第一次作业

- 作出插值点$\{(-2.00.0.00).(2.00.3.00).(5.00.6.00)\}$的二次 $Lagrange$插值多项式 $L_2(x)$，并计算 $L_2(-1.2), L_2(1.2)$.

  解:

$$
\begin{aligned}
L_2(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)\\
&= \frac{(x-2)(x-5)}{(-2-2)(-2-5)}\times 0+\frac{(x+2)(x-5)}{(2+2)(2-5)}\times 3+\frac{(x+2)(x-2)}{(5+2)(5-2)}\times 6\\
&= -\frac{1}{4}(x+2)(x-5)+\frac{2}{7}(x+2)(x-2)\\
&= \frac{1}{28}x^2+\frac{3}{4}x+\frac{19}{14}\\
L_2(-1.2) &= \frac{89}{175}\\
L_2(1.2) &= \frac{404}{175}
\end{aligned}
$$

- $f(x)=\sqrt{x}$在离散点有$\{f(81)=9, f(100)=10, f(121)=11\}$. 用插值方法计算$\sqrt{105}$的近似值，并由误差公式给出误差界，同时与实际误差作比较.

$$
\begin{aligned}
L_2(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)\\
&= \frac{(x-100)(x-121)}{(81-100)(81-121)}\times 9+\frac{(x-81)(x-121)}{(100-81)(100-121)}\times 10+\frac{(x-81)(x-100)}{(121-81)(121-100)}\times 11\\
&= -\frac{1}{7980}x^2+\frac{601}{7980}x+\frac{495}{133}
\end{aligned}
$$

  $L_2(105)=\frac{1363}{133}$

  $R_2(105)=\|\frac{f^{(3)}(\theta)}{3!}(105-81)(105-100)(105-121\|)\leq 2.032\times 10^{-3}$

  实际误差为1.1695×10^{-3},在误差界内

- 证明：$\{l_i(x)\}_{i=0}^n$线性无关

  解:

  不妨采用反证法，设$\{l_i(x)\}_{i=0}^n$线性相关.则存在$a_0, a_1, \ldots, a_n$不全为0，

  满足$\sum_{i=0}^n a_i l_i(x)=0$

  令x分别取$x_0, x_1, \ldots, x_n$

  可以得到 $a_i l_i(x_i)=0, i=0,1,\ldots,n$

  由于$l_i(x_i)=1$,可得 $a_i=0, i=0,1,\ldots,n$，这与假设矛盾！

  故$l_i(x)$线性无关.

- 记$P_{i_0,i_1,\ldots i_k}(x)$为节点$\{x_{i_0}, x_{i_1}\ldots x_{i_k}\}$上的插值多项式

  证明:

$$
L_n(x)=\frac{(x-x_j)P_{0,1,\ldots j-1,j+1,\ldots,n}(x)-(x-x_i)P_{0,1,\ldots i-1,i+1,\ldots,n}(x)}{x_i-x_j}
$$

解:

当$x_k \neq x_i, x_k \neq x_j$时

$$\begin{aligned}LHS = L_n(x_k) &= \frac{(x_k - x_j)P_{0,1,\ldots j-1,j+1,\ldots,n}(x_k) - (x_k - x_i)P_{0,1,\ldots i-1,i+1,\ldots,n}(x_k)}{x_i - x_j} \\ &= \frac{(x_k - x_j) - (x_k - x_i)}{x_i - x_j} \\ &= 1 = RHS\end{aligned}$$

当$x_k = x_i$时

$$\begin{aligned}LHS = L_n(x_i) &= \frac{(x_i - x_j)P_{0,1,\ldots j-1,j+1,\ldots,n}(x_i) - (x_i - x_i)P_{0,1,\ldots i-1,i+1,\ldots,n}(x_i)}{x_i - x_j} \\ &= \frac{(x_i - x_j) - 0}{x_i - x_j} \\ &= 1 = RHS\end{aligned}$$

当$x_k = x_j$时

$$\begin{aligned}LHS = L_n(x_j) &= \frac{(x_j - x_j)P_{0,1,\ldots j-1,j+1,\ldots,n}(x_j) - (x_j - x_i)P_{0,1,\ldots i-1,i+1,\ldots,n}(x_j)}{x_i - x_j} \\ &= \frac{0 - (x_j - x_i)}{x_i - x_j} \\ &= 1 = RHS\end{aligned}$$

由插值函数的定义可证得原命题.