8
Spontaneous symmetry breaking and the Weinberg–Salam model

The foregoing chapters have dealt with field theories, including gauge theories, and their quantisation. The stage is now almost set for applying this knowledge to particle physics. One crucial bit of scenery, however, is still missing – the idea of 'spontaneous breaking of symmetry'. About 1960 Nambu and Goldstone realised the significance of this notion in condensed matter physics, and Nambu in particular speculated on its application to particle physics. In 1964 Higgs pointed out that the consequences of spontaneous symmetry breaking in gauge theories are very different from those in non-gauge theories. Weinberg and Salam, building on earlier work of Glashow, then applied Higgs' ideas to an $SU(2) \times U(1)$ gauge theory, which they claimed described satisfactorily the weak and electromagnetic interactions together, in other words, in a *unified* way. Serious interest was shown in this theory when 't Hooft proved, in 1971, that is was renormalisable. It has met with notable experimental successes. These matters are the concern of this chapter (with the exception of renormalisation, which we deal with in the next chapter). We begin by explaining spontaneous symmetry breaking, which, when applied to field theory, is a concept that refines our notion of the vacuum.

8.1 What is the vacuum?

We begin by considering two simple physical examples. First, consider the situation illustrated in Fig. 8.1. Place a thin rod of circular cross section vertically on a table, and push down on it along its length, with a force F. If F

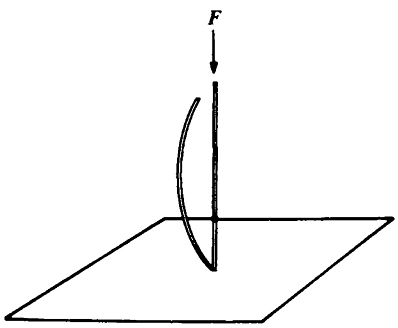

Fig. 8.1. A rod bent under pressure. The bent position exhibits spontaneous symmetry breaking.

is small, nothing happens. If F exceeds a critical value, F_{crit}, however, the rod bends, as shown, in a plane which it 'chooses at random'. The symmetric (unbent) configuration becomes unstable when $F > F_{crit}$, and the new ground state is unsymmetric. Also, there are infinitely many possible new (degenerate) ground states, which are related by a rotational symmetry. The rod can only, of course, choose one of them, but the others are all reached by a rotation. The salient points of this example are

(i) A parameter (in this case force F) assumes a critical value. Beyond that value,
(ii) the symmetric configuration becomes unstable, and
(iii) the ground state is degenerate.

The second example we consider is ferromagnetism. The atoms in the ferromagnet interact through a spin–spin interaction

$$H = -\sum_{i,j} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j$$

which is a scalar and therefore invariant under rotations. The *ground state*, however, is one in which all the spins (within a domain) are aligned, as in Fig. 8.2, and this is clearly not rotationally invariant. The direction of spontaneous magnetisation is random, and all the degenerate ground states may be reached from a given one by rotation. As Coleman (1985) points out, a 'little man' living in such a ferromagnet would have to be very clever to realise that the Hamiltonian H is rotation invariant! The spontaneous magnetisation disappears at high temperature T, when the ground state becomes symmetric (that is, the atoms become randomly oriented).

It is clear that the general situation here is the same as in our first example, the relevant parameter here being T. These two examples exhibit what is known as 'spontaneous breaking of symmetry'. In both cases the system possesses a symmetry (rotation symmetry) but the ground state is not invariant under that symmetry; rather, it changes into one of the other (degenerate) ground states.

One subtlety about the ferromagnet is that it must, in principle, be an infinite system. The magnetisation has singled out a particular direction, and a (quantum-mechanical) measurement of direction (angle) will give a sharp answer. But the conjugate variable to angle is angular momentum (recall that

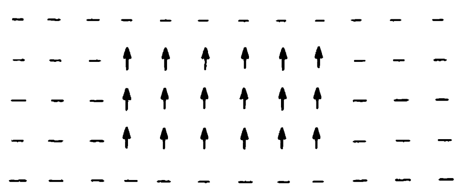

Fig. 8.2. Spin alignment in a ferromagnet.

$J_z = i\hbar\partial/\partial\phi$), so the angular momentum of the system is *completely undefined*,* and must therefore be an infinite sum of all possible values of J. The fact that the ferromagnet is infinite makes it an interesting system to compare with field theory, since a field, of course, is a system with an infinite number of degrees of freedom.

We therefore now look for a similar situation in scalar field theory, in which the symmetry of the Lagrangian is not shared by the ground state solution. In a field theory the ground state is regarded as being the vacuum, so we are in quest of a theory with a new type of vacuum. Since $\mathscr{L}$ must have a symmetry, we choose complex ϕ^4 theory:

$$\begin{aligned}\mathscr{L} &= (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \\ &= (\partial_\mu\phi)(\partial^\mu\phi^*) - V(\phi, \phi^*).\end{aligned} \tag{8.1}$$

The λ term is a self-interaction. In the usual scalar field theory, quantisation yields particles of mass m, but here m^2 is regarded as a *parameter* only, and not as a mass term. This is because we shall shortly let it become negative. $\mathscr{L}$ is invariant under the *global* gauge transformation

$$\phi \to e^{i\Lambda}\phi \quad (\Lambda \text{ const}). \tag{8.2}$$

The ground state is obtained by minimising the potential V. We have

$$\frac{\partial V}{\partial \phi} = m^2\phi^* + 2\lambda\phi^*(\phi^*\phi) \tag{8.3}$$

so that when $m^2 > 0$, the minimum occurs at $\phi^* = \phi = 0$. If $m^2 < 0$, however, there is a local maximum at $\phi = 0$, and a minimum at

$$|\phi|^2 = -\frac{m^2}{2\lambda} = a^2, \tag{8.4}$$

i.e. at $|\phi| = a$. In the quantum theory, where ϕ becomes an operator, this condition refers to the vacuum expectation value of ϕ

$$|\langle 0|\phi|0\rangle|^2 = a^2. \tag{8.5}$$

The function V is shown in Fig. 8.3, plotted against ϕ_1 and ϕ_2, where $\phi = \phi_1 + i\phi_2$ (though it should be borne in mind that ϕ is a *field*, not simply a pair of co-ordinates). The minima of V lie along the circle $|\phi| = a$, which form a set of degenerate vacua related to each other by rotation. The physical fields, which are excitations above the vacuum, are then realised by performing perturbations about $|\phi| = a$, not about $\phi = 0$. Let us work in polar co-ordinates, putting

$$\phi(x) = \rho(x)\, e^{i\theta(x)}, \tag{8.6}$$

so the complex field ϕ is expressed in terms of two real scalar fields ρ and θ.

* In fact, the uncertainty relation involving J_z and ϕ is somewhat problematic since ϕ has only a finite range and therefore a maximum uncertainty. See Constantinescu *et al.* (1976), pages 79, 91–2. I thank Dr B. Sheikholeslami-Sabzevari for bringing this to my attention.

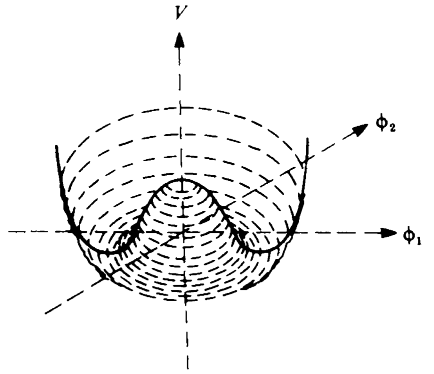

Fig. 8.3. The potential V has a minimum at $|\phi| = a$, and a local maximum at $\phi = 0$.

Let us *choose* the vacuum state

$$\langle 0|\phi|0\rangle = a \tag{8.7}$$

where a is real; then

$$\langle 0|\rho|0\rangle = a, \quad \langle 0|\theta|0\rangle = 0. \tag{8.8}$$

We see that this field theoretic example exhibits the same features as the ferromagnet. It has degenerate vacua, which are connected by the symmetry operations of the theory. A particular vacuum involves a particular choice for the values of the field ((8.8) in the field theory, direction of magnetisation for the ferromagnet), and is, of course, not invariant under the symmetry.

