

$$\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r}$$

is a free Abelian group of rank r. The group $\mathbb{Z}_2 = \{0, 1\}$ is finitely generated by 1 but is *not* free since 1 is not linearly independent (note $1 + 1 = 0$).

3.1.3 Cyclic groups

If G is generated by one element x, $G = \{0, \pm x, \pm 2x, \ldots\}$, G is called a **cyclic group**. If $nx \neq 0$ for any $n \in \mathbb{Z} - \{0\}$, it is an **infinite** cyclic group while if $nx = 0$ for some $n \in \mathbb{Z} - \{0\}$, a **finite cyclic group**. Let G be a cyclic group generated by x and let $f : \mathbb{Z} \to G$ be a homomorphism defined by $f(n) = nx$. f maps $\mathbb{Z}$ onto G but not necessarily one to one. From theorem 3.1, we have $G = \operatorname{im} f \cong \mathbb{Z}/\ker f$. Let N be the smallest positive integer such that $Nx = 0$. Clearly

$$\ker f = \{0, \pm N, \pm 2N, \ldots\} = N\mathbb{Z} \tag{3.8}$$

and we have

$$G \cong \mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}_N. \tag{3.9}$$

If G is an infinite cyclic group, then $\ker f = \{0\}$ and $G \cong \mathbb{Z}$. Any infinite cyclic group is isomorphic to $\mathbb{Z}$ while a finite cyclic group is isomorphic to some $\mathbb{Z}_N$.

We will need the following lemma and theorem in due course. We first state the lemma without proof.

Lemma 3.2. Let G be a free Abelian group of rank r and let H ($\neq\emptyset$) be a subgroup of G. We may always choose p generators $x_1, \ldots, x_p$, out of r generators of G so that $k_1 x_1, \ldots, k_p x_p$ generate H. Thus, $H \cong k_1\mathbb{Z} \oplus \ldots \oplus k_p\mathbb{Z}$ and H is of rank p.

Theorem 3.2. (**Fundamental theorem of finitely generated Abelian groups**) Let G be a finitely generated Abelian group (not necessarily free) with m generators. Then G is isomorphic to the direct sum of cyclic groups,

$$G \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_p} \tag{3.10}$$

where $m = r + p$. The number r is called the **rank** of G.

Proof. Let G be generated by m elements $x_1, \ldots, x_m$ and let

$$f : \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m} \to G$$

be a surjective homomorphism,

$$f(n_1, \ldots, n_m) = n_1 x_1 + \cdots + n_m x_m.$$

Theorem 3.1 states that

$$\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m} / \ker f \cong G.$$

Since $\ker f$ is a subgroup of

$$\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m}$$

lemma 3.2 claims that if we choose the generators properly, we have

$$\ker f \cong k_1\mathbb{Z} \oplus \cdots \oplus k_p\mathbb{Z}.$$

We finally obtain

$$\begin{aligned} G &\cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m} / \ker f \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m} / (k_1\mathbb{Z} \oplus \cdots \oplus k_p\mathbb{Z}) \\ &\cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m-p} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_p}. \end{aligned}$$

□

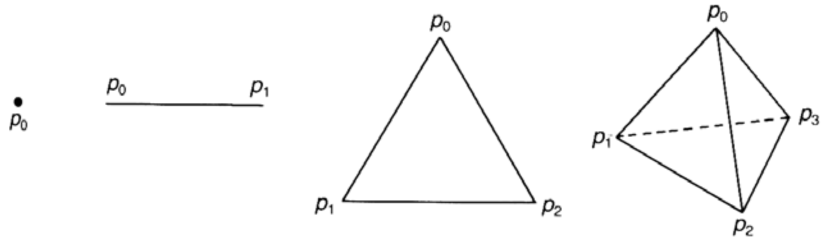

Figure 3.2. 0-, 1-, 2- and 3-simplexes.

3.2 Simplexes and simplicial complexes

Let us recall how the Euler characteristic of a surface is calculated. We first construct a polyhedron homeomorphic to the given surface, then count the numbers of vertices, edges and faces. The Euler characteristic of the polyhedron, and hence of the surface, is then given by equation (2.31). We abstract this procedure so that we may represent each part of a figure by some *standard* object. We take triangles and their analogues in other dimensions, called simplexes, as the standard objects. By this standardization, it becomes possible to assign to each figure Abelian group structures.

3.2.1 Simplexes

Simplexes are building blocks of a polyhedron. A 0-simplex $\langle p_0 \rangle$ is a point, or a vertex, and a 1-simplex $\langle p_0 p_1 \rangle$ is a line, or an edge. A 2-simplex $\langle p_0 p_1 p_2 \rangle$ is defined to be a triangle with its interior included and a 3-simplex $\langle p_0 p_1 p_2 p_3 \rangle$ is a solid tetrahedron (figure 3.2). It is common to denote a 0-simplex without the bracket; $\langle p_0 \rangle$ may be also written as p_0. It is easy to continue this construction to any r-simplex $\langle p_0 p_1 \ldots p_r \rangle$. Note that for an r-simplex to represent an r-dimensional object, the vertices p_i must be *geometrically independent*, that is, no $(r-1)$-dimensional hyperplane contains all the $r+1$ points. Let $p_0, \ldots, p_r$ be points geometrically independent in $\mathbb{R}^m$ where $m \geq r$. The r-simplex $\sigma_r = \langle p_0, \ldots, p_r \rangle$ is expressed as

$$\sigma^r = \left\{ x \in \mathbb{R}^m \,\middle|\, x = \sum_{i=0}^{r} c_i p_i, c_i \geq 0, \sum_{i=0}^{r} c_i = 1 \right\}. \tag{3.11}$$

$(c_0, \ldots, c_r)$ is called the **barycentric coordinate** of x. Since σ_r is a bounded and closed subset of $\mathbb{R}^m$, it is compact.

Let q be an integer such that $0 \leq q \leq r$. If we choose $q+1$ points $p_{i_0}, \ldots, p_{i_q}$ out of $p_0, \ldots, p_r$, these $q+1$ points define a q-simplex $\sigma_q = \langle p_{i_0}, \ldots, p_{i_q} \rangle$, which is called a ***q*-face** of σ_r. We write $\sigma_q \leq \sigma_r$ if σ_q is a face of

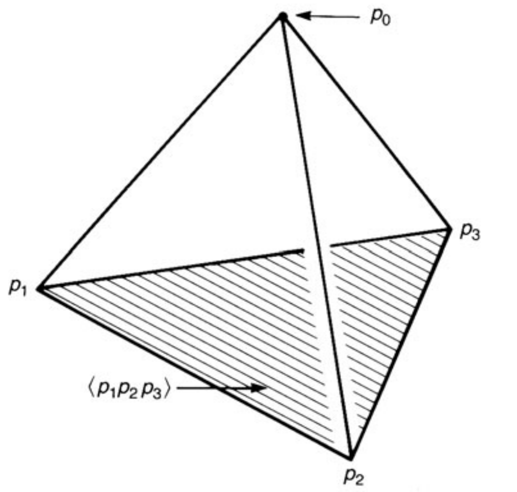

Figure 3.3. A 0-face p_0 and a 2-face $\langle p_1 p_2 p_3 \rangle$ of a 3-simplex $\langle p_0 p_1 p_2 p_3 \rangle$.

σ_r. If $\sigma_q \neq \sigma_r$, we say σ_q is a **proper face** of σ_r, denoted as $\sigma_q < \sigma_r$. Figure 3.3 shows a 0-face p_0 and a 2-face $\langle p_1 p_2 p_3 \rangle$ of a 3-simplex $\langle p_0 p_1 p_2 p_3 \rangle$. There are one 3-face, four 2-faces, six 1-faces and four 0-faces. The reader should verify that the number of q-faces in an r-simplex is $\begin{pmatrix} r+1 \\ q+1 \end{pmatrix}$. A 0-simplex is defined to have no proper faces.

