Spectrally-large scale geometry and symplectic squeezing in cotangent bundle of torus

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Spectrally-large scale geometry

Main objects

- Today we focus on two metric spaces with respect to cotangent bundles T^*N:

(1) **Hamiltonian diffeomorphism group** $\mathrm{Ham}_c(D_g^*N, \omega_{\mathrm{can}})$. Here D_g^*N is the unit codisk bundle with respect to some Riemannian metric g:

$$D_g^*N := \{(q,p) \in T^*N \mid \|p\|_g \leq 1\}.$$

(2) **The orbit space** of fibers under Hamiltonian deformations:

$$\mathscr{O}(F_q) := \{\phi(F_q) \mid \phi \in \mathrm{Ham}_c(D_g^*N, \omega)\}.$$

- There are various metrics on $\mathrm{Ham}_c(D_g^*N, \omega_{\mathrm{can}})$ and $\mathscr{O}(F_q)$:

(1) **Hofer metric** d_{Hofer} or δ_{CH} induced by Hofer norm $\|\cdot\|_{\mathrm{Hofer}}$.

(2) **Spectral metric** d_γ or δ_γ induced by spectral norm γ.

Spectral norm and Hofer norm

- Spectral norm γ is defined via Floer homologies (loop space $\mathscr{L}$, action functional $\mathscr{A}_H$, etc.) Viewed from a Morse homology, γ is defined as the largest gap between critical values of homologically essential classes of the Morse function $\mathscr{A}_H$.

- For any $\phi \in \mathrm{Ham}_c(D^*N, \omega)$, the spectral norm satisfies

$$\gamma(\phi) \leq \|\phi\|_{\mathrm{Hofer}}$$

Similarly, on the orbit space $\mathscr{O}(F_q)$, the induced metric

$$\delta_\gamma(L_1, L_2) \leq \delta_{\mathrm{Hofer}}(L_1, L_2).$$

- For an exact symplectic manifold (e.g. $(D^*_g Q, \omega_{\mathrm{can}})$), the calculation of δ_γ can be considerably simplified as follows,

$$\delta_\gamma(L_1, L_2) := \inf\{\gamma(\phi) \mid \phi(L_1) = L_2\} = \ell(\mathbb{1}_L, H) + \ell(\mathbb{1}_L, \overline{H})$$

for any H satisfying $\phi^1_H(L_1) = L_2$.

Quasi-flat

Definition

Let $(M_1, d_1), (M_2, d_2)$ be metric spaces. A map $f : M_1 \to M_2$ is called a **quasi-isometric embedding** if there exist constant $A \geq 1, B \geq 0$ such that for any $x, y \in M_1$,

$$\frac{1}{A} d_1(x,y) - B \leq d_2(f(x), f(y)) \leq A d_1(x,y) + B$$

We say that (M, d) contains a **rank-n quasi-flat** if there is a quasi-isometric embedding from $(\mathbb{R}^n, d_\infty)$ to (M, d), where $n \in \mathbb{N} \cup \{\infty\}$ and $d_\infty(x,y) = |x-y|_\infty$.

Remark

A metric space (M, d) that contains a rank-n quasi-flat roughly means that there are n-many linearly independent directions that go to infinity.

Summary of large scale geometry phenomenon

	d_{Hofer} and δ_{Hofer}	d_γ and δ_γ
absolute	Some $\text{Ham}(M, \omega)$ contain a rank-∞ quasi-flat	Some $\text{Ham}(M, \omega)$ contain a rank-∞ quasi-flat
	$\text{Ham}(D_g^* N, \omega_{\text{can}})$ contains a rank-∞ quasi-flat for some closed manifold N	$\text{Ham}(W, \omega)$ is unbounded for any Liouville domain W with $\text{SH}^*(W) \neq 0$
		$\text{Ham}(D_g^* N, \omega_{\text{can}})$ contains a rank-∞ quasi-flat for some closed manifold N (Theorem B)
relative (fiber F_x)	$\mathcal{O}(F_x)$ contains a rank-∞ quasi-flat for some closed manifold N	$\mathcal{O}(F_x)$ is unbounded for any closed N
		$\mathcal{O}(F_x)$ contains a rank-∞ quasi-flat for some closed N (Theorem A)

Theorem C

Spectrally-large scale geometry

Theorem A (F.-Zhang, 2024)

If $q \in S^{n\neq 2}$ and F_q is the fiber of q in $D_g^*S^n$, then metric space $\left(\mathscr{O}(F_q), \delta_\gamma\right)$ contains a rank-∞ quasi-flat.

