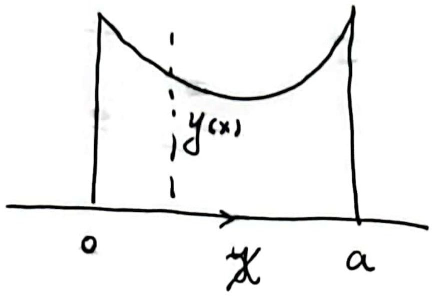
理论力学 ③

OIM 2023.9.

悬垂线问题:



$$ds = \sqrt{dx^2 + dy^2}$$
$$= \sqrt{1+y'^2}\, dx.$$

$$L_0 = \int_0^a ds = \int_0^a \sqrt{1+y'^2}\, dx = \text{Const}$$

Density ρ

$$U = \int_0^a \rho\, ds\, y = \rho \int_0^a \sqrt{1+y'^2}\, y\, dx$$

$$S = \int_0^a \left[\rho \sqrt{1+y'^2}\, y - \lambda \sqrt{1+y'^2}\right] dx = \int_0^a \mathcal{L}\, dx.$$
$$= \int_0^a \mathcal{L}(y, y')\, dx.$$

$$\boxed{\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = \frac{\partial \mathcal{L}}{\partial y}}$$

悬垂线问题

$$\mathcal{L} = \rho\sqrt{1+y'^2}\,y - \lambda\sqrt{1+y'^2}\,,\qquad \frac{d}{dz}\sqrt{1+z^2} = \frac{z}{\sqrt{1+z^2}}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial y'} = \dfrac{\rho y y'}{\sqrt{1+y'^2}} - \dfrac{\lambda y'}{\sqrt{1+y'^2}} = \dfrac{\rho y y' - \lambda y'}{\sqrt{1+y'^2}} \\[2ex] \dfrac{\partial \mathcal{L}}{\partial y} = \rho\sqrt{1+y'^2} \end{cases}$$

注意计算方法

$$\therefore \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = \frac{\rho y'^2 + \rho y y'' - \lambda y''}{\sqrt{1+y'^2}} - (\rho y y' - \lambda y')\frac{y' y''}{\left(\sqrt{1+y'^2}\right)^3}$$

$$= \rho\sqrt{1+y'^2}\,.$$

$$\therefore (\rho y'^2 + \rho y y'' - \lambda y'') - \frac{(\rho y y' - \lambda y')y' y''}{1+y'^2} = \rho(1+y'^2)\,.$$

以及

$$\boxed{(\rho y'^2 + \rho y y'' - \lambda y'')(1+y'^2) - (\rho y y' - \lambda y')y' y'' = \rho(1+y'^2)^2}$$

$$\therefore \cancel{\rho y'^2} + \rho y y'' - \lambda y'' + \cancel{\rho y'^4} + \rho y \cancel{y'' y'^2} - \lambda \cancel{y'' y'^2} - \rho y \cancel{y'^2} y'' + \lambda \cancel{y'^2} y''$$

$$= \rho + \cancel{2}\rho y'^2 + \cancel{\rho y'^4}$$

$$\therefore \boxed{\rho y y'' - \lambda y'' = \rho + \rho y'^2} \Rightarrow \boxed{y'' = \frac{\rho(1+y'^2)}{\rho y - \lambda}}$$

IF $\boxed{y = f + \frac{\lambda}{\rho}}$

$$\therefore \quad f'' = \frac{1+f'^2}{f}$$

$$\therefore \quad \boxed{f''f = 1 + f'^2}$$ → 数学中的重要方程..

其解为 $f = A_1 e^{\alpha x} + A_2 e^{-\alpha x}$.

$$\therefore \quad \alpha^2 [A_1 e^{\alpha x} + A_2 e^{-\alpha x}]^2$$
$$= 1 + \alpha^2 (A_1 e^{\alpha x} - A_2 e^{-\alpha x})^2$$

$$\therefore \quad 2\alpha^2 A_1 A_2 = 1 - 2\alpha^2 A_1 A_2$$

$$\therefore \quad \boxed{4\alpha^2 A_1 A_2 = 1}.$$

or

$$f = A\left[e^{\alpha x + \beta} + e^{-\alpha x - \beta}\right] = \frac{1}{2\alpha}\left[e^{\alpha x+\beta} + e^{-\alpha x-\beta}\right]$$

for any β.