3.2.2 Simplicial complexes and polyhedra

Let K be a set of finite number of simplexes in $\mathbb{R}^m$. If these simplexes are *nicely* fitted together, K is called a **simplicial complex**. By 'nicely' we mean:

(i) an arbitrary face of a simplex of K belongs to K, that is, if $\sigma \in K$ and $\sigma' \leq \sigma$ then $\sigma' \in K$; and
(ii) if σ and σ' are two simplexes of K, the intersection $\sigma \cap \sigma'$ is either empty or a common face of σ and σ', that is, if $\sigma, \sigma' \in K$ then either $\sigma \cap \sigma' = \emptyset$ or $\sigma \cap \sigma' \leq \sigma$ and $\sigma \cap \sigma' \leq \sigma'$.

For example, figure 3.4(a) is a simplicial complex but figure 3.4(b) is not. The dimension of a simplicial complex K is defined to be the largest dimension of simplexes in K.

Example 3.4. Let σ_r be an r-simplex and $K = \{\sigma' | \sigma' \leq \sigma_r\}$ be the set of faces of σ_r. K is an r-dimensional simplicial complex. For example, take

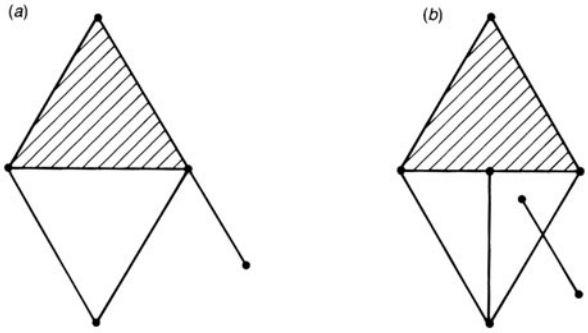

Figure 3.4. (*a*) is a simplicial complex but (*b*) is not.

$\sigma_3 = \langle p_0 p_1 p_2 p_3 \rangle$ (figure 3.3). Then

$$\begin{aligned} K = \{ & p_0, p_1, p_2, p_3, \langle p_0 p_1 \rangle, \langle p_0 p_2 \rangle, \langle p_0 p_3 \rangle, \\ & \langle p_1 p_2 \rangle, \langle p_1 p_3 \rangle, \langle p_2 p_3 \rangle, \langle p_0 p_1 p_2 \rangle, \langle p_0 p_1 p_3 \rangle, \\ & \langle p_0 p_2 p_3 \rangle, \langle p_1 p_2 p_3 \rangle, \langle p_0 p_1 p_2 p_3 \rangle \}. \end{aligned} \tag{3.12}$$

A simplicial complex K is a *set* whose elements are simplexes. If each simplex is regarded as a subset of $\mathbb{R}^m$ ($m \geq \dim K$), the union of all the simplexes becomes a subset of $\mathbb{R}^m$. This subset is called the **polyhedron** $|K|$ of a simplicial complex K. The dimension of $|K|$ as a subset of $\mathbb{R}^m$ is the same as that of K; $\dim |K| = \dim K$.

Let X be a topological space. If there exists a simplicial complex K and a homeomorphism $f : |K| \to X$, X is said to be **triangulable** and the pair (K, f) is called a **triangulation** of X. Given a topological space X, its triangulation is far from unique. We will be concerned with triangulable spaces only.

Example 3.5. Figure 3.5(*a*) is a triangulation of a cylinder $S^1 \times [0, 1]$. The reader might think that somewhat simpler choices exist, figure 3.5(*b*), for example. This is, however, not a triangulation since, for $\sigma_2 = \langle p_0 p_1 p_2 \rangle$ and $\sigma_2' = \langle p_2 p_3 p_0 \rangle$, we find $\sigma_2 \cap \sigma_2' = \langle p_0 \rangle \cup \langle p_2 \rangle$, which is neither empty nor a simplex.

3.3 Homology groups of simplicial complexes

3.3.1 Oriented simplexes

We may assign *orientations* to an r-simplex for $r \geq 1$. Instead of $\langle \ldots \rangle$ for an unoriented simplex, we will use $(\ldots)$ to denote an oriented simplex. The symbol σ_r is used to denote both types of simplex. An oriented 1-simplex $\sigma_1 = (p_0 p_1)$ is a directed line segment traversed in the direction $p_0 \to p_1$ (figure 3.6(*a*)). Now

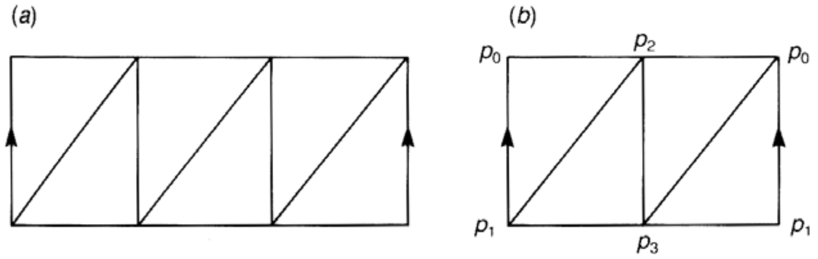

Figure 3.5. (*a*) is a triangulation of a cylinder while (*b*) is not.

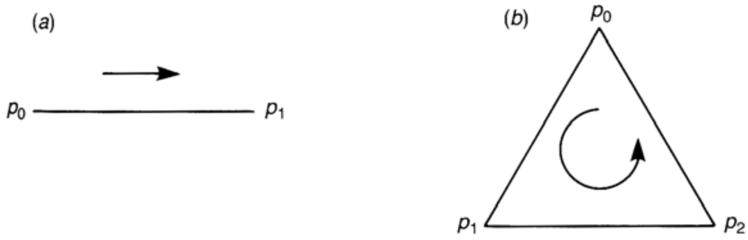

Figure 3.6. An oriented 1-simplex (*a*) and an oriented 2-simplex (*b*).

$(p_0 p_1)$ should be distinguished from $(p_1 p_0)$. We require that

$$(p_0 p_1) = -(p_1 p_0). \tag{3.13}$$

Here '$-$' in front of $(p_1 p_0)$ should be understood in the sense of a finitely generated Abelian group. In fact, $(p_1 p_0)$ is regarded as the *inverse* of $(p_0 p_1)$. Going from p_0 to p_1 followed by going from p_1 to p_0 means going nowhere, $(p_0 p_1) + (p_1 p_0) = 0$, hence $-(p_1 p_0) = (p_0 p_1)$.

Similarly, an oriented 2-simplex $\sigma_2 = (p_0 p_1 p_2)$ is a triangular region $p_0 p_1 p_2$ with a prescribed orientation along the edges (figure 3.6(*b*)). Observe that the orientation given by $p_0 p_1 p_2$ is the same as that given by $p_2 p_0 p_1$ or $p_1 p_2 p_0$ but opposite to $p_0 p_2 p_1$, $p_2 p_1 p_0$ or $p_1 p_0 p_2$. We require that

$$\begin{aligned}(p_0 p_1 p_2) &= (p_2 p_0 p_1) = (p_1 p_2 p_0) \\ &= -(p_0 p_2 p_1) = -(p_2 p_1 p_0) = -(p_1 p_0 p_2).\end{aligned}$$

Let P be a permutation of 0, 1, 2

$$P = \begin{pmatrix} 0 & 1 & 2 \\ i & j & k \end{pmatrix}.$$

These relations are summarized as

$$(p_i p_j p_k) = \mathrm{sgn}(P)(p_0 p_1 p_2)$$

where $\text{sgn}(P) = +1\ (-1)$ if P is an even (odd) permutation.

An oriented 3-simplex $\sigma_3 = (p_0 p_1 p_2 p_3)$ is an ordered sequence of four vertices of a tetrahedron. Let

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 \\ i & j & k & l \end{pmatrix}$$

be a permutation. We define

$$(p_i p_j p_k p_l) = \text{sgn}(P)(p_0 p_1 p_2 p_3).$$

It is now easy to construct an oriented r-simplex for any $r \geq 1$. The formal definition goes as follows. Take $r + 1$ geometrically independent points $p_0, p_1, \ldots, p_r$ in $\mathbb{R}^m$. Let $\{p_{i_0}, p_{i_1}, \ldots, p_{i_r}\}$ be a sequence of points obtained by a permutation of the points $p_0, \ldots, p_r$. We define $\{p_0, \ldots, p_r\}$ and $\{p_{i_0}, \ldots, p_{i_r}\}$ to be equivalent if

$$P = \begin{pmatrix} 0 & 1 & \ldots & r \\ i_0 & i_1 & \ldots & i_r \end{pmatrix}$$

is an even permutation. Clearly this is an equivalence relation, the equivalence class of which is called an **oriented *r*-simplex**. There are two equivalence classes, one consists of even permutations of $p_0, \ldots, p_r$, the other of odd permutations. The equivalence class (oriented r-simplex) which contains $\{p_0, \ldots, p_r\}$ is denoted by $\sigma_r = (p_0 p_1 \ldots p_r)$, while the other is denoted by $-\sigma_r = -(p_0 p_1 \ldots p_r)$. In other words,

$$(p_{i_0} p_{i_1} \ldots p_{i_r}) = \text{sgn}(P)(p_0 p_1 \ldots p_r). \tag{3.14}$$

For $r = 0$, we formally define an oriented 0-simplex to be just a point $\sigma_0 = p_0$.