Now let us put

$$\phi(x) = [\rho'(x) + a]\,\mathrm{e}^{\mathrm{i}\theta(x)}, \tag{8.9}$$

so that ρ' and θ both have vanishing vacuum expectation values. We regard them as the 'physical' fields, and express $\mathcal{L}$ in terms of them. We have, from (8.1),

$$\begin{aligned} V &= m^2\rho'^2 + 2m^2a\rho' + m^2a^2 + \lambda(\rho'^4 + 4\alpha\rho'^3 + 6a^2\rho'^2 + 4a^3\rho' + a^4) \\ &= \lambda\rho'^4 + 4a\lambda\rho'^3 + 4\lambda a^2\rho'^2 - \lambda a^4 \\ &= \lambda[(\rho' + a)^2 - a^2]^2 - \lambda a^4 \\ &= \lambda(\phi^*\phi - a^2)^2 - \lambda a^4 \end{aligned}$$

where (8.4) has been used. In addition,

$$(\partial_\mu\phi)(\partial^\mu\phi^*) = (\partial_\mu\rho')(\partial^\mu\rho') + (\rho' + a)^2(\partial_\mu\theta)(\partial^\mu\theta)$$

with $\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - V$. We see that there is a term in ρ'^2, so ρ' has a mass given by

$$m_{\rho'}^2 = 4\lambda a^2,$$

but there is no term in θ^2, so θ is a *massless* field. As a result of spontaneous symmetry breaking, what would otherwise be two massive fields (the real parts of ϕ), become one massive and one massless field. We may interpret this with reference to Fig. 8.3. It clearly costs energy to displace ρ' against the restoring forces of the potential, but there are no restoring forces corresponding to displacements along the circular valley $|\phi| = a$, in view of the vacuum degeneracy. Hence for the angular excitations θ, of wavelength λ, we have $\omega \to 0$ as $\lambda \to \infty$, so $\omega \propto \lambda^{-1}$, $E \propto p$, and the relativistic particles are massless. The θ particle is known as a *Goldstone boson*. The important point is that this phenomenon is *general*: spontaneous breaking of a (continuous) symmetry entails the existence of a massless particle, the Goldstone particle.‡ This statement, known as the Goldstone theorem, will be proved in the next section.

For future reference it is useful to display this result using a 'Cartesian', rather than a polar, decomposition of ϕ. If, instead of (8.9), we have

$$\phi(x) = a + \frac{\phi_1(x) + i\phi_2(x)}{\sqrt{2}} \tag{8.10}$$

so that $\langle \phi_1 \rangle_0 = \langle \phi_2 \rangle_0 = 0$, it is easy to see that (ignoring constant terms)

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi_1)^2 + \tfrac{1}{2}(\partial_\mu \phi_2)^2 - 2\lambda a^2 \phi_1^2 - \sqrt{2}\lambda \phi_1(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2. \tag{8.11}$$

Hence the ϕ_2 field is massless, but ϕ_1 has a (mass)^2 of $4\lambda a^2$, the same result as above.

We conclude by noting the analogy of the above with the ferromagnet. Consider a 'spin wave' of long wavelength λ. It induces a slow variation in the direction of magnetisation, over the specimen, as indicated in Fig. 8.4. Because the forces in a ferromagnet are of *short* range, it requires very little energy to excite this situation, so the frequency of the spin waves approaches zero with increasing λ, i.e. $\omega = ck$. In the relativistic domain, this is equivalent to a

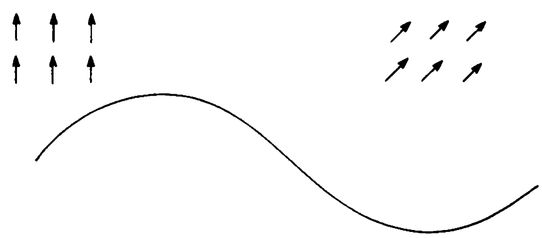

Fig. 8.4. A spin wave inducing a slow spatial variation in the direction of magnetisation in a ferromagnet.

‡ In this example it has spin zero, but this is not always the case. For example in theories of spontaneous breaking of supersymmetry, there are spin $\frac{1}{2}$ Goldstone particles.

massless particle. It is noteworthy, however, that this argument breaks down if there are *long*-range forces, like for example the $1/r$ Coulomb force; in other words if there is a *gauge* field present. In this case, it still costs a finite amount of energy to excite a spin wave of even very long wavelength, since work has to be done against the Coulomb force, so $\omega \to$ finite as $\lambda \to \infty$, $k \to 0$, and the corresponding excitations are *massive*. What is more, so are the photons! It was this situation in solid state physics, discussed in these terms by Anderson (1963), which led first Higgs (1964a, b; 1966) and then Weinberg (1967) and Salam (in Svartholm 1968) to consider the application of these ideas to the relativistic domain and particle physics. The situation in particle physics, on the face of it, seems to offer no fruitful ground for the application of gauge theories or theories of spontaneous symmetry breakdown. They both predict massless particles; the gauge particles, with spin 1, and the Goldstone bosons, with spin 0 – and, apart from the photon, there are no massless particles in existence. The observation on the ferromagnet above, that the presence of both effects together gets rid of *both* massless particles, is the key to the Weinberg–Salam model of electroweak interactions, described below.

8.2 The Goldstone theorem

In the example above, the Lagrangian has a $U(1)$ symmetry, and the two real fields in ϕ form a 2-dimensional representation of $U(1)$. One of these fields has a non-vanishing vacuum expectation value, and there turns out to be one massless particle (the Goldstone boson) and one massive one. It should also be emphasised that the argument above is classical. So two questions arise: first, in the general case where $\mathscr{L}$ is invariant under a symmetry group G, how many Goldstone bosons will there be? And second, what is the status of all this in quantum theory – in particular, how do we prove the existence of massless particles given degenerate vacua? We shall tackle these questions in this order.

The case of a general symmetry group is best approached by considering a specific non-Abelian group, say $SO(3)$. So let us consider an example like the above, except that ϕ_i ($i = 1, 2, 3$) is an isovector Lorentz-scalar field, and

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{m^2}{2}\phi_i\phi_i - \lambda(\phi_i\phi_i)^2 \tag{8.12}$$

(summation convention applies).

$\mathscr{L}$ is invariant under isospin rotations, which generate the symmetry group G (in this case $SO(3)$):

$$G\colon \phi_i \to \mathrm{e}^{\mathrm{i}Q_k\alpha_k}\phi_i\,\mathrm{e}^{-\mathrm{i}Q_k\alpha_k} = (\mathrm{e}^{-\mathrm{i}T_k\alpha_k})_{ij}\phi_j = U_{ij}\phi_j = [U(g)\phi]_i. \tag{8.13}$$

Here α_i are the angles of rotation in isospin space, Q_i are the generators of the group, and T_i a set of matrices obeying the Lie algebra of the group, of the same dimensionality as the representation to which ϕ belongs – in this case

3-dimensional. The matrix $U(g)$, corresponding to the group element g, is a unitary matrix (if T is Hermitian), hence we have a unitary representation. Although essential in quantum theory, this is not essential in the classical case, but there is no loss of generality in having one.

We look, as before, for the minimum of the potential $V(\phi_i)$

$$V = \frac{m^2}{2}\phi_i\phi_i + \lambda(\phi_i\phi_i)^2. \tag{8.14}$$

When the parameter $m^2 > 0$, this occurs at $\phi_i = 0$. When $m^2 < 0$, there is a minimum when

$$|\phi_0| = (\phi_1^2 + \phi_2^2 + \phi_3^2)^{1/2} = \left(\frac{-m^2}{4\lambda}\right)^{1/2} \equiv a. \tag{8.15}$$

We again have degenerate vacua and we are free to choose which one is the physical one. We choose

$$\vec{\phi}_0 = a\hat{e}_3. \tag{8.16}$$

The vacuum value of ϕ, ϕ_0, points in the 3 direction in isospin space. This is sketched in Fig. 8.5. It is clear that ϕ_0 is *not* invariant under the full group G, so that there are elements $g \in G$ for which

$$G: \quad \phi_0' = U(g)\phi_0 \neq \phi_0; \tag{8.17}$$

but it *is* invariant under a *subgroup* H of G, in this case rotations about the 3 axis:

$$\begin{aligned} H: \quad \phi_0' &= U(h)\phi_0 = \phi_0 \\ U(h) &= e^{iT_3\alpha_3}. \end{aligned} \tag{8.18}$$

On the other hand, of course, V is invariant under the whole of G:

$$V(\phi') = V(\phi), \quad \phi' = U(g)\phi, \tag{8.19}$$

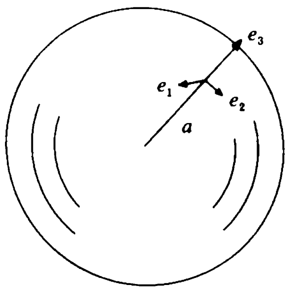

Fig. 8.5. The vacuum value of $\vec{\phi}$ points in the third direction in isospin space.

and it is this which gives rise to the Goldstone bosons. How many are there? Putting

$$\phi_3 = \chi + a, \tag{8.20}$$

the physical fields are ϕ_1, ϕ_2 and χ, and it is straightforward to verify (remembering (8.15)) that

$$\begin{aligned} V &= \frac{m^2}{2}[\phi_1^2 + \phi_2^2 + (\chi + a)^2] + \lambda[\phi_1^2 + \phi_2^2 + (\chi + a)^2]^2 \\ &= 4a^2\lambda\chi^2 + 4a\lambda\chi(\phi_1^2 + \phi_2^2 + \chi^2) + \lambda(\phi_1^2 + \phi_2^2 + \chi^2)^2 - \lambda a^4 \\ &= \lambda[(\phi_i\phi_i - a^2)^2 - a^4]. \end{aligned} \tag{8.21}$$

Only the field χ has a quadratic term, and therefore a mass

$$m_\chi^2 = 8a^2\lambda, \quad m_{\phi_1} = m_{\phi_2} = 0, \tag{8.22}$$

so, after spontaneous symmetry breaking, we have *two Goldstone bosons*, and one massive scalar field.