$$\therefore \quad \alpha^2 A^2 [e^{\alpha x+\beta} + e^{-\alpha x-\beta}]^2$$
$$= 1 + \alpha^2 A^2 [e^{\alpha x+\beta} - e^{-\alpha x-\beta}]^2$$

$$\therefore \quad 4\alpha^2 A^2 = 1$$

$$\therefore \quad \boxed{A = \frac{1}{2\alpha}}$$

直观图像
猜测

$x \Rightarrow ix$

$y''y = 1 + y'^2$ 变成

$y'' \to -y''$

$y'^2 \to -y'^2$

$\therefore -y''y = 1 - y'^2$

$$\begin{cases} y \sim \cos(x+\varphi) \\ y'' \sim -\cos(x+\varphi) \\ y' \sim \sin(x+\varphi) \end{cases}$$

$-yy'' = \cos^2(x+\varphi)$

$1 - y'^2 = 1 - \sin^2(x+\varphi)$

$= \cos^2(x+\varphi).$

$\therefore \boxed{y \sim \cos(ix+\varphi)}$

$\boxed{\cos(ix) \sim \cosh(x)}$

$\therefore \boxed{y \sim \cosh(ax+\beta)}$

最小作用量原理的启示和总结

科学中的许多问题，都可以用微分方程描述。我们看到，牛顿方程有最小作用量原理，光有最小作用量原理，悬垂线有最小作用量原理，那么可以期待，其它几乎所有的运动，都可能有类似的最小作用量原理。所以，这个原理也就超越了物理学的应用范畴，有了更加广泛的应用。

的确，这个原理后来用在热力学、电磁学、量子力学、相对论等几乎所有方程中，也用在数理方程中。

许多老师讲理论力学，会花大量时间在这个原理和变分原理上。但 Landau 的书的特点是将它当做一个基本方法看待，不会花大的篇幅在这个方法上。不同的教材对这些问题的处理是不同的。

理论力学 ②

used

GM 2023. 9. 7.
(At home, 重复)

$\boxed{\text{守衡量}}$ ⇒ 原因 ⇒ Symmetry.

{ 动量守衡 ⇒ 空间平移不变性
能量 …… ⇒ 时间 ……
角动量 … ⇒ 转动不变性 } 理论物理看问题的角度

$\boxed{\forall \text{一个对称性} \Rightarrow \text{有一个对应的守衡量}}$

eg 平移不变性 $\theta \Rightarrow \theta + C$, C is const, $\mathcal{L}$ 不变

$$\frac{1}{2}g(\theta)\dot{\theta}^2 - U(\theta) = g(\theta+C)\dot{\theta}^2 - U(\theta+C)$$

$$\therefore \begin{cases} g(\theta) = g \\ U(\theta) = U \end{cases} \quad \text{const}$$

$$\left.\begin{aligned} \mathcal{L} &= \frac{1}{2} g \dot{\theta}^2 \\ \frac{\partial \mathcal{L}}{\partial \theta} &= 0 \end{aligned}\right\} \quad \frac{d}{dt}(\underbrace{g\dot{\theta}}_{\text{动量}}) = 0$$

$$\therefore g\dot{\theta} = \text{Const}$$

$\boxed{\text{下面推}\ \mathcal{L}\ \text{方程}}$.

① D'Alambert 原则 (2022年讲)

② 从欧氏空间 Lagrange eq (2023年讲).

Einstein 求和规则

$$\boxed{\sum_{i} x_i y_i = x_i y_i}$$

eg

$$\frac{\partial L}{\partial \dot{x}_i} = \sum_{\alpha} \frac{\partial L}{\partial \theta_\alpha}\frac{\partial \theta_\alpha}{\partial \dot{x}_i} + \frac{\partial L}{\partial \dot{\theta}_\alpha}\frac{\partial \dot{\theta}_\alpha}{\partial \dot{x}_i}$$

$$= \frac{\partial L}{\partial \theta_\alpha}\frac{\partial \theta_\alpha}{\partial \dot{x}_i} + \frac{\partial L}{\partial \dot{\theta}_\alpha}\frac{\partial \dot{\theta}_\alpha}{\partial \dot{x}_i}$$

Einstin Sum Rule，省略。

从 Lagrange eq 推出位形空间的 Lagrange eq.