3.3.2 Chain group, cycle group and boundary group

Let $K = \{\sigma_\alpha\}$ be an n-dimensional simplicial complex. We regard the simplexes σ_α in K as oriented simplexes and denote them by the same symbols σ_α as remarked before.

Definition 3.2. The ***r*-chain group** $C_r(K)$ of a simplicial complex K is a free Abelian group generated by the oriented r-simplexes of K. If $r > \dim K$, $C_r(K)$ is defined to be 0. An element of $C_r(K)$ is called an ***r*-chain**.

Let there be I_r r-simplexes in K. We denote each of them by $\sigma_{r,i}$ $(1 \leq i \leq I_r)$. Then $c \in C_r(K)$ is expressed as

$$c = \sum_{i=1}^{I_r} c_i \sigma_{r,i} \qquad c_i \in \mathbb{Z}. \tag{3.15}$$

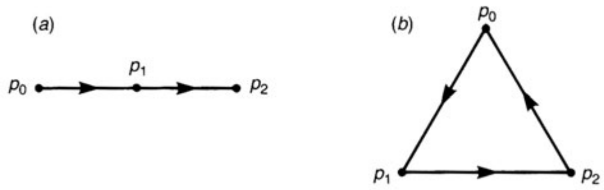

Figure 3.7. (*a*) An oriented 1-simplex with a fictitious boundary p_1. (*b*) A simplicial complex without a boundary.

The integers c_i are called the coefficients of c. The group structure is given as follows. The addition of two r-chains, $c = \sum_i c_i \sigma_{r,i}$ and $c' = \sum_i c'_i \sigma_{r,i}$, is

$$c + c' = \sum_i (c_i + c'_i)\sigma_{r,i}. \tag{3.16}$$

The unit element is $0 = \sum_i 0 \cdot \sigma_{r,i}$, while the inverse element of c is $-c = \sum_i (-c_i)\sigma_{r,i}$. [*Remark*: An oppositely oriented r-simplex $-\sigma_r$ is identified with $(-1)\sigma_r \in C_r(K)$.] Thus, $C_r(K)$ is a free Abelian group of rank I_r,

$$C_r(K) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{I_r}. \tag{3.17}$$

Before we define the cycle group and the boundary group, we need to introduce the boundary operator. Let us denote the boundary of an r-simplex σ_r by $\partial_r \sigma_r$. ∂_r should be understood as an *operator* acting on σ_r to produce its boundary. This point of view will be elaborated later. Let us look at the boundaries of lower-dimensional simplexes. Since a 0-simplex has no boundary, we define

$$\partial_0 p_0 = 0. \tag{3.18}$$

For a 1-simplex $(p_0 p_1)$, we define

$$\partial_1 (p_0 p_1) = p_1 - p_0. \tag{3.19}$$

The reader might wonder about the appearance of a minus sign in front of p_0. This is again related to the orientation. The following examples will clarify this point. In figure 3.7(*a*), an oriented 1-simplex $(p_0 p_2)$ is divided into two, $(p_0 p_1)$ and $(p_1 p_2)$. We agree that the boundary of $(p_0 p_2)$ is $\{p_0\} \cup \{p_2\}$ and so should be that of $(p_0 p_1) + (p_1 p_2)$. If $\partial_1(p_0 p_2)$ were defined to be $p_0 + p_2$, we would have $\partial_1(p_0 p_1) + \partial_1(p_1 p_2) = p_0 + p_1 + p_1 + p_2$. This is not desirable since p_1 is a *fictitious* boundary. If, instead, we take $\partial_1(p_0 p_2) = p_2 - p_0$, we will have $\partial_1(p_0 p_1) + \partial_1(p_1 p_2) = p_1 - p_0 + p_2 - p_1 = p_2 - p_0$ as expected. The next example is the triangle of figure 3.7(*b*). It is the sum of three oriented 1-simplexes,

$(p_0p_1) + (p_1p_2) + (p_2p_0)$. We agree that it has no boundary. If we insisted on the rule $\partial_1(p_0p_1) = p_0 + p_1$, we would have

$$\partial_1(p_0p_1) + \partial_1(p_1p_2) + \partial_1(p_2p_0) = p_0 + p_1 + p_1 + p_2 + p_2 + p_0$$

which contradicts our intuition. If, on the other hand, we take $\partial_1(p_0p_1) = p_1 - p_0$, we have

$$\partial_1(p_0p_1) + \partial_1(p_1p_2) + \partial_1(p_2p_0) = p_1 - p_0 + p_2 - p_1 + p_0 - p_2 = 0$$

as expected. Hence, we put a plus sign if the first vertex is omitted and a minus sign if the second is omitted. We employ this fact to define the boundary of a general r-simplex.

Let $\sigma_r(p_0 \dots p_r)$ $(r > 0)$ be an oriented r-simplex. The **boundary** $\partial_r\sigma_r$ of σ_r is an $(r-1)$-chain defined by

$$\partial_r\sigma_r \equiv \sum_{i=0}^{r}(-1)^i(p_0p_1 \dots \hat{p}_i \dots p_r) \tag{3.20}$$

where the point p_i under $\hat{}$ is omitted. For example,

$$\begin{aligned}\partial_2(p_0p_1p_2) &= (p_1p_2) - (p_0p_2) + (p_0p_1)\\ \partial_3(p_0p_1p_2p_3) &= (p_1p_2p_3) - (p_0p_2p_3) + (p_0p_1p_3) - (p_0p_1p_2).\end{aligned}$$

We formally define $\partial_0\sigma_0 = 0$ for $r = 0$.

The operator ∂_r acts linearly on an element $c = \sum_i c_i\sigma_{r,i}$ of $C_r(K)$,

$$\partial_r c = \sum_i c_i\partial_r\sigma_{r,i}. \tag{3.21}$$

The RHS of (3.21) is an element of $C_{r-1}(K)$. Accordingly, ∂_r defines a map

$$\partial_r : C_r(K) \to C_{r-1}(K). \tag{3.22}$$

∂_r is called the **boundary operator**. It is easy to see that the boundary operator is a homomorphism.

Let K be an n-dimensional simplicial complex. There exists a sequence of free Abelian groups and homomorphisms,

$$0 \xrightarrow{i} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0 \tag{3.23}$$

where $i : 0 \hookrightarrow C_n(K)$ is an inclusion map (0 is regarded as the unit element of $C_n(K)$). This sequence is called the **chain complex** associated with K and is denoted by $C(K)$. It is interesting to study the *image* and *kernel* of the homomorphisms ∂_r.

Definition 3.3. If $c \in C_r(K)$ satisfies

$$\partial_r c = 0 \tag{3.24}$$

c is called an ***r*-cycle**. The set of r-cycles $Z_r(K)$ is a subgroup of $C_r(K)$ and is called the ***r*-cycle group**. Note that $Z_r(K) = \ker \partial_r$. [*Remark*: If $r = 0$, $\partial_0 c$ vanishes identically and $Z_0(K) = C_0(K)$, see (3.23).]

Definition 3.4. Let K be an n-dimensional simplicial complex and let $c \in C_r(K)$. If there exists an element $d \in C_{r+1}(K)$ such that

$$c = \partial_{r+1} d \tag{3.25}$$

then c is called an ***r*-boundary**. The set of r-boundaries $B_r(K)$ is a subgroup of $C_r(K)$ and is called the ***r*-boundary group**. Note that $B_r(K) = \operatorname{im} \partial_{r+1}$. [*Remark*: $B_n(K)$ is defined to be 0.]