Now we can understand this in a very general way. Expanding $V(\phi)$ about its minimum, since

$$\left.\frac{\partial V}{\partial \phi_a}\right|_{\phi=\phi_0} = 0,$$

we have

$$V(\phi) = V(\phi_0) + \frac{1}{2}\left(\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right)_{\phi=\phi_0} \chi_i\chi_j + O(\chi^3) \tag{8.23}$$

where $\chi(x) = \phi(x) - \phi_0$, and hence the mass matrix is

$$M_{ij} = \left(\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right)_{\phi=\phi_0} \geqslant 0. \tag{8.24}$$

Since $V(\phi_0)$ is the minimum, M_{ij} must be positive or zero. To find out for which fields it is zero, we do a group transformation. The invariance of V, equation (8.19), gives

$$V(\phi_0) = V(U(g)\phi_0) = V(\phi_0) + \frac{1}{2}\left(\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right)_{\phi_0} \delta\phi_i\delta\phi_j + \cdots$$

and hence

$$\left(\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right)_{\phi_0} \delta\phi_i\delta\phi_j = 0, \tag{8.25}$$

where $\delta\phi_i$ is the variation in ϕ_i under a group transformation. What is this? From (8.17) and (8.18) it depends on whether the group element g belongs to

H or not. If g belongs to H, then $\phi_0' = \phi_0$ and $\delta\phi_i = 0$, or equivalently

$$\delta\phi = \left(\frac{\partial U}{\partial \alpha_3}\right)_{\alpha_3=0} \phi_0 \delta\alpha_3 = 0 \tag{8.26}$$

so (8.25) is satisfied already. If, however, g does not belong to H (if it is, in our example, a rotation about an axis in the 1, 2 plane), then

$$\delta\phi_m = \left[\left(\frac{\partial U}{\partial \alpha_i}\right)_{\alpha_i=0} \phi_0\right]_m \delta\alpha_i \neq 0. \tag{8.27}$$

(Recall that (8.17), and therefore the above equation, are matrix equations.) In this case, from (8.24) and (8.25),

$$M_{ij}[U'(0)\phi_0]_j = 0$$

and the fields $U'(0)\phi_0$ have *zero mass*. These are the Goldstone bosons. It is now clear that the question of the number of fields with non-zero mass and the number with zero mass is simply a matter of group theory. A field whose mass is not *required* to be zero (though, of course, it may be zero 'by accident') obeys (8.26), and the number of such fields is simply the dimension of the Lie algebra (or order of the Lie group) of H, the subgroup under which the vacuum is invariant. In our case, $H = SO(2) \sim U(1)$ with one generator (T_3), so one field remains massive. The elements of G which do not belong to H, do not form a subgroup (they cannot, since the identity is in H), but a coset G/H may be defined; and the number of Goldstone particles is the dimension of the coset space, which is the number of generators of G that are not also generators of H – in our case $3 - 1 = 2$. These results agree with the explicit calculation above. The interesting finding, however, is that this result does not depend on what representation of G the fields belong to – in our case it was the vector (regular) representation – nor on what form the potential V takes: the number of Goldstone bosons is simply the dimension of G/H. This fact is of great importance when we come to consider the spontaneous breaking of *gauge* symmetries.

Finally, to emphasise the generality of the conclusion, note that it also applies when the symmetry is *not* spontaneously broken. In this case there is a *unique* vacuum (a singlet under G) which is therefore invariant under G itself, so $H = G$, the coset is simply the identity, and there are no Goldstone bosons. At the other extreme, if the vacuum is such that there is *no* subgroup H which leaves one of the vacuum states ϕ_0 invariant, then H is the identity and $G/H = G$, and the number of Goldstone bosons is equal to the order of G. We have now answered the first question posed at the beginning of this section.

We therefore turn to the second question, and ask what the status of the above, classical, argument is in quantum theory. Here the Goldstone theorem states that if there is a field operator $\phi(x)$ with non-vanishing vacuum expectation value $\langle 0|\phi(x)|0\rangle \neq 0$, and which is *not* a singlet under the transformation of some symmetry group, then massless particles must exist in the spectrum of states. There are some rather subtle questions of existence raised by this topic, and for a careful treatment the reader is referred to Bernstein (1974) and Guralnik *et al.* (1964). Here we shall simply outline the proof.

We begin with some preliminary remarks about the symmetry group. If $\mathscr{L}$ is invariant under a group of transformations then (see §3.3) the currents

$$j_\mu^a(x) = \frac{\partial \mathscr{L}}{\partial(\partial^\mu \phi)} \frac{\delta \phi(x)}{\delta \alpha^a}$$

have zero divergence, $\partial^\mu j_\mu^a = 0$, and the corresponding charges

$$Q^a = \int d^3x j_0^a(x) \tag{8.28}$$

are conserved, $dQ^a/dt = 0$, and have the commutation relations of the symmetry group

$$[Q^a, Q^b] = C^{abc} Q^c$$

where C^{abc} are the structure constants of the Lie algebra. The unitary operator corresponding to a group transformation is

$$U = e^{iQ^a \alpha^a}. \tag{8.29}$$

If the vacuum is invariant under the group (i.e. a singlet), $U|0\rangle = |0\rangle$, hence

$$Q^a|0\rangle = 0 \tag{8.30}$$

and the charges annihilate the vacuum. This is the usual case for a symmetry. If this is *not* the case, we may say that we have 'degenerate vacua'; $U|0\rangle = |0\rangle' \neq |0\rangle$ or $Q^a|0\rangle \neq 0$. Strictly speaking we should say that $Q^a|0\rangle$ *does not exist* in Hilbert space – in other words, its norm is infinite (Fabri & Picasso 1966).

Returning to the operator $\phi(x)$, since it is not a singlet under the group, there must exist an operator $\phi'(x)$ such that, for some a,

$$[Q^a, \phi'(x)] = \phi(x) \tag{8.31}$$

and, since $\langle 0|\phi(x)|0\rangle \neq 0$,

$$\langle 0|[Q^a, \phi'(x)]|0\rangle = \langle 0|Q^a \phi'(x) - \phi'(x) Q^a|0\rangle \neq 0. \tag{8.32}$$

This means that (8.30) *cannot* apply, so we do not have a symmetry in the usual sense (of degenerate multiplets). (An example of one of the subtleties is that, from above $\langle 0|\phi'(x)Q^a|0\rangle$ exists, whereas $Q^a|0\rangle$ does not.) We now show that (8.32) implies the existence of massless particles. Substituting (8.28) into (8.32) and inserting a complete set of intermediate states we have

$$\sum_n \int d^3y [\langle 0|j_0^a(y)|n\rangle\langle n|\phi'(x)|0\rangle - \langle 0|\phi'(x)|n\rangle\langle n|j_0^a(y)|0\rangle]|_{x^0=y^0} \neq 0. \tag{8.33}$$