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial x_i}}\ ,\qquad \theta_\alpha \text{ 和速度无关、}$$

令 $\theta_\alpha = \theta_\alpha(x_1, \dots, x_N)$，其中 $\alpha = 1, \dots, N-M = Q \leq N$.

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \cancel{\frac{\partial \mathcal{L}}{\partial \theta_\alpha}\frac{\partial \theta_\alpha}{\partial \dot{x}_i}} + \frac{\partial \mathcal{L}}{\partial \dot{\theta}_\alpha}\frac{\partial \dot{\theta}_\alpha}{\partial \dot{x}_i}$$

$$= \frac{\partial \mathcal{L}}{\partial \dot{\theta}_\alpha}\frac{\partial \dot{\theta}_\alpha}{\partial \dot{x}_i}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_i} = \dfrac{\partial \mathcal{L}}{\partial \theta_\alpha}\left(\dfrac{\partial \theta_\alpha}{\partial x_i}\right) + \underline{\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}_\alpha}\left(\dfrac{\partial \dot{\theta}_\alpha}{\partial x_i}\right)}. \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}_\alpha}\right)\dfrac{\partial \dot{\theta}_\alpha}{\partial \dot{x}_i} + \underline{\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}_\alpha}}\dfrac{d}{dt}\left(\dfrac{\partial \dot{\theta}_\alpha}{\partial \dot{x}_i}\right) \end{cases}$$

$$\theta_\alpha = \theta_\alpha(x_1, \dots, x_N)$$

则 $d\theta_\alpha = \dfrac{\partial \theta_\alpha}{\partial x_i} dx_i$

$\therefore \boxed{\dot{\theta}_\alpha = \dfrac{\partial \theta_\alpha}{\partial x_i} \dot{x}_i}$ $\Bigg\}$ $\boxed{\dfrac{\partial \dot{\theta}_\alpha}{\partial \dot{x}_i} = \dfrac{\partial \theta_\alpha}{\partial x_i}}$.

求 $\dfrac{d}{dt}$ 即可.

证 $\boxed{\dfrac{\partial \dot{\theta}_\alpha}{\partial x_i} = \dfrac{\partial \ddot{\theta}_\alpha}{\partial \dot{x}_i}}$

关键公式: $\theta_\alpha = \theta_\alpha(x_1, \dots, x_N)$

$\boxed{\dfrac{\partial \dot{\theta}_\alpha}{\partial \dot{x}_i} = \dfrac{\partial \theta_\alpha}{\partial x_i}}$ if $\theta_\alpha = x_i$ 结论显而易见.

问：Fourier 变换是否可以？ Yes !

即 $x_i = \sum_k x_k e^{ik \cdot x_i}$

$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right) = \frac{\partial L}{\partial x_k}}$ 也成立.

在物理中这个方法用得多. 以后会讨论，这个公式的物理意义. 即对应什么？

$$\mathcal{L} = \frac{1}{2} g_{ij}(q) \dot{q}_i \dot{q}_j - U(q) + \frac{d}{dt}F \quad \text{1次}$$

2次 0次

$$= f_i(q)\dot{q}_i$$

$$= \delta F$$

并非所有关系都有 $\boxed{\frac{dF}{dt} = f_i \dot{q}_i}$

$$\frac{\partial}{\partial \dot{q}_i}\delta F = f_i(q)$$

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\delta F\right) &= \left(\frac{\partial f_i}{\partial q_j}\right)\dot{q}_j \\ \frac{\partial}{\partial q_i}\delta F &= \left(\frac{\partial f_j}{\partial q_i}\right)\dot{q}_j \end{aligned}\right\}$$

若 $\frac{\partial f_i}{\partial q_j} = \frac{\partial f_j}{\partial q_i}$ 则无贡献.

若 $\delta F = \sum_i f_i \dot{q}_i = \frac{dF}{dt}$

则 $f_i = \left(\frac{\partial F}{\partial q_i}\right)$

$$\therefore \frac{\partial f_i}{\partial q_j} = \frac{\partial^2 F}{\partial q_j \partial q_i} = \frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}F$$

$$= \frac{\partial f_j}{\partial q_i}$$

这部份内容展开讲有很多和规范场等都有关。

Integral of motion

运动积分

$\ddot{x} = F(x)$ 求解.