From lemma 3.1, it follows that $Z_r(K)$ and $B_r(K)$ are subgroups of $C_r(K)$. We now prove an important relation between $Z_r(K)$ and $B_r(K)$, which is crucial in the definition of homology groups.

Lemma 3.3. The composite map $\partial_r \circ \partial_{r+1} : C_{r+1}(K) \to C_{r-1}(K)$ is a zero map; that is, $\partial_r(\partial_{r+1} c) = 0$ for any $c \in C_{r+1}(K)$.

Proof. Since ∂_r is a linear operator on $C_r(K)$, it is sufficient to prove the identity $\partial_r \circ \partial_{r+1} = 0$ for the generators of $C_{r+1}(K)$. If $r = 0$, $\partial_0 \circ \partial_1 = 0$ since ∂_0 is a zero operator. Let us assume $r > 0$. Take $\sigma = (p_0 \dots p_r p_{r+1}) \in C_{r+1}(K)$. We find

$$\begin{aligned}
\partial_r(\partial_{r+1}\sigma) &= \partial_r \sum_{i=0}^{r+1} (-1)^i (p_0 \dots \hat{p}_i \dots p_{r+1}) \\
&= \sum_{i=0}^{r+1} (-1)^i \partial_r (p_0 \dots \hat{p}_i \dots p_{r+1}) \\
&= \sum_{i=0}^{r+1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j (p_0 \dots \hat{p}_j \dots \hat{p}_i \dots p_{r+1}) \right. \\
&\quad \left. + \sum_{j=i+1}^{r+1} (-1)^{j-1} (p_0 \dots \hat{p}_i \dots \hat{p}_j \dots p_{r+1}) \right) \\
&= \sum_{j<i} (-1)^{i+j} (p_0 \dots \hat{p}_j \dots \hat{p}_i \dots p_{r+1}) \\
&\quad - \sum_{j>i} (-1)^{i+j} (p_0 \dots \hat{p}_i \dots \hat{p}_j \dots p_{r+1}) = 0
\end{aligned} \tag{3.26}$$

which proves the lemma. □

Theorem 3.3. Let $Z_r(K)$ and $B_r(K)$ be the r-cycle group and the r-boundary group of $C_r(K)$, then

$$B_r(K) \subset Z_r(K) \qquad (\subset C_r(K)). \tag{3.27}$$

Proof. This is obvious from lemma 3.3. Any element c of $B_r(K)$ is written as $c = \partial_{r+1}d$ for some $d \in C_{r+1}(K)$. Then we find $\partial_r c = \partial_r(\partial_{r+1}d) = 0$, that is, $c \in Z_r(K)$. This implies $Z_r(K) \supset B_r(K)$. □

What are the geometrical pictures of r-cycles and r-boundaries? With our definitions, ∂_r picks up the boundary of an r-chain. If c is an r-cycle, $\partial_r c = 0$ tells us that c has no boundary. If $c = \partial_{r+1}d$ is an r-boundary, c is the boundary of d whose dimension is higher than c by one. Our intuition tells us that a boundary has no boundary, hence $Z_r(K) \supset B_r(K)$. Those elements of $Z_r(K)$ that are *not* boundaries play the central role in this chapter.

3.3.3 Homology groups

So far we have defined three groups $C_r(K)$, $Z_r(K)$ and $B_r(K)$ associated with a simplicial complex K. How are they related to topological properties of K or to the topological space whose triangulation is K? Is it possible for $C_r(K)$ to express any property which is conserved under homeomorphism? We all know that the edges of a triangle and those of a square are homeomorphic to each other. What about their chain groups? For example, the 1-chain group associated with a triangle is

$$\begin{aligned} C_1(K_1) &= \{i(p_0p_1) + j(p_1p_2) + k(p_2p_0) | i, j, k \in \mathbb{Z}\} \\ &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

while that associated with a square is

$$C_1(K_2) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Clearly $C_1(K_1)$ is not isomorphic to $C_1(K_2)$, hence $C_r(K)$ cannot be a candidate of a topological invariant. The same is true for $Z_r(K)$ and $B_r(K)$. It turns out that the homology groups defined in the following provide the desired topological invariants.

Definition 3.5. Let K be an n-dimensional simplicial complex. The ***r*th homology group** $H_r(K)$, $0 \le r \le n$, associated with K is defined by

$$H_r(K) \equiv Z_r(K)/B_r(K). \tag{3.28}$$

[*Remarks*: If necessary, we define $H_r(K) = 0$ for $r > n$ or $r < 0$. If we want to stress that the group structure is defined with integer coefficients, we write $H_r(K;\mathbb{Z})$. We may also define the homology groups with $\mathbb{R}$-coefficients, $H_r(K;\mathbb{R})$ or those with $\mathbb{Z}_2$-coefficients, $H_r(K;\mathbb{Z}_2)$.]

Since $B_r(K)$ is a subgroup of $Z_r(K)$, $H_r(K)$ is well defined. The group $H_r(K)$ is the set of equivalence classes of r-cycles,

$$H_r(K) \equiv \{[z]|z \in Z_r(K)\} \tag{3.29}$$

where each equivalence class $[z]$ is called a **homology class**. Two r-cycles z and z' are in the same equivalence class if and only if $z - z' \in B_r(K)$, in which case z is said to be **homologous** to z' and denoted by $z \sim z'$ or $[z] = [z']$. Geometrically $z - z'$ is a boundary of some space. By definition, any boundary $b \in B_r(K)$ is homologous to 0 since $b - 0 \in B_r(K)$. We accept the following theorem without proof.

Theorem 3.4. Homology groups are topological invariants. Let X be homeomorphic to Y and let (K, f) and (L, g) be triangulations of X and Y respectively. Then we have

$$H_r(K) \cong H_r(L) \qquad r = 0, 1, 2, \ldots. \tag{3.30}$$

In particular, if (K, f) and (L, g) are two triangulations of X, then

$$H_r(K) \cong H_r(L) \qquad r = 0, 1, 2, \ldots. \tag{3.31}$$

Accordingly, it makes sense to talk of homology groups of a topological space X which is not necessarily a polyhedron but which is triangulable. For an arbitrary triangulation (K, f), $H_r(X)$ is defined to be

$$H_r(X) \equiv H_r(K) \qquad r = 0, 1, 2, \ldots. \tag{3.32}$$

Theorem 3.4 tells us that this is independent of the choice of the triangulation (K, f).

Example 3.6. Let $K = \{p_0\}$. The 0-chain is $C_0(K) = \{ip_0|i \in \mathbb{Z}\} \cong \mathbb{Z}$. Clearly $Z_0(K) = C_0(K)$ and $B_0(K) = \{0\}$ ($\partial_0 p_0 = 0$ and p_0 cannot be a boundary of anything). Thus

$$H_0(K) \equiv Z_0(K)/B_0(K) = C_0(K) \cong \mathbb{Z}. \tag{3.33}$$

Exercise 3.1. Let $K = \{p_0, p_1\}$ be a simplicial complex consisting of two 0-simplexes. Show that

$$H_r(K) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & (r = 0) \\ \{0\} & (r \neq 0). \end{cases} \tag{3.34}$$

Example 3.7. Let $K = \{p_0, p_1, (p_0 p_1)\}$. We have

$$C_0(K) = \{ip_0 + jp_1 | i, j \in \mathbb{Z}\}$$
$$C_1(K) = \{k(p_0 p_1) | k \in \mathbb{Z}\}.$$

Since $(p_0 p_1)$ is not a boundary of any simplex in K, $B_1(K) = \{0\}$ and

$$H_1(K) = Z_1(K)/B_1(K) = Z_1(K).$$

If $z = m(p_0 p_1) \in Z_1(K)$, it satisfies

$$\partial_1 z = m\partial_1(p_0 p_1) = m\{p_1 - p_0\} = mp_1 - mp_0 = 0.$$

Thus, m has to vanish and $Z_1(K) = 0$, hence

$$H_1(K) = 0. \tag{3.35}$$

As for $H_0(K)$, we have $Z_0(K) = C_0(K) = \{ip_0 + jp_1\}$ and

$$B_0(K) = \operatorname{im} \partial_1 = \{\partial_1 i(p_0 p_1) | i \in \mathbb{Z}\} = \{i(p_0 - p_1) | i \in \mathbb{Z}\}.$$