(The restriction $x^0 = y^0$ arises because this is necessary to prove (8.31).) Now translation invariance implies that

$$j_0^a(y) = \mathrm{e}^{-\mathrm{i}py} j_0^a(0)\,\mathrm{e}^{\mathrm{i}py}$$

so (8.33) becomes

$$\begin{aligned}
\sum_n \int \mathrm{d}^3 y[\langle 0|j_0^a(0)|n\rangle\langle n|\phi'(x)|0\rangle\,\mathrm{e}^{\mathrm{i}p_n y} &- \langle 0|\phi'(x)|n\rangle\langle n|j_0^a(0)|0\rangle\,\mathrm{e}^{-\mathrm{i}p_n y}]|_{x^0=y^0} \\
&= (2\pi)^3 \sum_n \delta^3(\mathbf{p}_n)[\langle 0|j_0^a(0)|n\rangle\langle n|\phi'(x)|0\rangle\,\mathrm{e}^{\mathrm{i}p_{n0}y_0} \\
&\quad - \langle 0|\phi'(x)|n\rangle\langle n|j_0^a(0)|0\rangle\,\mathrm{e}^{-\mathrm{i}p_{n0}y_0}]|_{x^0=y^0} \\
&= (2\pi)^3 \sum_n \delta^3(\mathbf{p}_n)[\langle 0|j_0^a(0)|n\rangle\langle n|\phi'(x)|0\rangle\,\mathrm{e}^{\mathrm{i}M_n y_0} \\
&\quad - \langle 0|\phi'(x)|n\rangle\langle n|j_0^a(0)|0\rangle\,\mathrm{e}^{-\mathrm{i}M_n y_0}]|_{x^0=y^0} \\
&\neq 0
\end{aligned} \tag{8.34}$$

where we have performed the spatial integral, and, in view of the delta function in $\mathbf{p}_n$, put $p_{n0} = M_n$, the mass of the intermediate state n. It remains only to show that (8.34) must be *independent of* y_0; if we can show this, we conclude that $M_n = 0$, so all the intermediate states have zero mass, which is the Goldstone theorem. Moreover, these intermediate states *must* exist, in order that (8.34) be non-zero: it will be noticed that the vacuum ($|n\rangle = |0\rangle$) gives no contribution to the sum. To show that the above expression is independent of y_0, we start from the fact that $j_\mu^a(y)$ is divergenceless:

$$\partial^\mu j_\mu^a(y) = \partial_0 j_0^a(y) + \boldsymbol{\nabla}\cdot\mathbf{j}^a(y) = 0,$$

which on integration gives

$$\frac{\partial}{\partial y_0}\int \mathrm{d}^3 y\, j_0^a(y) = -\int \mathrm{d}^3 y\, \boldsymbol{\nabla}\cdot\mathbf{j}^a(y).$$

Hence, since (8.34) is the same as (8.32),

$$\begin{aligned}
\frac{\partial}{\partial y_0}\langle 0|[Q^a, \phi'(x)]|0\rangle &= \frac{\partial}{\partial y_0}\int \mathrm{d}^3 y\,\langle 0|[j_0^a(y), \phi'(x)]|0\rangle \\
&= -\int \mathrm{d}^3 y\,\langle 0|[\boldsymbol{\nabla}\cdot\mathbf{j}^a(y), \phi'(x)]|0\rangle \\
&= -\int \mathrm{d}\mathbf{S}\cdot\langle 0|[\mathbf{j}^a(y), \phi'(x)]|0\rangle
\end{aligned}$$

and under fairly orthodox assumptions (Guralnik, Hagen & Kibble in Cool & Marshak 1968) this surface integral may be shown to vanish. Hence the Goldstone theorem is proved.

In the 1960s a lot of effort was devoted to searching for a role for Goldstone's theorem in high energy physics. Although there are no zero mass hadrons, the pion has a seductively low mass and might be *nearly* a Goldstone boson. This would account for the success of the PCAC hypothesis (partially conserved axial current). For accounts of this topic, the reader is referred elsewhere (Lee 1981, chs. 16, 22, 24; Taylor 1976, ch. 5).

8.3 Spontaneous breaking of gauge symmetries

Now let us see what happens when the symmetry in question is a *gauge* symmetry. Consider the simplest model, that of equation (8.1), but now let us demand invariance under

$$\phi \to e^{i\Lambda(x)}\phi. \tag{8.35}$$

This results (see (3.84)) in the introduction of the electromagnetic field through a covariant derivative, and (8.1) is replaced by

$$\mathcal{L} = (\partial_\mu + ieA_\mu)\phi(\partial^\mu - ieA^\mu)\phi^* - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{8.36}$$

As before, we consider m^2 as a (positive or negative) parameter, so that in the case $m^2 < 0$, and in the absence of a gauge field, the vacuum is at

$$|\phi| = a = \left(\frac{-m^2}{2\lambda}\right)^{1/2} \tag{8.4}$$

Then, as in (8.10), setting

$$\phi(x) = a + \frac{\phi_1(x) + i\phi_2(x)}{\sqrt{2}} \tag{8.10}$$

gives for the Lagrangian, in terms of the physical fields ϕ_1 and ϕ_2:

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2a^2A_\mu A^\mu + \tfrac{1}{2}(\partial_\mu\phi_1)^2 + \tfrac{1}{2}(\partial_\mu\phi_2)^2 \\ &- 2\lambda a^2\phi_1^2 + \sqrt{2}eaA^\mu\partial_\mu\phi_2 + \text{cubic + quartic terms}\end{aligned} \tag{8.37}$$

where we have taken account of (8.4). The interesting term is the second one, proportional to A_μ^2. It indicates that the photon has become *massive*. The scalar field ϕ_1 is a massive field, and ϕ_2 appears to be a massless one, but there is the odd mixed term in $A^\mu\partial_\mu\phi_2$, which would seem to indicate that a propagating photon could turn into a ϕ_2 field, so ϕ_2 does not seem to be a very physical field. In fact, it can be eliminated by a gauge transformation. For infinitesimal Λ in (8.35), (8.10) gives

$$\left.\begin{aligned}\phi_1' &= \phi_1 - \Lambda\phi_2, \\ \phi_2' &= \phi_2 + \Lambda\phi_1 + \sqrt{2}\Lambda a.\end{aligned}\right\} \tag{8.38}$$

This shows that ϕ_2, like A_μ, undergoes an *inhomogeneous* transformation corresponding to a rotation *and translation* in the (ϕ_1, ϕ_2) plane, and so does not have a direct physical interpretation. We may then choose Λ to make $\phi_2 = 0$, and the mixed term above disappears. In this gauge, the Lagrangian (8.37) becomes (putting ϕ_1 rather than ϕ_1')

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2a^2A_\mu A^\mu + \tfrac{1}{2}(\partial_\mu\phi_1)^2 - 2\lambda a^2\phi_1^2 + \text{coupling terms.} \tag{8.39}$$

This Lagrangian contains two fields only, the photon with spin 1, and ϕ_1 with spin 0, and they are *both massive*. The ϕ_2 field, which in the case of spontaneous breaking of the global symmetry became massless (Goldstone boson), has in this case *disappeared*. And in addition, the gauge field whose presence is due to the fact that we have a *local* symmetry, has now acquired a mass – the '*photon*' *has become massive*. This phenomenon is called the *Higgs phenomenon*. In this Abelian model it is summarised by saying that spontaneous breaking of a gauge symmetry results, not in the presence of a massless Goldstone boson, but in the disappearance of that field altogether, and the appearance, instead, of a massive, rather than a massless, gauge field. Spontaneous breaking of $U(1)$ symmetry, then, yields the following particle spectrum, depending on whether the symmetry is global or local.

Goldstone mode (SB of global $U(1)$ symmetry):
2 massive scalar fields $\rightarrow$ 1 massive scalar field
+ 1 massless scalar field. (8.40)

Higgs mode (SB of gauge $U(1)$ symmetry):

$$\left.\begin{array}{l}\text{2 massive scalar fields}\\ \quad +\text{ 1 photon}\end{array}\right\} \rightarrow \left\{\begin{array}{l}\text{1 massive scalar field}\\ \quad +\text{ 1 massive photon.}\end{array}\right. \tag{8.41}$$

Note that the number of degrees of freedom is preserved under these transformations. In the Goldstone case, this is trivial, since massless and massive scalar fields each have one degree of freedom: $2 = 1 + 1$. In the Higgs case, a massless photon has two degrees of freedom, but a massive one has three, since it has a physical longitudinal polarisation state: $2 + 2 = 1 + 3$. In a manner of speaking, we can say that the photon has eaten a scalar field and acquired a mass; or, more properly, we may compare the situation with the Gupta–Bleuler mechanism (§4.4). In that mechanism, the longitudinal and timelike components of the photon cancel each other, leaving the two transverse components. Here the timelike component of the photon is cancelled by the scalar field, leaving three polarisation states for the photon, rendering it massive.

It should be remarked that although these conclusions about the *particle spectrum* were reached by considering the Lagrangian of (8.39), there is nothing otherwise special about this particular form. In fact, when we come to consider *renormalisation*, it is much more convenient (and, of course, physically equivalent) to consider (8.37). The gauge defined by (8.39) is called the *physical* or *unitary gauge* (or *U gauge*), since in this gauge, only physical particles (i.e. those which would appear in the unitarity condition) appear.