方法

$$\boxed{m\ddot{x} = -\partial_x U} \longrightarrow \boxed{E = \frac{1}{2}m\dot{x}^2 + U(x)}$$

$$\downarrow$$

$$\sqrt{\frac{2}{m}(E - U(x))} = \dot{x}$$

$$\Downarrow$$

$$\therefore \frac{dx}{dt} = \sqrt{\frac{2}{m}(E - U(x))}$$

$$\Downarrow$$

$$\frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} = dt$$

$$\Downarrow$$

$$\boxed{\int_a^b \frac{dx}{\sqrt{\frac{2}{m}(E-U(x))}} = \int_{t_a}^{t_b} dt = t_b - t_a}$$

另一种方法.

$$\boxed{\ddot{x} = F(x)} \Leftrightarrow \boxed{\dot{x}^2 = G(x)}$$

$$\Downarrow$$

$$2\dot{x}\ddot{x} = \frac{\partial G}{\partial x}\dot{x}$$

$$F = \frac{1}{2}\frac{\partial G}{\partial x}$$

$$\boxed{\ddot{x} = \frac{1}{2}\frac{\partial G}{\partial x}}$$

难点 Nonlinear differential eq

Integral of Motion // 运动积分，或者运动方程的积分

有时也叫 Integration of equation of motion

A function of the coordinates which is constant along a trajectory in phase space. The number of degrees of freedom of a dynamical system such as the Duffing differential equation can be decreased by one if an integral of motion can be found. In general, it is very difficult to discover integrals of motion.

很多书都没有对 integral of motion 给出严格的定义，一般把它当做一个名词。从字面意思，它表示通过积分的方式得到运动轨迹。这样的做法，要求我们把二次方程变为一次方程。这些积分往往涉及到椭圆积分。

对于高维系统，一般都非常困难，但是如果存在守恒量，就简单得多。这本书中的例子包括能量守恒、角动量守恒等。所以学习理论力学的时候要注意：

1. 守恒量
2. 椭圆积分
3. 二次函数到一次函数的转变

求解 r''[t] + a r'[t] + ... = f[r[t]]这样的方程，往往需要很多方法/变换，这里我们用的是积分方法求解。所以 Integral of motion，表示如何求解上面的微分方程 --- 我们用的是积分的方法求解。但是需要用到守恒量才行。后来它发展为可积系统，即可以用积分方法求解的系统，Integral systems 以及 integrability/可积性。

Integ

Integral of Motion // 运动积分，或者运动方程的积分

有时也叫 Integration of equation of motion

难点：经典力学中碰到的都是非线性方程，不能求解，不能用教材的方法。

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微分方程的处理方法//见微积分（下）

1. 多个变量，分离变量法
2. 常系数方程（damped oscillator），平面波
3. 平面波方法 f(x - v t)

例子：Schrodinger 方程，以及数理方程中的特殊函数。通过定义特殊函数来求解某些方程。

非线性方程，比如经典力学中碰到的方程，一般很难求解。从这个角度可以理解为什么积分法是一个特殊的方法。不要认为它可以处理任何问题，其实不行。

教材中涉及几个求解这些方程的方法：

1. Integral of motion
2. Perturbation theory//微扰理论、或者摄动理论

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First and Second Order Differential Equations

First Order Differential equations

A first order differential equation is of the form:

$$\frac{dy}{dx} = f(x,y)$$

Linear Equations:

$$\frac{dy}{dx} + p(x)y = q(x).$$

The general general solution is given by

$$y = \frac{\int u(x)q(x)dx + C}{u(x)},$$

where

$$u(x) = \exp\left(\int p(x)dx\right),$$

is called the **integrating factor**.

Separable Equations:

$$\frac{dy}{dx} = h(x)g(y)$$

(1)
Solve the equation $g(y) = 0$ which gives the constant solutions.

(2)
The non-constant solutions are given by

$$\int \frac{1}{g(y)}dy = \int h(x)dx.$$

Bernoulli Equations:

$$\frac{dy}{dx} + p(x)y = q(x)y^{n}.$$

(1)
Consider the new function $v = y^{1-n}$.

(2)
The new equation satisfied by v is

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x).$$

(3)
Solve the new linear equation to find v.

(4)

Back to the old function y through the substitution $y = v^{1/(1-n)}$.

(5)

If $n > 1$, add the solution y=0 to the ones you got in (4).