Define a surjective (onto) homomorphism $f : Z_0(K) \to \mathbb{Z}$ by

$$f(ip_0 + jp_1) = i + j.$$

Then we find

$$\ker f = f^{-1}(0) = B_0(K).$$

Theorem 3.1 states that $Z_0(K)/\ker f \cong \operatorname{im} f = \mathbb{Z}$, or

$$H_0(K) = Z_0(K)/B_0(K) \cong \mathbb{Z}. \tag{3.36}$$

Example 3.8. Let $K = \{p_0, p_1, p_2, (p_0 p_1), (p_1 p_2), (p_2 p_0)\}$, see figure 3.7($b$). This is a triangulation of S^1. Since there are no 2-simplexes in K, we have $B_1(K) = 0$ and $H_1(K) = Z_1(K)/B_1(K) = Z_1(K)$. Let $z = i(p_0 p_1) + j(p_1 p_2) + k(p_2 p_0) \in Z_1(K)$ where $i, j, k \in \mathbb{Z}$. We require that

$$\begin{aligned}\partial_1 z &= i(p_1 - p_0) + j(p_2 - p_1) + k(p_0 - p_2) \\ &= (k - i)p_0 + (i - j)p_1 + (j - k)p_2 = 0.\end{aligned}$$

This is satisfied only when $i = j = k$. Thus, we find that

$$Z_1(K) = \{i\{(p_0 p_1) + (p_1 p_2) + (p_2 p_0)\} | i \in \mathbb{Z}\}.$$

This shows that $Z_1(K)$ is isomorphic to $\mathbb{Z}$ and

$$H_1(K) = Z_1(K) \cong \mathbb{Z}. \tag{3.37}$$

Let us compute $H_0(K)$. We have $Z_0(K) = C_0(K)$ and

$$B_0(K) = \{\partial_1[l(p_0p_1) + m(p_1p_2) + n(p_2p_0)] | l, m, n \in \mathbb{Z}\}$$
$$= \{(n-l)p_0 + (l-m)p_1 + (m-n)p_2 \,|\, l, m, n \in \mathbb{Z}\}.$$

Define a surjective homomorphism $f : Z_0(K) \to \mathbb{Z}$ by

$$f(ip_0 + jp_1 + kp_2) = i + j + k.$$

We verify that

$$\ker f = f^{-1}(0) = B_0(K).$$

From theorem 3.1 we find $Z_0(K)/\ker f \cong \operatorname{im} f = \mathbb{Z}$, or

$$H_0(K) = Z_0(K)/B_0(K) \cong \mathbb{Z}. \tag{3.38}$$

K is a triangulation of a circle S^1, and (3.37) and (3.38) are the homology groups of S^1.

Exercise 3.2. Let $K = \{p_0, p_1, p_2, p_3, (p_0p_1), (p_1p_2), (p_2p_3), (p_3p_0)\}$ be a simplicial complex whose polyhedron is a square. Verify that the homology groups are the same as those of example 3.8 above.

Example 3.9. Let $K = \{p_0, p_1, p_2, (p_0p_1), (p_1p_2), (p_2p_0), (p_0p_1p_2)\}$; see figure 3.6($b$). Since the structure of 0-simplexes and 1-simplexes is the same as that of example 3.8, we have

$$H_0(K) \cong \mathbb{Z}. \tag{3.39}$$

Let us compute $H_1(K) = Z_1(K)/B_1(K)$. From the previous example, we have

$$Z_1(K) = \{i\{(p_0p_1) + (p_1p_2) + (p_2p_0)\} | i \in \mathbb{Z}\}.$$

Let $c = m(p_0p_1p_2) \in C_2(K)$. If $b = \partial_2 c \in B_1(K)$, we have

$$b = m\{(p_1p_2) - (p_0p_2) + (p_0p_1)\}$$
$$= m\{(p_0p_1) + (p_1p_2) + (p_2p_0)\} \qquad m \in \mathbb{Z}.$$

This shows that $Z_1(K) \cong B_1(K)$, hence

$$H_1(K) = Z_1(K)/B_1(K) \cong \{0\}. \tag{3.40}$$

Since there are no 3-simplexes in K, we have $B_2(K) = \{0\}$. Then $H_2(K) = Z_2(K)/B_2(K) = Z_2(K)$. Let $z = m(p_0p_1p_2) \in Z_2(K)$. Since $\partial_2 z = m\{(p_1p_2) - (p_0p_2) + (p_0p_1)\} = 0$, m must vanish. Hence, $Z_1(K) = \{0\}$ and we have

$$H_2(K) \cong \{0\}. \tag{3.41}$$

Exercise 3.3. Let

$$K = \{p_0, p_1, p_2, p_3, (p_0p_1), (p_0p_2), (p_0p_3), (p_1p_2), (p_1p_3), (p_2p_3), (p_0p_1p_2), (p_0p_1p_3), (p_0p_2p_3), (p_1p_2p_3)\}$$

be a simplicial complex whose polyhedron is the surface of a tetrahedron. Verify that

$$H_0(K) \cong \mathbb{Z} \qquad H_1(K) \cong \{0\} \qquad H_2(K) \cong \mathbb{Z}. \tag{3.42}$$

K is a triangulation of the sphere S^2 and (3.42) gives the homology groups of S^2.

3.3.4 Computation of $H_0(K)$

Examples 3.6–3.9 and exercises 3.2, 3.3 share the same zeroth homology group, $H_0(K) \cong \mathbb{Z}$. What is common to these simplicial complexes? We have the following answer.

Theorem 3.5. Let K be a *connected* simplicial complex. Then

$$H_0(K) \cong \mathbb{Z}. \tag{3.43}$$

Proof. Since K is connected, for any pair of 0-simplexes p_i and p_j, there exists a sequence of 1-simplexes $(p_ip_k), (p_kp_l), \ldots, (p_mp_j)$ such that $\partial_1((p_ip_k) + (p_kp_l) + \cdots + (p_mp_j)) = p_j - p_i$. Then it follows that p_i is homologous to p_j, namely $[p_i] = [p_j]$. Thus, any 0-simplex in K is homologous to p_1 say. Suppose

$$z = \sum_{i=1}^{I_0} n_i p_i \in Z_0(K)$$

where I_0 is the number of 0-simplexes in K. Then the homology class $[z]$ is generated by a single point,

$$[z] = \left[\sum_i n_i p_i\right] = \sum_i n_i [p_i] = \sum_i n_i [p_1].$$

It is clear that $[z] = 0$, namely $z \in B_0(K)$, if $\sum n_i = 0$.

Let $\sigma_j = (p_{j,1}p_{j,2})$ $(1 \leq j \leq I_1)$ be 1-simplexes in K, I_1 being the number of 1-simplexes in K, then

$$\begin{aligned} B_0(K) &= \operatorname{im} \partial_1 \\ &= \{\partial_1(n_1\sigma_1 + \cdots + n_{I_1}\sigma_{I_1}) | n_1, \ldots, n_{I_1} \in \mathbb{Z}\} \\ &= \{n_1(p_{1,2} - p_{1,1}) + \cdots + n_{I_1}(p_{I_1,2} - p_{I_1,1}) | n_1, \ldots, n_{I_1} \in \mathbb{Z}\}. \end{aligned}$$

Note that n_j $(1 \leq j \leq I_1)$ always appears as a pair $+n_j$ and $-n_j$ in an element of $B_0(K)$. Thus, if

$$z = \sum_j n_j p_j \in B_0(K) \qquad \text{then } \sum_j n_j = 0.$$

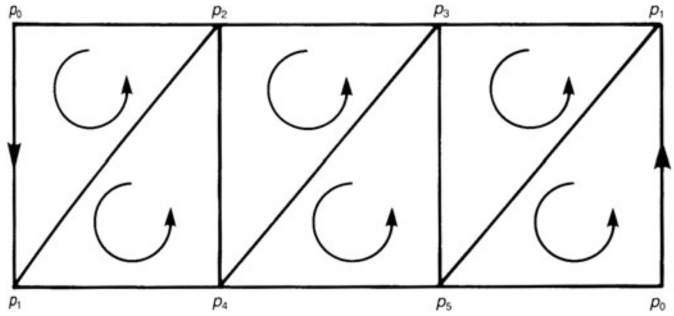

Figure 3.8. A triangulation of the Möbius strip.