We now turn to the non-Abelian Higgs phenomenon (Kibble 1967) and for definiteness consider the $O(3)$ model discussed in the previous section. The Lagrangian (8.12), then, needs to be modified by substituting a covariant derivative for the ordinary one, and adding the gauge field term. This gives

$$\mathcal{L} = \tfrac{1}{2}(D_\mu\phi_i)(D^\mu\phi_i) - \frac{m^2}{2}\phi_i\phi_i - \lambda(\phi_i\phi_i)^2 - \tfrac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}. \tag{8.42}$$

where, from (3.122) and (3.131) (making no distinction between upper and lower internal indices, so that $\varepsilon^{ijk} = \varepsilon_{ijk}$),

$$\begin{aligned} D_\mu\phi_i &= \partial_\mu\phi_i + g\varepsilon_{ijk}A^j_\mu\phi_k \\ F^i_{\mu\nu} &= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon^{ijk}A^j_\mu A^k_\nu. \end{aligned} \tag{8.43}$$

The potential V has a minimum (for $m^2 < 0$) at

$$|\phi_0| = \left(\frac{-m^2}{4\lambda}\right)^{1/2} = a \tag{8.15}$$

and, as before, we choose the vacuum which points in the 3 direction

$$\vec{\phi}_0 = a\hat{e}_3. \tag{8.16}$$

The physical fields are then ϕ_1, ϕ_2 and $\chi = \phi_3 - a$. After some algebra, it follows easily that

$$\begin{aligned} \mathcal{L} = {} & \tfrac{1}{2}[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 + (\partial_\mu\chi)^2] + ag[(\partial_\mu\phi_1)A^\mu_2 - (\partial_\mu\phi_2)A^\mu_1] \\ & + \frac{a^2g^2}{2}[(A^1_\mu)^2 + (A^2_\mu)^2] - \tfrac{1}{4}(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu)^2 \\ & - 4a^2\lambda\chi^2 + \text{cubic} + \text{quartic terms}. \end{aligned} \tag{8.44}$$

We have shown explicitly only the terms quadratic in the fields, since for the present purpose they are the only important ones. This Lagrangian is analogous to (8.37); it contains a mixed term in A^μ and ϕ, so is not easy to interpret. To obtain a more 'physical' Lagrangian, we may now make use of the fact that we have a *local* symmetry, so may perform independent gauge transformations at each point in space-time. We therefore select a gauge – the unitary gauge – so that at *every* point in space-time $\vec{\phi}$ lies along the third isospin axis:

$$\vec{\phi}(x) = \hat{e}_3\phi_3 = \hat{e}_3(a + \chi). \tag{8.45}$$

This gets rid of the fields ϕ_1 and ϕ_2, and we have

$$\begin{aligned} D_\mu\phi_1 &= g(a + \chi)A^2_\mu, \\ D_\mu\phi_2 &= -g(a + \chi)A^1_\mu, \\ D_\mu\phi_3 &= \partial_\mu\chi, \end{aligned}$$

so that

$$(D_\mu \phi_i)^2 = a^2 g^2[(A^1_\mu)^2 + (A^2_\mu)^2] + (\partial_\mu \chi)^2 + \cdots$$

and

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu)^2 - \tfrac{1}{2}a^2 g^2[(A^1_\mu)^2 + (A^2_\mu)^2] + \tfrac{1}{2}(\partial_\mu \chi)^2 - 4a^2\lambda\chi^2 + \text{cubic} + \text{quartic terms.} \qquad (8.46)$$

The remaining particles, then, are one massive scalar, two massive vectors and one massless vector particle. In particular, the Goldstone bosons, present in the spontaneously broken *global* symmetry model, have both disappeared in the *local* symmetry model, and two of the massless gauge fields have become massive. Thus, analogously to (8.40) and (8.41), we may summarise the results for spontaneous breaking of an $O(3)$ symmetric model as follows:

Goldstone mode (global $O(3)$ symmetry):
3 massive scalar fields $\rightarrow$ 1 massive scalar field
+ 2 massless scalar fields. (8.47)

Higgs mode (local $O(3)$ symmetry):

$$\left.\begin{matrix}\text{3 massive scalar fields}\\ \\ \text{+ 3 massless vector fields}\end{matrix}\right\} \rightarrow \left\{\begin{matrix}\text{1 massive scalar field}\\ \text{+ 2 massive vector fields}\\ \text{+ 1 massless vector field.}\end{matrix}\right. \qquad (8.48)$$

We may also check that the number of independent modes is preserved: in the Higgs case $3 + 3 \times 2 = 9 = 1 + 2 \times 3 + 2$.

This $O(3)$ model contains all the features of the general non-Abelian case. It should be clear that one massless vector field remains because the subgroup H ($= U(1)$) under which the vacuum remains invariant has one generator – it was this circumstance that allowed one scalar field, in the Goldstone case, to remain massive. Thus, the number of massless vector fields is $\dim H$. And on the other hand, the two vector fields which have become massive have done so by absorbing the two Goldstone modes; so the number of massive vector fields is $\dim G/H$. Thus the total number of gauge particles (massive or massless) is $\dim G$, as expected, since the gauge field transforms according to the regular representation of the group. The fact that in the model above, there is also a scalar field remaining is because we chose the scalar fields to belong to an isotriplet. We see again that the outcome of the Higgs mechanism is dictated largely by group theory.

8.4 Superconductivity

Superconductivity provides a nice illustration of the Abelian Higgs model. As everyone knows, superconductivity is the phenomenon, shown by many metals, of having no resistance at very low temperatures. Such metals are therefore capable of carrying persistent currents. These currents effectively screen out magnetic flux, which is therefore zero in a superconductor (the Meissner effect). Another way of stating the Meissner effect is to say that the photons are effectively massive, as in the Higgs phenomenon discussed above. We shall show very briefly how these conclusions follow from the Lagrangian (8.36).

To begin, we consider a *static* situation, so $\partial_0 \phi = 0$, etc., and (8.36) takes the form

$$\mathscr{L} = -(\boldsymbol{\nabla} - \mathrm{i}e\mathbf{A})\phi \cdot (\boldsymbol{\nabla} + \mathrm{i}e\mathbf{A})\phi^* - m^2|\phi|^2 - \lambda|\phi|^4 - \tfrac{1}{2}(\boldsymbol{\nabla} \times \mathbf{A})^2$$

or

$$-\mathscr{L} = \tfrac{1}{2}(\boldsymbol{\nabla} \times \mathbf{A})^2 + |(\boldsymbol{\nabla} - \mathrm{i}e\mathbf{A})\phi|^2 + m^2|\phi|^2 + \lambda|\phi|^4. \tag{8.49}$$

Now $-\mathscr{L}$ is the *Landau–Ginzburg free energy*, where $m^2 = a(T - T_c)$ near the critical temperature $T = T_c$; ϕ is the macroscopic many-particle wave function, and its use is justified by the Bardeen–Cooper–Schrieffer (BCS) theory, according to which, under certain conditions, there is an *attractive* force between electrons, and the field quanta are electron pairs, which, of course, are bosons. At low temperatures, these fall into the same quantum state (Bose–Einstein condensation), and because of this, a many-particle wave function ϕ may be used to describe the macroscopic system. Now, at $T > T_c$, $m^2 > 0$ and the minimum free energy is at $|\phi| = 0$. But when $T < T_c$, $m^2 < 0$ and the minimum free energy is at

$$|\phi|^2 = -\frac{m^2}{2\lambda} > 0; \tag{8.50}$$

which is, of course, an example of spontaneous symmetry breaking. Now $\mathscr{L}$ is invariant under the usual phase transformation

$$\phi \to \mathrm{e}^{\mathrm{i}\Lambda(x)}\phi, \quad \mathbf{A} \to \mathbf{A} + \frac{1}{e}\boldsymbol{\nabla}\Lambda(x)$$

and the associated conserved current is

$$\mathbf{j} = -\mathrm{i}(\phi^* \boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\phi^*) - 2e|\phi|^2\mathbf{A}. \tag{8.51}$$

When $T < T_c$, and ϕ varies only very slightly over the sample, the second of these terms dominates over the first, and

$$\mathbf{j} = \frac{em^2}{\lambda}\mathbf{A} = -k^2\mathbf{A} \tag{8.52}$$

where k is a positive constant. This is the *London equation*. The electric field $\mathbf{E} = -\partial\mathbf{A}/\partial t = 0$, and Ohm's law defines resistance by $\mathbf{E} = R\mathbf{j}$, so

$$R = 0$$

and we have superconductivity.

The Meissner effect (expulsion of magnetic flux) is easily derived. Ampère's equation is

$$\nabla \times \mathbf{B} = \mathbf{j}.$$

Taking its curl, remembering that $\nabla \cdot \mathbf{B} = 0$, and using (8.52) gives

$$\nabla^2 \mathbf{B} = k^2 \mathbf{B}. \tag{8.53}$$

Confining ourselves for simplicity to one spatial dimension, (8.53) has the solution

$$B_x = B_0 \, \mathrm{e}^{-kx},$$

so that magnetic field only penetrates the specimen to a characteristic depth $1/k$. When the numerical factors are properly taken into account, we get $1/k \approx 10^{-6}$ cm. Finally, it is clear that (8.53) implies that $\nabla^2 \mathbf{A} = k^2 \mathbf{A}$, or, in a Lorentz covariant form,

$$\Box A_\mu = -k^2 A_\mu$$

which means the 'photons' have a mass k, which is the characteristic feature of the Higgs phenomenon.