Homogenous Equations:

$$\frac{dy}{dx} = f(x,y)$$

is **homogeneous** if the function $f(x,y)$ is homogeneous, that is

$$f(tx,ty) = f(x,y) \text{ for any number } t.$$

By substitution, we consider the new function

$$z = \frac{y}{x} \text{ (which is equivalent to } y = xz).$$

The new differential equation satisfied by z is

$$x\frac{dz}{dx} + z = f(1,z)$$

which is a separable equation. The solutions are the constant ones $f(1,z) - z = 0$ and the non-constant ones given by

$$\ln|x| = \int \frac{dz}{f(1,z) - z} + C$$

Do not forget to go back to the old function $y = xz$.

Exact Equations:

$$M(x,y)dx + N(x,y)dy =$$

is **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The condition of exactness insures the existence of a function $F(x,y)$ such that

$$\begin{cases} \dfrac{\partial F}{\partial x} = M(x,y), \\ \dfrac{\partial F}{\partial y} = N(x,y). \end{cases}$$

All the solutions are given by the implicit equation

$$F(x,y) = C.$$

Second Order Differential equations

Homogeneous Linear Equations with constant coefficients:

$$ay'' + by' + cy = 0 \qquad (a \neq 0)$$

Write down the **characteristic equation**

$$ar^2 + br + c = 0.$$

(1)

If r_1 and r_2 are distinct real numbers (this happens if $b^2 - 4ac > 0$), then the general solution

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

不能用积分求解

CHAPTER 12

Second Order Linear Differential Equations

§12.1. Homogeneous Equations

A **differential equation** is a relation involving variables $x, y, y', y'', \ldots$. A **solution** is a function $f(x)$ such that the substitution $y = f(x)$, $y' = f'(x)$, $y'' = f''(x)$, $\ldots$ gives an identity. The differential equation is said to be **linear** if it is linear in the variables $y, y', y'', \ldots$. We have already seen (in section 6.4) how to solve first order linear equations; in this chapter we turn to second order linear equations with constant coefficients. The general form of such an equation is

$$y'' + ay' + by = g(x), \tag{12.1}$$

常系数可以求解.

where a and b are constants, and $g(x)$ is a differentiable function of x. In chapter 6.4, we saw that a first order equation has a one-parameter family of solutions, and that the specification of an initial condition $y(x_0) = y_0$ uniquely determines a solution. In the case of second order equations, the basic theorem is this:

Theorem 12.1 *Given x_0 in the domain of the differentiable function g, and numbers y_0, y_0', there is a unique function $f(x)$ which solves the differential equation (12.1) and satisfies the initial conditions $f(x_0) = y_0$, $f'(x_0) = y_0'$.*

In this section we shall see how to completely solve equation (12.1) when the function on the right hand side is zero:

$$y'' + ay' + by = 0. \tag{12.2}$$

This is called the **homogeneous equation**. An important first step is to notice that if $f(x)$ and $g(x)$ are two solutions, then so is the sum; in fact, so is any linear combination $Af(x) + Bg(x)$. Thus, once we know two solutions (they must be *independent* in the sense that one isn't a constant multiple of the other) we can solve the initial value problem in theorem 12.1 by solving for A and B.

Example 12.1 Solve $y'' + y = 0$, $y(0) = 4$, $y'(0) = -1$.

Now, we know that $\cos x$ and $\sin x$ are solutions of the equation, so we try a solution of the form $y(x) = A\cos x + B\sin x$. Evaluating at $x = 0$, we find that $A = 4$. Differentiate, getting $y'(x) = -A\sin x + B\cos x$, and evaluating at $x = 0$, we find $B = -1$. Thus the solution is $y(x) = 4\cos x - \sin x$.

The reason the answer worked out so easily is that $y_1 = \cos x$ is the solution with the particular initial values $y_1(0) = 1$, $y_1'(0) = 0$ and $y_1 = \sin x$ is the solution with $y_1(0) = 0$. $y_1'(0) = 1$. Then the solution with initial values $y(0)$ and $y'(0)$ is

(12.3) $$y(x) = y(0)\cos x + y'(0)\sin x$$

Example 12.2 Solve $y'' - y = 0$, with given initial values $y(0), y'(0)$.

