$$\exp\left(\frac{\mathrm{i}}{\hbar}\hat{\theta}_n\right)(\hat{N}_n)^m \exp\left(-\frac{\mathrm{i}}{\hbar}\hat{\theta}_n\right) = \left[\exp\left(\frac{\mathrm{i}}{\hbar}\hat{\theta}_n\right)\hat{N}_n \exp\left(-\frac{\mathrm{i}}{\hbar}\hat{\theta}_n\right)\right]^m$$
$$= (\hat{N}_n + 1)^m \tag{1.2.25}$$

holds. For a general function $g(N)$ we obtain

$$\exp\left(\frac{\mathrm{i}}{\hbar}\hat{\theta}_n\right) g(\hat{N}_n) \exp\left(-\frac{\mathrm{i}}{\hbar}\hat{\theta}_n\right) = g(\hat{N}_n + 1) \ . \tag{1.2.26}$$

This means that $\hat{U} = \exp(\frac{\mathrm{i}}{\hbar}\hat{\theta}_n)$ is a linear operator acting on $\hat{N}_n$, just like $\hat{U}(\boldsymbol{a})$ in (1.1.60). If $\hat{\theta}_n$ were Hermitian, then $\hat{U}$ would be a unitary operator with $\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = 1$. However, this identity is not true. This can be seen from (1.2.26): $\hat{U}^\dagger$ increases $\hat{N}_n$ by one, $\hat{U}$ decreases $\hat{N}_n$ by one. Acting with $\hat{U}$ on the vacuum state with no particles $|N_n = 0\rangle$ we obtain $\hat{U}|N_n = 0\rangle = 0$. Acting on this equation with $\hat{U}^\dagger$ we obtain of course $\hat{U}^\dagger\hat{U}|N_n = 0\rangle = 0$. However, because of $\hat{U}\hat{U}^\dagger|N_n = 0\rangle = \hat{U}|N_n = 1\rangle = |N_n = 0\rangle \neq 0$ we have just demonstrated that $\hat{U}^\dagger\hat{U} \neq \hat{U}\hat{U}^\dagger$. Therefore, we conclude that because the particle number N_n is bounded from below, $\hat{\theta}$ is not Hermitian. However, when only states with $N_n \gg 1$ are considered, the existence of a lower bound can be neglected, and $\hat{\theta}$ can be regarded to be Hermitian.

Next, we deduce the Hamiltonian occurring after second quantization. Continuing in a heuristic manner as above, we declare in (1.2.8) $\langle\hat{H}\rangle$ to be an operator again and write

$$\hat{H} = \sum_{n,m} \hat{A}_n^\dagger \langle\phi_n|\hat{H}_1|\phi_m\rangle \hat{A}_m \ . \tag{1.2.27}$$

Here, $\hat{H}_1$ is the single-particle Hamiltonian, being an operator in the sense that it acts on single-particle wave functions $\phi_n^*(\boldsymbol{r})$ and $\phi_m(\boldsymbol{r})$. $\hat{H}$ is an operator because $\hat{A}_n^\dagger$ and $\hat{A}_m$ are operators; however, $\langle\phi_n|\hat{H}_1|\phi_m\rangle$ is a simple complex number.

Equation (1.2.27) can also be expressed in terms of the field operators $\hat{\psi}^\dagger(\boldsymbol{r})$ and $\hat{\psi}(\boldsymbol{r})$:

$$\hat{H} = \int \mathrm{d}^3\boldsymbol{r}\, \hat{\psi}^\dagger(\boldsymbol{r})\hat{H}_1\hat{\psi}(\boldsymbol{r}) \ . \tag{1.2.28}$$

The Heisenberg equation of motion of $\hat{\psi}$ is given by

$$\mathrm{i}\hbar\frac{\partial\hat{\psi}(\boldsymbol{r},t)}{\partial t} = [\hat{\psi}(\boldsymbol{r},t), \hat{H}] = \hat{H}_1\hat{\psi}(\boldsymbol{r},t) \ . \tag{1.2.29}$$

If $\hat{\psi}(\boldsymbol{r})$ were a single-particle wave function, then this equation would be the Schrödinger equation (1.1.1); however, again we mention that $\hat{\psi}(\boldsymbol{r})$ is an operator, and the above equation describes the time evolution of this operator in the Heisenberg picture, which leads to a totally different meaning.