Following this application of the Higgs phenomenon to a low energy example, we now describe its application to weak interactions.

8.5 The Weinberg–Salam model

Spontaneous breaking of gauge symmetries was the crucial new ingredient in the model of unified weak and electromagnetic interactions constructed independently by Weinberg and Salam. The general idea was that weak interactions should be mediated by gauge bosons ($W^\pm$), which are, 'to begin with', massless. The Lagrangian for the theory also contains terms for *massless* electrons, muons and neutrinos, and is invariant under an internal symmetry group, which is a gauge symmetry. A scalar field (the Higgs field) is then introduced with a non-vanishing vacuum-expectation-value. The resulting spontaneous breakdown of symmetry gives masses to e, μ and τ and to the gauge bosons, but *not* to the photon and neutrino. It is therefore 'realistic', and has indeed met with a good degree of success in describing weak interactions. The model may also be extended to hadrons, but this will not be described here.

We start with the spinor fields. The Dirac Lagrangian

$$\mathcal{L} = \mathrm{i}\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

becomes simply $\mathrm{i}\bar{\psi}\gamma \cdot \partial\psi$ if $m = 0$. Writing, as in Chapter 2,

$$\psi_\mathrm{L} = \left(\frac{1-\gamma_5}{2}\right)\psi, \quad \psi_\mathrm{R} = \left(\frac{1+\gamma_5}{2}\right)\psi$$

it follows that

$$\mathrm{i}\bar{\psi}\gamma\cdot\partial\psi = \mathrm{i}\bar{\psi}_{\mathrm{R}}\gamma\cdot\partial\psi_{\mathrm{R}} + \mathrm{i}\bar{\psi}_{\mathrm{L}}\gamma\cdot\partial\psi_{\mathrm{L}}$$

since γ_5 anticommutes with γ_μ. The electron and muon have L and R components, but according to the 2-component neutrino theory, ν_e and ν_μ have L components only, so the lepton Lagrangian is

$$\mathscr{L} = \mathrm{i}\bar{e}_{\mathrm{R}}\gamma\cdot\partial e_{\mathrm{R}} + \mathrm{i}\bar{e}_{\mathrm{L}}\gamma\cdot\partial e_{\mathrm{L}} + \mathrm{i}\bar{\nu}_e\gamma\cdot\partial\nu_e + (e \to \mu). \tag{8.54}$$

It is clear that terms for the τ and its neutrino may also be added if desired; there would then be three 'generations' of leptons. From now on we shall forget about the μ and τ generations, which may trivially be added at any stage.

What internal symmetry does (8.54) possess? The transformations must be between particles whose space–time properties are the same, so the only possibility is a mixing of e_{L} and ν_e. We therefore write the 'isospinor'

$$L = \begin{pmatrix} \nu_e \\ e_{\mathrm{L}} \end{pmatrix} \tag{8.55}$$

and assign to this doublet a non-Abelian charge $I_{\mathrm{w}} = \frac{1}{2}$($I_{\mathrm{w}}$ is 'weak isospin'). ν_e has a third component $I^3_{\mathrm{w}} = \frac{1}{2}$, and e_{L} has $I^3_{\mathrm{w}} = -\frac{1}{2}$. This is obviously in straight analogy with (strong) isospin. The remaining particle

$$R = e_{\mathrm{R}} \tag{8.56}$$

is an isosinglet: $I_{\mathrm{w}} = 0$. We have

$$\mathscr{L} = \mathrm{i}\bar{R}\gamma\cdot\partial R + \mathrm{i}\bar{L}\gamma\cdot\partial L \tag{8.57}$$

and $\mathscr{L}$ is invariant under

$$\left.\begin{aligned} L &\to \mathrm{e}^{-(\mathrm{i}/2)\boldsymbol{\tau}\cdot\boldsymbol{\alpha}}L, \\ R &\to R, \end{aligned}\right\} \tag{8.58}$$

which are rotations in weak isospin space. They generate the group $SU(2)$. More explicitly, the transformations are

$$SU(2)\colon \begin{pmatrix} \nu_e \\ e_{\mathrm{L}} \\ e_{\mathrm{R}} \end{pmatrix} \to \left(\begin{array}{cc:c} \multicolumn{2}{c:}{\mathrm{e}^{-(\mathrm{i}/2)\boldsymbol{\tau}\cdot\boldsymbol{\alpha}}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & 1 \end{array}\right) \begin{pmatrix} \nu_e \\ e_{\mathrm{L}} \\ e_{\mathrm{R}} \end{pmatrix}. \tag{8.59}$$

The relation between electric charge Q, and I^3_{w} is

$$L\colon Q = I^3_{\mathrm{w}} - \tfrac{1}{2};\quad R\colon Q = I^3_{\mathrm{w}} - 1. \tag{8.60}$$

When we come to gauge this symmetry (i.e. make $\boldsymbol{\alpha}$ a space–time function) we will acquire three massless gauge fields. The photon, however, will *not* be one of them, since e_{R}, being a singlet, will not interact with our gauge fields, but does interact with the photon.

Note, however, that $SU(2)$ is not the maximal symmetry of $\mathscr{L}$. We could also have a simple $U(1)$ transformation on e_R

$$U(1)\colon e_R \to e^{i\beta} e_R. \tag{8.61}$$

How will this affect L? It can only be by an *overall* phase; in other words, ν_e and e_L must pick up the *same* phase (as each other), since otherwise it would be a special case of an $SU(2)$ transformation. This phase, however, is not necessarily the same as that of R. We write, therefore,

$$U(1)\colon \begin{pmatrix} \nu_e \\ e_L \\ e_R \end{pmatrix} \to \begin{pmatrix} e^{in\beta} & 0 & 0 \\ 0 & e^{in\beta} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} \nu_e \\ e_L \\ e_R \end{pmatrix} \tag{8.62}$$

where n is a number, which we must now find. This $U(1)$ symmetry leads to a conserved charge, of which e_R possesses one value, and ν_e and e_L another value. It is clearly not Q, since ν_e and e_L have different values of Q. (In other words, the gauge field we get on gauging $U(1)$ is also not the photon field.)

Weinberg suggested that this charge is 'weak hypercharge' Y_w defined by a quasi-Gell-Mann–Nishijima relation

$$Q = I_w^3 + \frac{Y_w}{2}. \tag{8.63}$$

Comparing with (8.60), it is clear that

$$\begin{aligned} L \text{ has } Y_w &= -1, \\ R \text{ has } Y_w &= -2, \end{aligned} \tag{8.64}$$

so, in (8.62), $n = \frac{1}{2}$; the left-handed fields couple, with half the strength of the right-handed field, to the hypercharge gauge field. The $U(1)$ transformation is then

$$U(1)\colon \begin{pmatrix} \nu_e \\ e_L \\ e_R \end{pmatrix} \to \begin{pmatrix} e^{i\beta/2} & 0 & 0 \\ 0 & e^{i\beta/2} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix} \begin{pmatrix} \nu_e \\ e_L \\ e_R \end{pmatrix}. \tag{8.65}$$

The Lagrangian (8.54) is then invariant under $SU(2) \otimes U(1)$. (Note that the $U(1)$ could be identified with lepton number, giving $n = 1$. This would then result in a different theory.)

We now gauge the theory. Gauging $SU(2)$ means that we introduce three gauge potentials W_μ^i so that, acting on the isospinor L, the ordinary derivative is replaced by the covariant derivative (3.153):

$$D_\mu L = \partial_\mu L - \frac{i}{2} g \boldsymbol{\tau} \cdot \mathbf{W}_\mu L. \tag{8.66}$$

Here g is the $SU(2)$ coupling constant. Gauging $U(1)$ introduces another potential X_μ and coupling constant g', and, from (8.65), since L has half the hypercharge of R, the covariant derivatives are

$$\left.\begin{aligned} D_\mu L &= \partial_\mu L + \frac{\mathrm{i}}{2}g' X_\mu L, \\ D_\mu R &= \partial_\mu R + \mathrm{i}g' X_\mu R. \end{aligned}\right\} \tag{8.67}$$

Putting (8.66) and (8.67) into (8.57), and adding the gauge-field terms (see (3.131)) gives the Lagrangian

$$\mathscr{L}_1 = \mathrm{i}\bar{R}\gamma^\mu(\partial_\mu + \mathrm{i}g' X_\mu)R + \mathrm{i}\bar{L}\gamma^\mu\left(\partial_\mu + \frac{\mathrm{i}}{2}g' X_\mu - \frac{\mathrm{i}}{2}g\boldsymbol{\tau}\cdot\mathbf{W}_\mu\right)L$$
$$- \tfrac{1}{4}(\partial_\mu \mathbf{W}_\nu - \partial_\nu \mathbf{W}_\mu + g\mathbf{W}_\mu \times \mathbf{W}_\nu)^2 - \tfrac{1}{4}(\partial_\mu X_\nu - \partial_\nu X_\mu)^2. \tag{8.68}$$

