理论力学 ②

GM 2023.9.7.

附 reading material or reference.

- Lagrange 方程的推导
- Lagrange 的意义 ⇒ 最小作用量原理.
- Lagrange 和对称性
- 光和物质的相似性.

Lagrange 推导

① D' Alambert 原理 (科大书，金尚年书)

② 从欧空间 ⇒ $\forall$ 空间

上一个 Note. 用 θ_α 表示广义坐标 ⇒ 现用 q_α 表示，和 Landau 书一

Assume $q_\alpha = q_\alpha(x_1, \dots, x_N)$

$\alpha = 1, 2, \dots, N-M$

M: 受限方向数.

☆ q_α 只和广义坐标有关，和速度无关。

=0 和 $\dot{x}_i$ 无关

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial q_\alpha}\cancel{\frac{\partial q_\alpha}{\partial \dot{x}_i}} + \frac{\partial L}{\partial \dot{q}_\alpha}\frac{\partial \dot{q}_\alpha}{\partial \dot{x}_i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)\frac{\partial \dot{q}_\alpha}{\partial \dot{x}_i} + \frac{\partial L}{\partial \dot{q}_\alpha}\frac{d}{dt}\left(\frac{\partial \dot{q}_\alpha}{\partial \dot{x}_i}\right)$$

$$= \left(\frac{\partial L}{\partial q_\alpha}\right)\frac{\partial q_\alpha}{\partial x_i} + \frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\partial \dot{q}_\alpha}{\partial x_i}\right).$$

$\because\ q_\alpha = q_\alpha(x_1, \dots, x_N, t)$

$$\frac{dq_\alpha}{dt} = \frac{\partial q_\alpha}{\partial t} + \frac{\partial q_\alpha}{\partial x_i}\dot{x}_i$$

它只和 $x_1 \dots, x_N$ 有关，和 $\dot{x}_i$ 无关

所以

$$\dot{q}_\alpha = \frac{\partial q_\alpha}{\partial t} + \frac{\partial q_\alpha}{\partial x_i}\dot{x}_i \quad \text{线性方程}$$

$$\therefore\ \boxed{\frac{\partial \dot{q}_\alpha}{\partial \dot{x}_i} = \frac{\partial q_\alpha}{\partial x_i}} \quad \text{关键式}$$

交换性

$$\therefore\ \frac{d}{dt}\left(\frac{\partial \dot{q}_\alpha}{\partial \dot{x}_i}\right) = \frac{d}{dt}\left(\frac{\partial q_\alpha}{\partial x_i}\right) \qquad \text{using } \frac{\partial}{\partial x}\frac{\partial}{\partial y}f = \frac{\partial}{\partial y}\frac{\partial}{\partial x}f$$

$$\therefore\ \boxed{\frac{\partial \ddot{q}_\alpha}{\partial \dot{x}_i} = \frac{\partial \dot{q}_\alpha}{\partial x_i}}$$

这样

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)}\frac{\partial \dot{q}_\alpha}{\partial \dot{x}_i} + \frac{\partial L}{\partial \dot{q}_\alpha}\frac{\partial \ddot{q}_\alpha}{\partial \dot{x}_i}$$

$$= \boxed{\frac{\partial L}{\partial q_\alpha}}\left(\frac{\partial q_\alpha}{\partial x_i}\right) + \frac{\partial L}{\partial \dot{q}_\alpha}\frac{\partial \dot{q}_\alpha}{\partial x_i}$$

Lagrege eq.

证明结束.

grange eq 的性表义

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha}$$

广义动量　　　广义力

$$\boxed{\frac{d}{dt}(\text{广义动量}) = \text{广义力}}$$

⇒ 牛二定律推广到任意坐标.