Now e^x and e^{-x} are solutions of this differential equation, so the general solution is a linear combination of these. But we won't have as easy a time finding a solution like (12.3), since these functions do not have the initial values $1,0; 0,1$ respectively. However if we introduce the functions

(12.4) $$\cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \sinh x = \frac{1}{2}(e^x - e^{-x})$$

these do have the right initial values:

(12.5) $$\cosh 0 = 1\ , \quad \sinh 0 = 0$$

(12.6) $$\frac{d}{dx}(\cosh x) = \sinh x\ , \quad \frac{d}{dx}(\sinh x) = \cosh x$$

so $(\cosh)'(0) = 0$, $(\sinh)'(0) = 1$. Thus, the solution to our problem is

(12.7) $$y(x) = y(0)\cosh x + y'(0)\sinh x\ .$$

This particular differential equation comes up so often that it is important to remember these functions, $\cosh x$, $\sinh x$, called the **hyperbolic functions** and their basic properties: equation (12.6) and

(12.8) $$\cosh^2 x - \sinh^2 x = 1\ .$$

Because of (12.8) these functions parametrize the standard hyperbola (and it is for this reason that they are called hyperbolic functions).

We now return to the general second order equation.

Proposition 12.1 *Let r be a root of the equation*

(12.9) $$r^2 + ar + b = 0\ .$$

Then e^{rx} is a solution to the homogeneous equation:

(12.10) $$y'' + ay' + by = 0\ .$$

Equation (12.9) is called the ***auxiliary equation*** *of the differential equation (12.10). To verify the proposition, let $y = e^{rx}$ so that $y' = re^{rx}$, $y'' = r^2e^{rx}$. Substituting into equation (12.10):*

(12.11) $$r^2e^{rx} + are^{rx} + be^{rx} = e^{rx}(r^2 + ar + b) = 0$$

f and only if r is a root of the auxiliary equation.

线性方程还可以求解
非线性方程不能求解

DEFINITION 1.20 Two functions u and v are said to be ***linearly independent*** if neither is a constant multiple of the other. If one is a constant multiple of the other they are said to be ***linearly dependent***.

Thus, the functions $u(t) = t$ and $v(t) = t^2$ are linearly independent. It is true that $v(t) = tu(t)$, but the factor t is not a constant. On the other hand $u(t) = \sin t$ and $v(t) = -4\sin t$ are obviously linearly dependent.

We are going to use Theorem 1.17 to prove the following result, which will provide us with our solution strategy for homogeneous equations.

THEOREM 1.21 Suppose that y_1 and y_2 are linearly independent solutions to the equation

$$y'' + p(t)y' + q(t)y = 0. \tag{1.22}$$

Then the general solution to (1.1[illegible]s

$$y = C_1 y_1 + C_2 y_2,$$

where C_1 and C_2 are arbitrary constants.

Theorem 1.21 will be proved after some discussion of the result. We will find it advantageous to define some more terminology.

DEFINITION 1.23 A ***linear combination*** of the two functions u and v is any function of the form

$$w = Au + Bv,$$

where A and B are constants.

With this definition we can express Proposition 1.18 by saying that a linear combination of two solutions is also a solution. Theorem 1.21 says that the general solution is the general linear combination of the solutions y_1 and y_2, provided that y_1 and y_2 are linearly independent. Because of this result we will say that two linearly independent solutions form a ***fundamental set of solutions***.

Notice that Theorem 1.21 defines a strategy to be used in solving homogeneous equations. It says that it is only necessary to find two linearly independent solutions to find the general solution. That is what we will do in what follows.

EXAMPLE 1.24 ◆ Find a fundamental set of solutions to the equation for simple harmonic motion,

$$x'' + \omega^2 x = 0.$$

It can be shown by substitution that

$$x_1(t) = \cos\omega t \quad \text{and} \quad x_2(t) = \sin\omega t$$

are solutions. (See equation (1.14) and what follows.) It is clear that these functions are not multiples of each other, so they are linearly independent. It follows from

Theorem 1.21 that x_1 and x_2 are a fundamental set of solutions. Therefore, every solution to the equation for simple harmonic motion is a linear combination of x_1 and x_2. ◆

To prove Theorem 1.21, we need to know a little more about the impact of linear independence. The best way to determine if two given functions are linearly independent is by simple observation. For example, in Example 1.24 it is pretty obvious that $\cos\omega t$ and $\sin\omega t$ are not multiples of each other and therefore are linearly independent. However, we will need a way of making this determination in more difficult cases. The ***Wronskian*** of two functions u and v is defined to be

$$W(t) = \det\begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix} = u(t)v'(t) - v(t)u'(t).$$

The relationship of the Wronskian to linear independence is summed up in the next two propositions.