Now we have proved for a connected complex K that $z = \sum n_i p_i \in B_0(K)$ if and only if $\sum n_i = 0$.

Define a surjective homomorphism $f : Z_0(K) \to \mathbb{Z}$ by

$$f(n_1 p_1 + \cdots + n_{I_0} p_{I_0}) = \sum_{i=1}^{I_0} n_i.$$

We then have $\ker f = f^{-1}(0) = B_0(K)$. It follows from theorem 3.1 that $H_0(K) = Z_0(K)/B_0(K) = Z_0(K)/\ker f \cong \operatorname{im} f = \mathbb{Z}$. □

3.3.5 More homology computations

Example 3.10. This and the next example deal with homology groups of non-orientable spaces. Figure 3.8 is a triangulation of the Möbius strip. Clearly $B_2(K) = 0$. Let us take a cycle $z \in Z_2(K)$,

$$\begin{aligned} z = &\, i(p_0 p_1 p_2) + j(p_2 p_1 p_4) + k(p_2 p_4 p_3) \\ &+ l(p_3 p_4 p_5) + m(p_3 p_5 p_1) + n(p_1 p_5 p_0). \end{aligned}$$

z satisfies

$$\begin{aligned} \partial_2 z = &\, i\{(p_1 p_2) - (p_0 p_2) + (p_0 p_1)\} \\ &+ j\{(p_1 p_4) - (p_2 p_4) + (p_2 p_1)\} \\ &+ k\{(p_4 p_3) - (p_2 p_3) + (p_2 p_4)\} \\ &+ l\{(p_4 p_5) - (p_3 p_5) + (p_3 p_4)\} \\ &+ m\{(p_5 p_1) - (p_3 p_1) + (p_3 p_5)\} \\ &+ n\{(p_5 p_0) - (p_1 p_0) + (p_1 p_5)\} = 0. \end{aligned}$$

Since each of $(p_0 p_2)$, $(p_1 p_4)$, $(p_2 p_3)$, $(p_4 p_5)$, $(p_3 p_1)$ and $(p_5 p_0)$ appears once and only once in $\partial_2 z$, all the coefficients must vanish, $i = j = k = l = m = n =$

0. Thus, $Z_2(K) = \{0\}$ and

$$H_2(K) = Z_2(K)/B_2(K) \cong \{0\}. \tag{3.44}$$

To find $H_1(K)$, we use our intuition rather than doing tedious computations. Let us find the loops which make complete circuits. One such loop is

$$z = (p_0 p_1) + (p_1 p_4) + (p_4 p_5) + (p_5 p_0).$$

Then all the other complete circuits are homologous to multiples of z. For example, let us take

$$z' = (p_1 p_2) + (p_2 p_3) + (p_3 p_5) + (p_5 p_1).$$

We find that $z \sim z'$ since

$$z - z' = \partial_2\{(p_2 p_1 p_4) + (p_2 p_4 p_3) + (p_3 p_4 p_5) + (p_1 p_5 p_0)\}.$$

If, however, we take

$$z'' = (p_1 p_4) + (p_4 p_5) + (p_5 p_0) + (p_0 p_2) + (p_2 p_3) + (p_3 p_1)$$

we find that $z'' \sim 2z$ since

$$\begin{aligned} 2z - z'' &= 2(p_0 p_1) + (p_1 p_4) + (p_4 p_5) + (p_5 p_0) - (p_0 p_2) \\ &\quad - (p_2 p_3) - (p_3 p_1) \\ &= \partial_2\{(p_0 p_1 p_2) + (p_1 p_4 p_2) + (p_2 p_4 p_3) + (p_3 p_4 p_5) \\ &\quad + (p_3 p_5 p_1) + (p_0 p_1 p_5)\}. \end{aligned}$$

We easily verify that all the closed circuits are homologous to nz, $n \in \mathbb{Z}$. $H_1(K)$ is generated by just one element $[z]$,

$$H_1(K) = \{i[z] | i \in \mathbb{Z}\} \cong \mathbb{Z}. \tag{3.45}$$

Since K is connected, it follows from theorem 3.5 that $H_0(K) = \{i[p_a] | i \in \mathbb{Z}\} \cong \mathbb{Z}$, p_a being any 0-simplex of K.

Example 3.11. The projective plane $\mathbb{R}P^2$ has been defined in example 2.5(c) as the sphere S^2 whose antipodal points are identified. As a coset space, we may take the hemisphere (or the disc D^2) whose opposite points on the boundary S^1 are identified, see figure 2.5(*b*). Figure 3.9 is a triangulation of the projective plane. Clearly $B_2(K) = \{0\}$. Take a cycle $z \in Z_2(K)$,

$$\begin{aligned} z &= m_1(p_0 p_1 p_2) + m_2(p_0 p_4 p_1) + m_3(p_0 p_5 p_4) \\ &\quad + m_4(p_0 p_3 p_5) + m_5(p_0 p_2 p_3) + m_6(p_2 p_4 p_3) \\ &\quad + m_7(p_2 p_5 p_4) + m_8(p_2 p_1 p_5) + m_9(p_1 p_3 p_5) + m_{10}(p_1 p_4 p_3). \end{aligned}$$

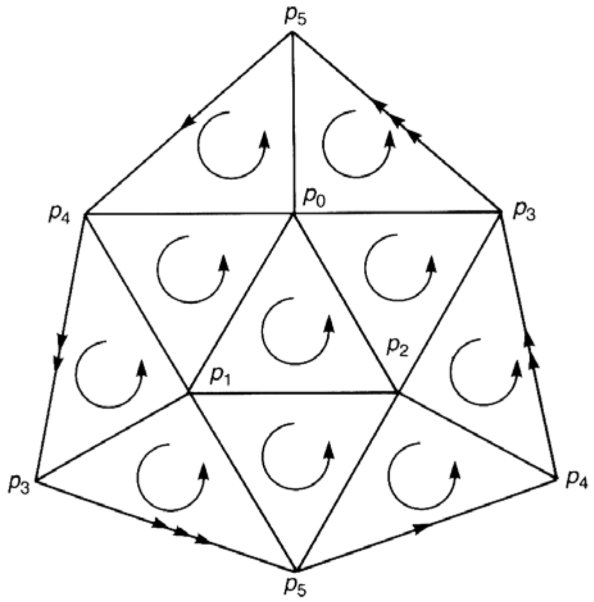

Figure 3.9. A triangulation of the projective plane.

The boundary of z is

$$\begin{aligned}
\partial_2 z = & m_1\{(p_1p_2) - (p_0p_2) + (p_0p_1)\} \\
& + m_2\{(p_4p_1) - (p_0p_1) + (p_0p_4)\} \\
& + m_3\{(p_5p_4) - (p_0p_4) + (p_0p_5)\} \\
& + m_4\{(p_3p_5) - (p_0p_5) + (p_0p_3)\} \\
& + m_5\{(p_2p_3) - (p_0p_3) + (p_0p_2)\} \\
& + m_6\{(p_4p_3) - (p_2p_3) + (p_2p_4)\} \\
& + m_7\{(p_5p_4) - (p_2p_4) + (p_2p_5)\} \\
& + m_8\{(p_1p_5) - (p_2p_5) + (p_2p_1)\} \\
& + m_9\{(p_3p_5) - (p_1p_5) + (p_1p_3)\} \\
& + m_{10}\{(p_4p_3) - (p_1p_3) + (p_1p_4)\} = 0.
\end{aligned}$$

Let us look at the coefficient of each 1-simplex. For example, we have $(m_1 - m_2)(p_0p_1)$, hence $m_1 - m_2 = 0$. Similarly,

$$\begin{aligned}
& -m_1 + m_5 = 0, m_4 - m_5 = 0, m_2 - m_3 = 0, m_1 - m_8 = 0, \\
& m_9 - m_{10} = 0, -m_2 + m_{10} = 0, m_5 - m_6 = 0, m_6 - m_7 = 0, \\
& m_6 + m_{10} = 0.
\end{aligned}$$