Next, we introduce an isospinor scalar field (the Higgs field),

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{8.69}$$

From (8.63), it carries the quantum numbers

$$\phi\colon I_\mathrm{w} = \tfrac{1}{2}, \quad Y_\mathrm{w} = 1 \tag{8.70}$$

so that both ϕ^+ and ϕ^0 are *complex* fields (the particle and antiparticle are distinct), and we may put

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_3 + \mathrm{i}\phi_4) \\ \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2) \end{pmatrix} \tag{8.71}$$

where $\phi_1, \ldots, \phi_4$ are real. In addition, by virtue of (8.70), the covariant derivative of ϕ is

$$D_\mu\phi = \left(\partial_\mu - \frac{\mathrm{i}}{2}g\boldsymbol{\tau}\cdot\mathbf{W}_\mu - \frac{\mathrm{i}}{2}g' X_\mu\right)\phi. \tag{8.72}$$

The Higgs field ϕ also interacts with e^- and ν_e with strength G_e, so the overall Lagrangian containing ϕ is

$$\mathscr{L}_2 = (D_\mu\phi)^\dagger(D_\mu\phi) - \frac{m^2}{2}\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 - G_e(\bar{L}\phi R + \bar{R}\phi^\dagger L). \tag{8.73}$$

The interaction term in $\mathscr{L}_2$, written out fully, is

$$-G_e(\bar{\nu}_e e_\mathrm{R}\phi^+ + \bar{e}_\mathrm{L} e_\mathrm{R}\phi^0 + \bar{e}_\mathrm{R}\nu_e\phi^- + \bar{e}_\mathrm{R} e_\mathrm{L}\overline{\phi^0}), \tag{8.74}$$

and

$$\phi^\dagger\phi = (\phi^+)^*\phi^+ + (\phi^0)^*\phi^0 = \tfrac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2). \tag{8.75}$$

As usual, in the case $m^2 > 0$, this describes a scalar field with mass m, and lowest energy state corresponding to $\phi = 0$. But if $m^2 < 0$, the lowest energy state is not at $\phi = 0$, but at

$$(\phi^\dagger \phi)_0 = -\frac{m^2}{\lambda}. \tag{8.76}$$

We choose the isospin frame so that

$$(\phi_1^2)_0 = -\frac{2m^2}{\lambda}, \quad (\phi_2)_0 = (\phi_3)_0 = (\phi_4)_0 = 0,$$

or

$$(\phi_1)_0 = \left(-\frac{2m^2}{\lambda}\right)^{1/2} \equiv \sqrt{2}\eta \tag{8.77}$$

and

$$(\phi)_0 = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \quad \eta \text{ real}. \tag{8.78}$$

We now have a degenerate vacuum, and spontaneous breaking of the gauge symmetry. For excitations of ϕ above the vacuum, we might expect to have

$$\phi(x) = \begin{pmatrix} \dfrac{\phi_3(x)}{\sqrt{2}} + \mathrm{i}\dfrac{\phi_4(x)}{\sqrt{2}} \\ \eta + \dfrac{\phi_1(x)}{\sqrt{2}} + \mathrm{i}\dfrac{\phi_2(x)}{\sqrt{2}} \end{pmatrix}$$

but this is not the case. The fact that the symmetry is *local* means that we may perform a *different* isospin rotation at each point in space, so $\phi(x)$ may be reduced to the form

$$\phi(x) = \begin{pmatrix} 0 \\ \eta + \dfrac{\sigma(x)}{\sqrt{2}} \end{pmatrix} \tag{8.79}$$

at *every* point. (This is the same argument that led to (8.45).) From (8.72), $D_\mu \phi$ is then

$$D_\mu \phi = \begin{pmatrix} 0 \\ \dfrac{1}{\sqrt{2}}\partial_\mu \sigma \end{pmatrix} - \left[\frac{\mathrm{i}g}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - \mathrm{i}W_\mu^2 \\ W_\mu^1 + \mathrm{i}W_\mu^2 & -W_\mu^3 \end{pmatrix} + \frac{\mathrm{i}g'}{2}X_\mu\right]$$
$$\times \begin{pmatrix} 0 \\ \eta + \dfrac{\sigma}{\sqrt{2}} \end{pmatrix}$$

$$= -\frac{i}{2}\begin{pmatrix} g\eta(W_\mu^1 - iW_\mu^2) + \frac{g\sigma}{\sqrt{2}}(W_\mu^1 - iW_\mu^2) \\ i\sqrt{2}\partial_\mu\sigma + \eta(-gW_\mu^3 + g'X_\mu) + \frac{\sigma}{\sqrt{2}}(-gW_\mu^3 + g'X_\mu) \end{pmatrix}.$$

Hence

$$(D_\mu\phi)^\dagger(D_\mu\phi) = \tfrac{1}{2}(\partial_\mu\sigma)^2 + \frac{g^2\eta^2}{4}[(W_\mu^1)^2 + (W_\mu^2)^2] + \frac{\eta^2}{4}(gW_\mu^3 - g'X_\mu)^2 + \text{cubic} + \text{quartic terms.} \tag{8.80}$$

Now we define

$$Z_\mu = \frac{gW_\mu^3 - g'X_\mu}{(g^2 + g'^2)^{1/2}} \equiv \cos\theta_W W_\mu^3 - \sin\theta_W X_\mu \tag{8.81}$$

and the orthogonal field

$$A_\mu = \frac{g'W_\mu^3 + gX_\mu}{(g^2 + g'^2)^{1/2}} = \sin\theta_W W_\mu^3 + \cos\theta_W X_\mu \tag{8.82}$$

where the 'Weinberg angle' θ_W is given by

$$\frac{g}{(g^2 + g'^2)^{1/2}} = \cos\theta_W, \quad \frac{g'}{g} = \tan\theta_W. \tag{8.83}$$

We see, from (8.80), that W_μ^1, W_μ^2 and Z_μ pick up masses

$$M_{W_1}^2 = M_{W_2}^2 = \frac{g^2\eta^2}{2}, \quad M_Z^2 = \frac{g^2\eta^2}{2\cos^2\theta_W} = \frac{M_W^2}{\cos^2\theta_W} \tag{8.84}$$

and A_μ is massless. A_μ is therefore provisionally identified with the electromagnetic field. The identification may be justified by noting that the lepton gauge-field coupling is, from (8.68) and (8.81)–(8.83),

$$\begin{aligned} &i\bar{R}\gamma^\mu(\partial_\mu + ig'X_\mu)R + i\bar{L}\gamma^\mu\left(\partial_\mu + \frac{i}{2}g'X_\mu - \frac{i}{2}g\boldsymbol{\tau}\cdot\mathbf{W}_\mu\right)L \\ &= i\bar{e}\gamma^\mu\partial_\mu e + i\bar{\nu}\gamma^\mu\partial_\mu\nu - g\sin\theta_W\bar{e}\gamma^\mu eA_\mu \\ &\quad + \frac{g}{\cos\theta_W}(\sin^2\theta_W\bar{e}_R\gamma^\mu e_R - \tfrac{1}{2}\cos 2\theta_W\bar{e}_L\gamma^\mu e_L + \tfrac{1}{2}\bar{\nu}\gamma^\mu\nu)Z_\mu \\ &\quad + \frac{g}{\sqrt{2}}[(\bar{\nu}\gamma^\mu e_L W_\mu^\dagger) + \text{h.c.}] \end{aligned} \tag{8.85}$$

where h.c. stands for Hermitian conjugate, and $W_\mu = (W_\mu^1 + iW_\mu^2)/\sqrt{2}$. Note that the A_μ field couples *only* to the electrons, and not to the neutrinos, and that it couples to left and right components equally strongly. We are thus justified in identifying A_μ with the electromagnetic potential, and it follows immediately that the pertinent coupling constant should be e, the proton charge:

■ $$e = g \sin \theta_W. \tag{8.86}$$

From the last term in (8.85) g is the coupling of the weak field to the electron–neutrino (and muon–neutrino) current. To second order, therefore, this interaction will account for muon decay,

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu,$$

through the diagram of Fig. 8.6, in which the gauge field W is propagated between the two vertices. From (8.85), then, the effective interaction is

$$\begin{aligned} H_{\text{int}} &= \frac{g^2}{2}\bar{\nu}_\mu \gamma^\kappa \mu_L (\text{Prop})_{\kappa\lambda} \bar{e}_L \gamma^\lambda \nu_e \\ &= \frac{g^2}{8}\bar{\nu}_\mu \gamma^\kappa (1 - \gamma_5)\mu (\text{Prop})_{\kappa\lambda} \bar{e} \gamma^\lambda (1 - \gamma_5)\nu_e. \end{aligned} \tag{8.87}$$

At low q, however, the propagator simply becomes $g_{\kappa\lambda}/M_W^2$, so (8.87) becomes

$$H_{\text{int}} = \frac{g^2}{8M_W^2} j_\mu^{\lambda\dagger} j_{e\lambda} \tag{8.88}$$

where

$$j_{l\lambda} = \bar{l}\gamma_\lambda (1 - \gamma_5)\nu_l$$

is the lepton (l) current. (8.88) is, however, exactly the Fermi current–current interaction

$$H_{\text{int}} = \frac{G}{\sqrt{2}} j_\mu^{\lambda\dagger} j_{e\lambda} \tag{8.89}$$

where G is the Fermi constant. Hence we have the equality

$$G = \frac{g^2}{4\sqrt{2}M_W^2}. \tag{8.90}$$

From muon decay (see, for example, Lee 1981, ch. 21), G has the value

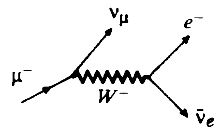

Fig. 8.6. Muon decay in an intermediate vector boson theory.