Theorem (Gong, 2024)

Let $(W, d\theta)$ be a Liouville domain, and let $L \subset M$ be an admissible Lagrangian submanifold. If $\mathrm{HW}^*(L) \neq 0$, then the metric space $(\mathscr{O}(L), \delta_\gamma)$ is ***unbounded***.

Remark

If $W = D_g^*N$ and $L = F_x$, then $\mathrm{HW}^*(F_x) \cong H_{-*}(\Omega N) \neq 0$ due to Viterbo's theorem, where ΩN denotes the based loop space.

Relative implies absolute

Similar to the relative case (wrapped Floer theory), we can consider the absolute case (Hamiltonian Floer theory).

Theorem B (F.-Zhang, 2024)

Let (N,g) be a closed Riemannian manifold. Suppose there is no non-constant contractible closed geodesic in (N,g), then metric space $\left(\mathrm{Ham}(D_g^*N, \omega_{\mathrm{can}}), d_\gamma\right)$ contains a rank-∞ quasi-flat.

Remark

In particular, $N = S^n$ for $n \geq 2$ are **excluded**.

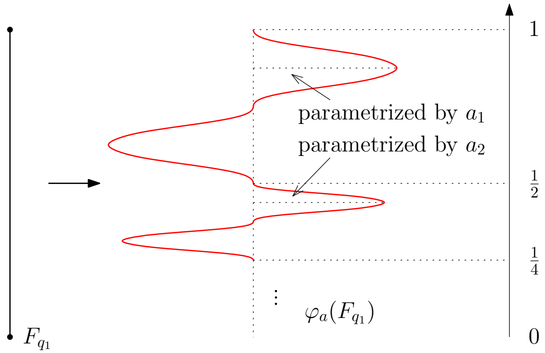
In fact, we can recover the relative results to the absolute case due to the $H^*(W)$-module structure of $H^*(L)$.

Theorem C (F.-Zhang, 2024)

For any $n \neq 2$, the metric space $\left(\mathrm{Ham}(D_g^*S^n, \omega_{\mathrm{can}}), d_\gamma\right)$ contains a rank-∞ quasi-flat.

Construction

- Given any $a = (a_1, a_2, \cdots) \in \mathbb{R}^\infty$, the Hamiltonian deformation $\phi_a(F_{q_1}) \in \mathscr{O}(F_{q_1})$ is given by the following picture,



- Morse index conditions help us to control the Floer boundary map.

Proof based on boundary depth

• The key step in proving Theorem A and Theorem B is seeking for a invariant that serves for the following two purposes:

(1) this invariant provides a lower bound of spectral norm γ;

(2) this invariant detects large-scale geometric properties.

• This invariant is the **boundary depth**. Roughly speaking, it measures the longest time interval that a homological invisible generator can persistent.

Theorem (Usher, 2014)

Given $a \in \mathbb{R}^{\infty}$ and $\phi(a)$ as above and under the Morse index conditions, β read off from the wrapped Floer chain complex $CW_{\mathfrak{c}}(F_{q_0}, F_{q_1}, \phi_a)$ satisfies $\beta \geq \|a\|_{\infty} - C$.

• Motived by Kislev-Shelukhin's inequality, we can show that

$$\beta(F_{q_0}, F_{q_1}, \phi_a) \leq \gamma(F_{q_1}, \phi_a).$$

Work in progress

Using the concept of **heavy** sets (due to Entov-Polterovich), we recently obtained the following result (cf. Yuhan Sun, 2024).

Theorem (F.-Zhang, 2024)

For any Liouville domain (W, ω), the followings are equivalent:

1. $\mathrm{SH}^*(W, \omega) \neq 0$.
2. *skeleton* $\mathrm{Sk}(W)$ *is heavy.*
3. $\exists\, (C_c^\infty([0,1]), d_\infty) \hookrightarrow (\mathrm{Ham}(W, \omega), d_\gamma)$.

Corollary

For any closed Riemannian manifold (N, g), there exists

$$(C_c^\infty([0,1]), d_\infty) \hookrightarrow (\mathrm{Ham}(D_g^* N, \omega_{\mathrm{can}}), d_\gamma).$$

Question: what will happen if one twists the coefficient of $\mathrm{SH}^*(D_g^* N)$ by a local system ν on ΛN (so that $\mathrm{SH}^*(D_g^* N; \nu) = 0$)?