These ten conditions are satisfied if and only if $m_i = 0,\ 1 \le i \le 10$. This means that the cycle group $Z_2(K)$ is trivial and we have

$$H_2(K) = Z_2(K)/B_2(K) \cong \{0\}. \tag{3.46}$$

Before we calculate $H_1(K)$, we examine $H_2(K)$ from a slightly different viewpoint. Let us add all the 2-simplexes in K with the same coefficient,

$$z \equiv \sum_{i=1}^{10} m\sigma_{2,i} \qquad m \in \mathbb{Z}.$$

Observe that each 1-simplex of K is a common face of exactly two 2-simplexes. As a consequence, the boundary of z is

$$\partial_2 z = 2m(p_3 p_5) + 2m(p_5 p_4) + 2m(p_4 p_3). \tag{3.47}$$

Thus, if $z \in Z_2(K)$, m must vanish and we find $Z_2(K) = \{0\}$ as before. This observation remarkably simplifies the computation of $H_1(K)$. Note that any 1-cycle is homologous to a multiple of

$$z = (p_3 p_5) + (p_5 p_4) + (p_4 p_3)$$

cf example 3.10. Furthermore, equation (3.47) shows that an even multiple of z is a boundary of a 2-chain. Thus, z is a cycle and $z + z$ is homologous to 0. Hence, we find that

$$H_1(K) = \{[z] | [z] + [z] \sim [0]\} \cong \mathbb{Z}_2. \tag{3.48}$$

This example shows that a homology group is not necessarily free Abelian but may have the full structure of a finitely generated Abelian group. Since K is connected, we have $H_0(K) \cong \mathbb{Z}$.

It is interesting to compare example 3.11 with the following examples. In these examples, we shall use the intuition developed in this section on boundaries and cycles to obtain results rather than giving straightforward but tedious computations.

Example 3.12. Let us consider the torus T^2. A formal derivation of the homology groups of T^2 is left as an exercise to the reader: see Fraleigh (1976), for example. This is an appropriate place to recall the intuitive meaning of the homology groups. The rth homology group is generated by those boundaryless r-chains that are not, by themselves, boundaries of some $(r + 1)$-chains. For example, the surface of the torus has no boundary but it is not a boundary of some 3-chain. Thus, $H_2(T^2)$ is freely generated by one generator, the surface itself, $H_2(T^2) \cong \mathbb{Z}$. Let us look at $H_1(T^2)$ next. Clearly the loops a and b in figure 3.10 have no boundaries but are not boundaries of some 2-chains. Take another loop a'. a' is homologous to a since $a' - a$ bounds the shaded area of figure 3.10.

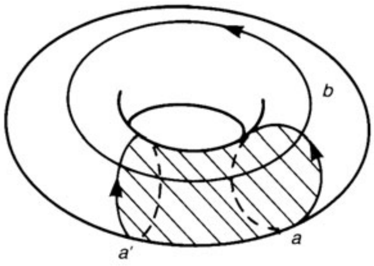

Figure 3.10. a' is homologous to a but b is not. a and b generate $H_1(T^2)$.

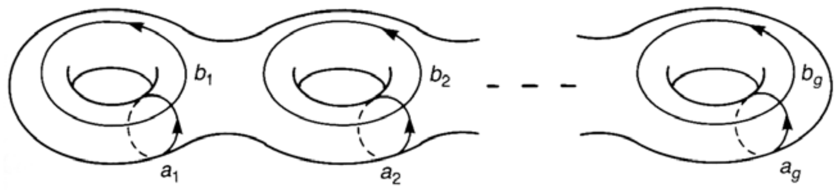

Figure 3.11. a_i and b_i $(1 \le i \le g)$ generate $\mathrm{H}_1(\Sigma_g)$.

Hence, $H_1(T^2)$ is freely generated by a and b and $H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since T^2 is connected, we have $H_0(T^2) \cong \mathbb{Z}$.

Now it is easy to extend our analysis to the torus Σ_g of genus g. Since Σ_g has no boundary and there are no 3-simplexes, the surface Σ_g itself freely generates $H_2(T^2) \cong \mathbb{Z}$. The first homology group $H_1(\Sigma_g)$ is generated by those loops which are not boundaries of some area. Figure 3.11 shows the standard choice for the generators. We find

$$\begin{aligned} H_1(\Sigma_g) &= \{i_1[a_1] + j_1[b_1] + \cdots + i_g[a_g] + j_g[b_g]\} \\ &\cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2g}. \end{aligned} \tag{3.49}$$

Since Σ_g is connected, $H_0(\Sigma_g) \cong \mathbb{Z}$. Observe that $a_i(b_i)$ is homologous to the edge $a_i(b_i)$ of figure 2.12. The $2g$ curves $\{a_i, b_i\}$ are called the **canonical system of curves** on Σ_g.

Example 3.13. Figure 3.12 is a triangulation of the Klein bottle. Computations of the homology groups are much the same as those of the projective plane. Since $B_2(K) = 0$, we have $H_2(K) = Z_2(K)$. Let $z \in Z_2(K)$. If z is a combination of all the 2-simplexes of K with the same coefficient, $z = \sum m\sigma_{2,i}$, the inner 1-simplexes cancel out to leave only the outer 1-simplexes

$$\partial_2 z = -2ma$$

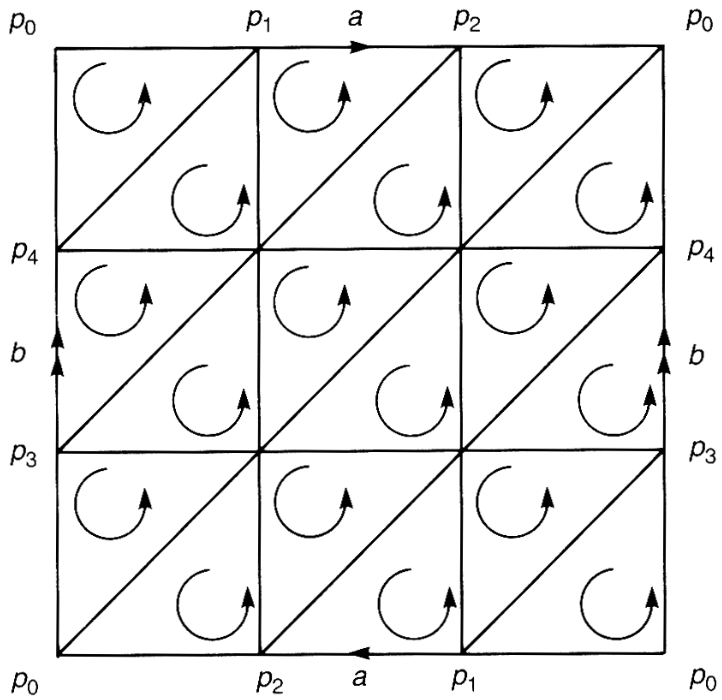

Figure 3.12. A triangulation of the Klein bottle.

where $a = (p_0 p_1) + (p_1 p_2) + (p_2 p_0)$. For $\partial_2 z$ to be 0, the integer m must vanish and we have

$$H_2(K) = Z_2(K) \cong \{0\}. \tag{3.50}$$

To compute $H_1(K)$ we first note, from our experience with the torus, that every 1-cycle is homologous to $ia + jb$ for some $i, j \in \mathbb{Z}$. For a 2-chain to have a boundary consisting of a and b only, all the 2-simplexes in K must be added with the same coefficient. As a result, for such a 2-chain $z = \sum m\sigma_{2,i}$, we have $\partial z = 2ma$. This shows that $2ma \sim 0$. Thus, $H_1(K)$ is generated by two cycles a and b such that $a + a = 0$, namely

$$H_1(K) = \{i[a] + j[b] | i, j \in \mathbb{Z}\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}. \tag{3.51}$$

We obtain $H_0(K) \cong \mathbb{Z}$ since K is connected.

3.4 General properties of homology groups

3.4.1 Connectedness and homology groups

Let $K = \{p_0\}$ and $L = \{p_0, p_1\}$. From example 3.6 and exercise 3.1, we have $H_0(K) = \mathbb{Z}$ and $H_0(L) = \mathbb{Z} \oplus \mathbb{Z}$. More generally, we have the following theorem.