PROPOSITION 1.25 Suppose the functions u and v are solutions to the linear, homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

in the interval (α, β). Then the Wronskian of u and v is either identically equal to zero on (α, β) or it is never equal to zero there.

Proof To prove this result, we differentiate the Wronskian $W = uv' - vu'$. We get

$$W' = u'v' + uv'' - v'u' - vu'' = uv'' - vu''.$$

Since u and v are solutions to $y'' + py' + qy = 0$, we can solve for their second derivatives and substitute. We get

$$\begin{aligned} W' &= u\left(-pv' - qv\right) - v\left(-pu' - qu\right) \\ &= -p(uv' - vu') \\ &= -pW. \end{aligned}$$

This is a separable first-order equation for W. If t_0 is a point in (α, β), the solution is

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p(s)\,ds} \quad \text{for } \alpha < t < \beta.$$

If $W(t_0) = 0$, then $W(t) = 0$ for $\alpha < t < \beta$. On the other hand, if $W(t_0) \neq 0$, then $W(t) \neq 0$, since the exponential term is never zero.

Consider the solutions $x_1(t) = \cos\omega t$ and $x_2(t) = \sin\omega t$ we found in Example 1.24. The Wronskian of x_1 and x_2 is

$$W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t) = \omega_0\cos^2\omega_0 t + \omega_0\sin^2\omega_0 t = \omega_0.$$

Thus for these two solutions the Wronskian is never equal to zero. This is always the case for a fundamental set of solutions, as we will prove in the next result.

讲述中心势运动的时候难点/有趣的点

1. 为什么万有引力下的轨道不可能是圆轨道？
2. 万有引力下，一般的运动轨迹可能是什么样子？
3. 导致这些不同轨迹的物理图像是什么？【要画很多图】有周期解吗？ No.
4. 轨迹的稳定性、封闭性问题，以及在地球物理中的应用
5. 讨论和分析： 为什么可以简化为两体问题，为什么需要考虑多体问题，以及太阳系的稳定性（以及 open questions)

如何地球自转、公转、以及其它行星的运动全部考虑进去，同时考虑气体或微行星交互作用，那么这个问题就非常有趣了，而且没有简单答案。可能是地球物理的研究范围，用的是经典力学方法。

历史介绍：太阳系的稳定性 【百度百科】

在二阶摄动基础上，泊松得到一个定理:大行星轨道半长径没有长期变化，即不会随时间增加而无限增大或缩小.这只是从一个侧面说明太阳系可能是稳定的.在高阶摄动基础上，不少人曾得出行星道半长径有长期变化的结果.但到 1982 年，梅塞奇终于证明了在任意阶摄动下，大行星轨道半长径仍没有长期变化.尽管这个定理很重要，但也只能说明太阳系可能是稳定的，因为若某行星轨道偏心率如增加到大 1，仍会出现逃逸而不稳定.卡姆理论出现

几亿年来，地球的公转周期一直没有太大变化吗？ //知乎

我知道地球自转速度是一直在变慢的，7000 万年前的一天只有 23.5 个小时左右，但一年有 372 天，算下来一年是的长度是 8700 多个小时，和现在是一样的，难道地球的公转周期一直没有太大变化吗？

行星迁移（Planetary migration）是行星或其他恒星旁的天体和恒星周围的盘内的气体或微行星交互作用时发生的现象；该现象会改变行星等天体的轨道半长轴等轨道参数。现在广被接受的行星形成理论内容指出，原行星盘内行星不会在相当接近恒星的区域形成，因为太过靠近恒星的区域内的天体质量不足以形成行星，并且温度过高无法让主要含岩石或冰的微行星存在。恒星旁气体盘还存在时，质量与地球相当行星可能会向内快速靠近恒星；这也可能会影响巨大行星（质量高于 10 倍地球质量）的核心形成，如果它们的形成是经由核心吸积机制的话。行星迁移是太阳系外行星中巨大质量且公转周期极短的热木星形成最可能的解释。