适用性：任意坐标

eg
- $(r\ \theta\ \phi)$
- $(r\ \theta\ z)$
- Fourier transformation （场论中）.
- …

特别强调这一点.

$$\delta S = \int_a^b [\mathcal{L}(q+\varepsilon,\ \dot{q}+\dot{\varepsilon}) - \mathcal{L}(q,\dot{q})]\,dt$$

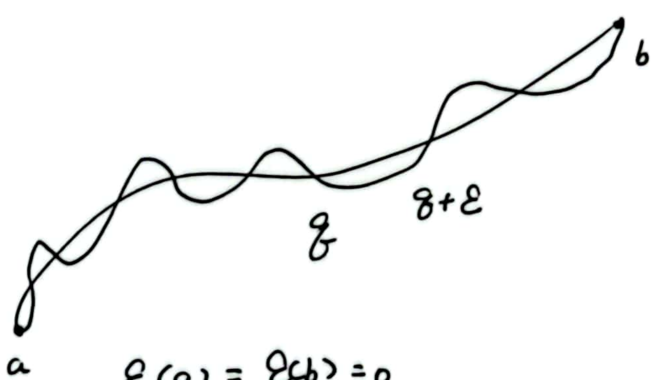
$$= \int_a^b \left[\frac{\partial f}{\partial q}\varepsilon + \frac{\partial f}{\partial \dot{q}}\dot{\varepsilon}\right]dt + O(\varepsilon^2)$$

$$= \underbrace{\int_a^b \left[\frac{\partial f}{\partial q} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}}\right)\right]\varepsilon\, dt}_{=0} + \underbrace{\int_a^b d\left(\frac{\partial f}{\partial \dot{q}}\varepsilon\right)}_{= \left.\frac{\partial f}{\partial \dot{q}}\varepsilon\right|_a^b = 0} + O(\varepsilon^2)$$

类比 $f(x+\varepsilon) - f(x)$

$$= \underbrace{\left(\frac{\partial f}{\partial x}\right)}_{=0}\varepsilon + O(\varepsilon^2)$$

$$\boxed{\delta S = O(\varepsilon^2)}$$



$\varepsilon(a) = \varepsilon(b) = 0$

论)

$$\boxed{\vec{F} = m\vec{a} \quad \text{牛二定律}}$$

||

$$\boxed{\text{Lagrange eq}}$$

||

$$\boxed{\delta S = 0}$$

最根本的，最本质的。在所有力学中都存在的最普遍的原理。

称性 → $\mathcal{L}$ 不变 ⇒ 守恒量

★ 每个对称性 ⇒ 对应一个守恒量.

空间平移 $q \to q + c$ 不变 ⇒ $U(q) = U(q+c) = \text{const}$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - U$$

$$\therefore \frac{\partial L}{\partial x} = 0 \iff \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0.$$

时间平移不变性

$E = T + V$ 不变

$L = T - V$

$$\therefore E + L = 2T = \sum_i m_i \dot{x}_i^2 = \sum_i p_i \dot{x}_i$$

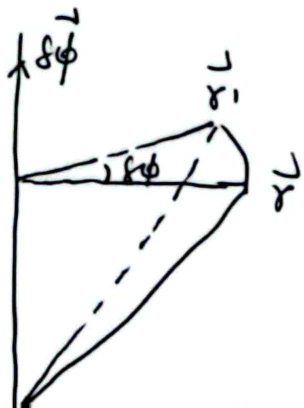
$$\therefore E = \sum_i p_i \dot{x}_i - \mathcal{L}$$

可以证明 $\frac{d}{dt}\left(\sum_\alpha p_\alpha \dot{q}_\alpha - L\right) = 0.$

Legendre 变换

Define $\boxed{H = \sum_\alpha p_\alpha \dot{q}_\alpha - L}$

Rotation symmetry



$$\vec{r}\,' = \vec{r} + \delta\vec{\phi} \times \vec{r}$$

$$\vec{v}\,' = \vec{v} + \delta\vec{\phi} \times \vec{v}$$

所以

$$\delta L = \frac{\partial L}{\partial \vec{r}_i}\cdot\delta\vec{r}_i + \frac{\partial L}{\partial \dot{\vec{r}}_i}\,\delta\dot{\vec{r}}_i$$