$G = 1.43 \times 10^{-49}$ erg cm^3, which in 'natural' units $\hbar = c = 1$ is $G \sim 10^{-5}\, m_p^{-2}$, a 'small' number (if the proton mass sets the scale). It is because G is small that the weak interactions are called weak. In the Weinberg–Salam theory, however, the fundamental coupling constant is g, and, from (8.86), if θ_W is not too small, neither is g. The weak interaction is then not *intrinsically* weak, but, from (8.90), only *appears* weak because M_W is large. In fact, numbers may be put to these quantities. From (8.85) both the neutrinos and charged leptons (e and μ) couple to the *neutral* field Z, so we expect, for example, to observe $\nu_\mu - e$ scattering:

$$\nu_\mu + e^- \rightarrow \nu_\mu + e^-,$$

through the Feynman diagram of Fig. 8.7. This is an example of a 'neutral current' interaction typically predicted by the Weinberg–Salam theory. This scattering has been observed, with a cross section $\sigma = (1.6 \pm 0.9) \times 10^{-42}$ (E_ν/GeV) cm^2 (see, for example, Lee 1981, p. 599). It is clear that this cross section gives a value for θ_W (or, equivalently, for g). This is (Lee 1981, p. 686)

$$\sin^2 \theta_W = 0.225 \begin{matrix} + 0.06 \\ - 0.05 \end{matrix}. \tag{8.91}$$

Equations (8.86) and (8.90) then yield

$$M_W^2 = \frac{e^2}{4\sqrt{2} G \sin^2 \theta_W} = (78.6\ \mathrm{GeV}/c^2)^2 \tag{8.92}$$

and (8.84) gives for the Z mass

$$M_Z = 89.3\ \mathrm{GeV}/c^2. \tag{8.93}$$

The existence of the charged and neutral vector bosons with the above masses is a crucial prediction of the electroweak theory, so the verification is all the more spectacular since these particles have now been found (UA1 Collaboration, *Physics Letters*, **122B**, 103; **126B**, 398 (1983)) at the correct mass. It is worth pointing out that neutral current processes (also observed) follow from any theory whose Lagrangian is invariant under a *global* $SU(2)$ symmetry. It is only when the theory is *gauged* that it becomes renormalisable and that intermediate vector bosons are predicted. For further details of the electroweak theory and its experimental implications, the reader is referred elsewhere.

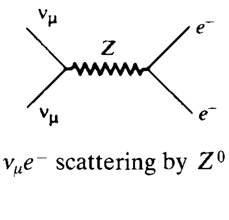

Fig. 8.7. $\nu_\mu e^-$ scattering by Z^0 exchange.

Finally, an interesting speculation concerns the high temperature behaviour of the electroweak interaction. If the analogy with ferromagnetism holds good, the symmetry, which is spontaneously broken at low temperatures should be restored at high temperatures, and above the critical temperature T_c the W and Z bosons, like the photon, would be massless, and the weak force would become long range like electromagnetism. Kirzhnits & Linde (1972) have speculated that this happens, with $T_c \sim 10^{16}$ K, and more formal calculations seem to vindicate this idea. It is based on the prescription that the finite temperature scalar field 2-point function (for example) is obtained by taking the Gibbs average of the 'ordinary' 2-point function:

$$i\Delta_{F\beta}(x - y) = \frac{\mathrm{Tr}\, e^{-\beta H} \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle}{\mathrm{Tr}\, e^{-\beta H}} \tag{8.94}$$

where $\beta = (kT)^{-1}$. If this hypothesis is correct, it will have cosmological implications, since, on the standard 'big bang' model, there was certainly a time when the temperature of the universe was higher than T_c. One interesting question is, when T cools below T_c, how does the spontaneous breaking set in? In a ferromagnet, for example, the direction of spontaneous magnetisation characterises not the whole sample, but only a *domain*, and domain walls separate regions magnetised in different directions. Does this happen in the universe? Another interesting question is whether, when $T > T_c$, the long-range weak force leads to large-scale repulsive effects between particles with the same Y_w and I_w^3 (just as it does between particles with the same Q in electromagnetism). For these and other related questions we refer the reader to the literature (Linde 1979).

Summary

[1]Ferromagnetism is shown to illustrate the phenomenon of spontaneous symmetry breaking (SSB), and a model scalar field theory is exhibited with SSB, which results in the presence of a massless particle, the Goldstone boson. [2]In the general case, where the Lagrangian is invariant under a symmetry group G, but the ground state (vacuum) is only invariant under a subgroup H, the number of Goldstone particles is shown to be $\dim G/H$. It is also shown that the above (classical) arguments also hold in quantum theory: that spontaneous symmetry breaking implies the existence of massless particles. This is the Goldstone theorem. [3]SSB of gauge theories, on the other hand, exhibits quite different phenomena: there are no Goldstone particles, and some (or all) gauge fields become massive. This is shown in Abelian and non-Abelian models. [4]It is shown that superconductivity is a theory with SSB of electromagnetism, an Abelian gauge theory, and the [5]Weinberg–Salam model of unified weak and electromagnetic interactions is described. It exhibits SSB of an $SU(2) \otimes U(1)$ gauge symmetry. Some experimental implications are discussed.

Guide to further reading

The possibility of an analogy between the ground state in many-body physics and the vacuum in quantum field theory was first entertained by Nambu (1960); Nambu & Jona-Lasinio (1961), and is reviewed by Nambu, in Gürsey (1964). Goldstone's model appeared in Goldstone (1961) and is proved in Goldstone, Salam & Weinberg (1962); see also Weinberg, in Deser, Grisaru & Pendleton (1970). Good discussions of the Goldstone theorem appear in Bernstein (1974), Guralnik, Hagen & Kibble in Cool & Marshak (1968), Lee (1981, chs. 16, 22, 24), Taylor (1976, ch. 5), O'Raigheartaigh (1979). For the Higgs phenomenon, see Higgs (1964a, b; 1966), Englert & Brout (1964), Guralnik, Hagen & Kibble (1964). Good reviews are to be found in Bernstein (1974), Guralnik, Hagen & Kibble in Cool & Marshak (1968), Lee (1981), Taylor (1976), O'Raigheartaigh (1979) and S. Coleman, in Zichichi (1975). For the non-Abelian case, see Kibble (1967), also Guralnik, Hagen & Kibble in Cool & Marshak (1968), Taylor (1976) and O'Raigheartaigh (1979). For readable accounts of superconductivity, see, for example, Tilley & Tilley (1974), de Gennes (1966). A good review of the Ginzburg–Landau theory is Cyrot (1973). Good accounts of superconductivity for particle physicists are Kirzhnits (1978), Aitchison & Hey (1982, ch. 9). The electroweak unified theory originates in Weinberg (1967), A. Salam in Svartholm (1968); see also Glashow (1961). For reviews, see Bernstein (1974), Lee (1981), Taylor (1976), O'Raigheartaigh (1979), and the 1979 Nobel prize acceptance speeches; Weinberg (1980), Salam (in Svartholm 1968, p. 525), Glashow (1961, p. 539). Up-to-date reviews, particularly of experimental data, are to be found in conference proceedings. The following papers deal with symmetry restoration at high temperatures: Kirzhnits & Linde (1972), Weinberg (1974), Dolan & Jackiw (1974). For a review, see Linde (1979). For finite temperature Green's functions, see Kirzhnits & Linde (1972), Weinberg (1974), Shuryak (1980), S.W. Hawking, in Hawking & Israel (1979, pp. 754–5) and S. Weinberg in Zichichi (1978).