Theorem 3.6. Let K be a disjoint union of N connected components, $K = K_1 \cup K_2 \cup \cdots \cup K_N$ where $K_i \cap K_j = \emptyset$. Then

$$H_r(K) = H_r(K_1) \oplus H_r(K_2) \oplus \cdots \oplus H_r(K_N). \tag{3.52}$$

Proof. We first note that an r-chain group is consistently separated into a direct sum of N r-chain subgroups. Let

$$C_r(K) = \left\{ \sum_{i=1}^{I_r} c_i \sigma_{r,i} \middle| c_i \in \mathbb{Z} \right\}$$

where I_r is the number of linearly independent r-simplexes in K. It is always possible to rearrange σ_i so that those r-simplexes in K_1 come first, those in K_2 next and so on. Then $C_r(K)$ is separated into a direct sum of subgroups,

$$C_r(K) = C_r(K_1) \oplus C_r(K_2) \oplus \cdots \oplus C_r(K_N).$$

This separation is also carried out for $Z_r(K)$ and $B_r(K)$ as

$$Z_r(K) = Z_r(K_1) \oplus Z_r(K_2) \oplus \cdots \oplus Z_r(K_N)$$
$$B_r(K) = B_r(K_1) \oplus B_r(K_2) \oplus \cdots \oplus B_r(K_N).$$

We now define the homology groups of each component K_i by

$$H_r(K_i) = Z_r(K_i)/B_r(K_i).$$

This is well defined since $Z_r(K_i) \supset B_r(K_i)$. Finally, we have

$$\begin{aligned} H_r(K) &= Z_r(K)/B_r(K) \\ &= Z_r(K_1) \oplus \cdots \oplus Z_r(K_N)/B_r(K_1) \oplus \cdots \oplus B_r(K_N) \\ &= \{Z_r(K_1)/B_r(K_1)\} \oplus \cdots \oplus \{Z_r(K_N)/B_r(K_N)\} \\ &= H_r(K_1) \oplus \cdots \oplus H_r(K_N). \end{aligned}$$ □

Corollary 3.1. (a) Let K be a disjoint union of N connected components, $K_1, \ldots, K_N$. Then it follows that

$$H_0(K) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{N \text{ factors}}. \tag{3.53}$$

(b) If $H_0(K) \cong \mathbb{Z}$, K is connected. [Together with theorem 3.5 we conclude that $H_0(K) \cong \mathbb{Z}$ if and only if K is connected.]

3.4.2 Structure of homology groups

$Z_r(K)$ and $B_r(K)$ are free Abelian groups since they are subgroups of a free Abelian group $C_r(K)$. It does not mean that $H_r(K) = Z_r(K)/B_r(K)$ is also free Abelian. In fact, according to theorem 3.2, the most general form of $H_r(K)$ is

$$H_r(K) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{f} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_p}. \tag{3.54}$$

It is clear from our experience that the number of generators of $H_r(K)$ counts the number of $(r+1)$-dimensional holes in $|K|$. The first f factors form a free Abelian group of rank f and the next p factors are called the **torsion subgroup** of $H_r(K)$. For example, the projective plane has $H_1(K) \cong \mathbb{Z}_2$ and the Klein bottle has $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. In a sense, the torsion subgroup detects the 'twisting' in the polyhedron $|K|$. We now clarify why the homology groups with $\mathbb{Z}$-coefficients are preferable to those with $\mathbb{Z}_2$- or $\mathbb{R}$-coefficients. Since $\mathbb{Z}_2$ has no non-trivial subgroups, the torsion subgroup can never be recognized. Similarly, if $\mathbb{R}$-coefficients are employed, we cannot see the torsion subgroup either, since $\mathbb{R}/m\mathbb{R} \cong \{0\}$ for any $m \in \mathbb{Z} - \{0\}$. [For any $a, b \in \mathbb{R}$, there exists a number $c \in \mathbb{R}$ such that $a - b = mc$.] If $H_r(K; \mathbb{Z})$ is given by (3.54), $H_r(K; \mathbb{R})$ is

$$H_r(K; \mathbb{R}) \cong \underbrace{\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{f}. \tag{3.55}$$

3.4.3 Betti numbers and the Euler–Poincaré theorem

Definition 3.6. Let K be a simplicial complex. The rth **Betti number** $b_r(K)$ is defined by

$$b_r(K) \equiv \dim H_r(K; \mathbb{R}). \tag{3.56}$$

In other words, $b_r(K)$ is the rank of the free Abelian part of $H_r(K; \mathbb{Z})$.

For example, the Betti numbers of the torus T^2 are (see example 3.12)

$$b_0(K) = 1, \qquad b_1(K) = 2, \qquad b_2(K) = 1$$

and those of the sphere S^2 are (exercise 3.3)

$$b_0(K) = 1, \qquad b_1(K) = 0, \qquad b_2(K) = 1.$$

The following theorem relates the Euler characteristic to the Betti numbers.

Theorem 3.7. (**The Euler–Poincaré theorem**) Let K be an n-dimensional simplicial complex and let I_r be the number of r-simplexes in K. Then

$$\chi(K) \equiv \sum_{r=0}^{n} (-1)^r I_r = \sum_{r=0}^{n} (-1)^r b_r(K). \tag{3.57}$$

[*Remark:* The first equality *defines* the Euler characteristic of a general polyhedron $|K|$. Note that this is the generalization of the Euler characteristic defined for surfaces in section 2.4.]

Proof. Consider the boundary homomorphism,

$$\partial_r : C_r(K; \mathbb{R}) \to C_{r-1}(K; \mathbb{R})$$

where $C_{-1}(K; \mathbb{R})$ is defined to be $\{0\}$. Since both $C_{r-1}(K; \mathbb{R})$ and $C_r(K; \mathbb{R})$ are vector spaces, theorem 2.1 can be applied to yield

$$\begin{aligned} I_r &= \dim C_r(K; \mathbb{R}) = \dim(\ker \partial_r) + \dim(\operatorname{im} \partial_r) \\ &= \dim Z_r(K; \mathbb{R}) + \dim B_{r-1}(K; \mathbb{R}) \end{aligned}$$

where $B_{-1}(K)$ is defined to be trivial. We also have

$$\begin{aligned} b_r(K) &= \dim H_r(K; \mathbb{R}) = \dim(Z_r(K; \mathbb{R})/B_r(K; \mathbb{R})) \\ &= \dim Z_r(K; \mathbb{R}) - \dim B_r(K; \mathbb{R}). \end{aligned}$$

From these relations, we obtain

$$\begin{aligned} \chi(K) &= \sum_{r=0}^{n} (-1)^r I_r = \sum_{r=0}^{n} (-1)^r (\dim Z_r(K; \mathbb{R}) + \dim B_{r-1}(K; \mathbb{R})) \\ &= \sum_{r=0}^{n} \{(-1)^r \dim Z_r(K; \mathbb{R}) - (-1)^r \dim B_r(K; \mathbb{R})\} \\ &= \sum_{r=0}^{n} (-1)^r b_r(K). \end{aligned}$$ □

Since the Betti numbers are topological invariants, $\chi(K)$ is also conserved under a homeomorphism. In particular, if $f : |K| \to X$ and $g : |K'| \to X$ are two triangulations of X, we have $\chi(K) = \chi(K')$. Thus, it makes sense to define the Euler characteristic of X by $\chi(K)$ for any triangulation (K, f) of X.

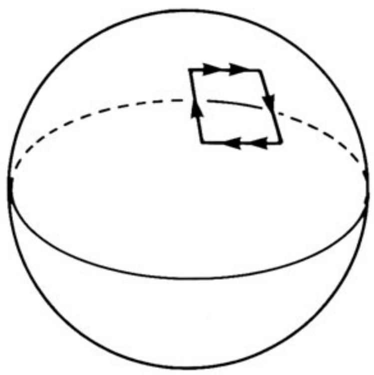

Figure 3.13. A hole in S^2, whose edges are identified as shown. We may consider S^2 with q such holes.

Problems

3.1 The most general orientable two-dimensional surface is a 2-sphere with h handles and q holes. Compute the homology groups and the Euler characteristic of this surface.

3.2 Consider a sphere with a hole and identify the edges of the hole as shown in figure 3.13. The surface we obtained was simply the projective plane $\mathbb{R}P^2$. More generally, consider a sphere with q such 'crosscaps' and compute the homology groups and the Euler characteristic of this surface.