$$= \dot{\vec{p}}_i\cdot\delta\vec{r}_i + \vec{p}_i\cdot\delta\dot{\vec{r}}_i$$

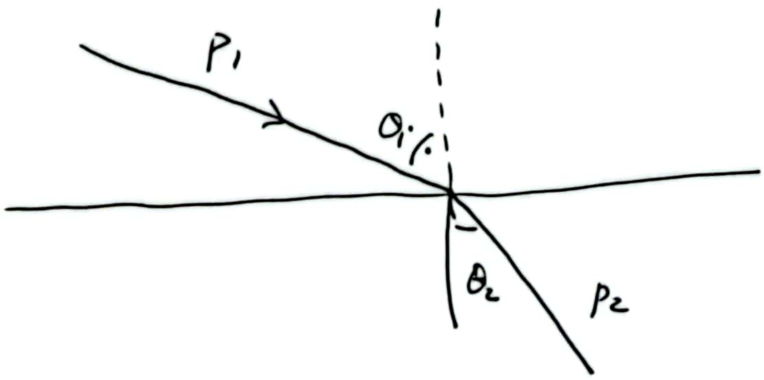
$$= \dot{\vec{p}}_i\cdot(\delta\vec{\phi}\times\vec{r}_i) + \vec{p}_i\cdot(\delta\vec{\phi}\times\dot{\vec{r}}_i)$$

$$= \delta\vec{\phi}\cdot(\vec{r}_i\times\dot{\vec{p}}_i + \dot{\vec{r}}_i\times\vec{p}_i)$$

$$= \delta\vec{\phi}\cdot\frac{d\vec{L}}{dt}, \quad \vec{L} = \sum_i \vec{r}_i\times\vec{p}_i \text{ 角动量}$$

$$= 0 \text{ for any } \delta\vec{\phi} \Rightarrow \boxed{\vec{L} = \text{const}}$$

光和物质的相似性



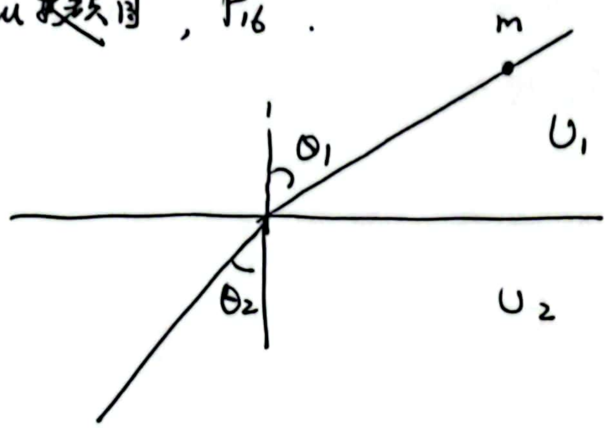
$$\therefore\ p_1\sin\theta_1 = p_2\sin\theta_2$$

$$p = h/\lambda = \frac{h}{vT} = \frac{hn}{cT} \propto n \text{ if } v = c/n.$$

$$\therefore\ \boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{p_2}{p_1} = \frac{n_2}{n_1}}$$

最小作用量原理 $\int n\,ds \propto \int p\,ds \sim \boxed{\int p\dot{x}\,dt}$

ndau 校题目，P_{16}.



potential

$$\begin{cases} \frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2 \Rightarrow T_1 + U_1 = T_2 + U_2 \\ m v_1 \sin\theta_1 = m v_2 \sin\theta_2 \quad // \text{方向动量守衡} \end{cases}$$

$$\therefore v_2 = \sqrt{\frac{U_1 - U_2 + \frac{1}{2} m v_1^2}{\frac{1}{2} m}}$$

$$\therefore \frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{\frac{U_1 - U_2 + \frac{1}{2} m v_1^2}{\frac{1}{2} m v_1^2}}$$

$$= \sqrt{1 + \frac{U_1 - U_2}{T_1}} = \frac{n_2}{n_1}$$

此题有深刻的指示。

业：Landau书 P10–P12 4题，不求解方程

思考/描述运动轨道/形式。

物理的直觉，思考能力很重要。当方程不能求解时，只能用直觉和思考去工作。

<u>总结</u>

$$\boxed{\vec{F} = m\vec{a}}$$

$$\parallel$$

$$\boxed{\text{Lagrange eq}} \qquad L$$

$$\parallel$$

$$\boxed{\delta S = 0} \qquad S = \int \mathcal{L}\, dt$$