Symplectic squeezing in $T^*\mathbb{T}^n$

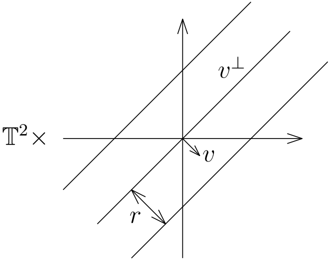
Denote the standard simplex in $\mathbb{R}^n_{\geq 0}$ by $\Delta^n(r) := \{x_1 + \cdots + x_n \leq r\}$. Introduce two domains in $T^*\mathbb{T}^n$:

$$P^{2n}(r) := \mathbb{T}^n \times \Delta^n(r)$$

and

$$Y^{2n}(r, v) := \mathbb{T}^n \times \left((-r, r)v \times v^\perp\right)$$

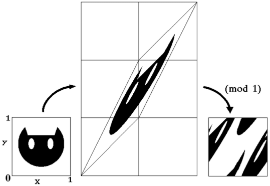
Here, v is a unit vector in $\mathbb{R}^n$ and $v^\perp$ denotes the **hyperplane** in fiber $\mathbb{R}^n$ that is perpendicular to v. In particular, when $n = 2$, $v^\perp$ is simply a line that is perpendicular to v.



Arnold cat map

- Consider a symplectomorphism in the form of of $\Phi_A = (A^{-1}, A)$ on $T^*\mathbb{T}^2$, where A is the famous **Arnold's cat map**,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$



- Matrix A has two irrational eigenvalues $0 < \lambda_1 < 1 < \lambda_2$ with eigenvectors v_1 and v_2, respectively. In particular, iterates of A stretch any domain in $\mathbb{R}^2$ along one v_1-direction while shrink along v_2-direction. This implies that $P^4(1,r) \hookrightarrow Y^4(1,v_1)$ for any $r > 0$.

- Note that embeddings produced in this way can only be induced by a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ with $\mathrm{tr}(A) > 2$. Its eigenvalue has to be an algebraic number solving $x^2 - \mathrm{tr}(A)x + 1 = 0$. The eigenvector v can be expressed in terms of the radical expression.

Main result

• Our Theorem answers Gong-Xue's question (2020) on arbitrarily large embeddings into $Y^4(1, v)$ for other irrational v's that are **NOT** the eigenvectors of any $A \in \mathrm{SL}_2(\mathbb{Z})$ as above, in an affirmative way.

Theorem (F.-Zhang, 2024)

Let v be an irrational unit vector in $\mathbb{R}^{n\geq 2}$, then there exist a symplectic embedding from $P^{2n}(r)$ to $Y^{2n}(1, v)$ for any $r > 0$.

• In particular, any subbundle of $T^*\mathbb{T}^n$ with bounded fibers in $\mathbb{R}^n$ can be symplectically embedded into $Y^{2n}(1, v)$.

• The method for proving this Theorem is not applicable when v is rational. And when v is rational, there exists some $r > 0$ such that there is **NO** symplectic embedding from $P^{2n}(r)$ to $Y^{2n}(1, v)$.

New question

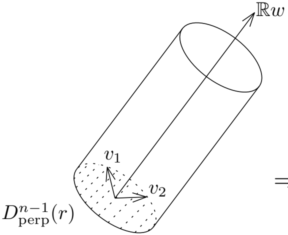
When the dimension $2n \geq 6$ (so $n \geq 3$), instead of the **"fat cylinder"**

$$Y^{2n}(r, v) := \mathbb{T}^n \times (-r, r)v \times v^{\perp}$$

one can consider the following **"thin cylinder"**,

$$X^{2n}(r, w) := \mathbb{T}^n \times D^{n-1}_{\mathrm{perp}}(r) \times \mathbb{R}w$$

where $D^{n-1}_{\mathrm{perp}}(r)$ is a disk of radius r in $\mathbb{R}^n$, $(n-1)$-dimensional, and perpendicular to the w-direction.



tilted "thin" cylinder $X^6(r, w)$
unbounded on the direction w

$v_1 \perp w, v_2 \perp w$
$\Rightarrow X^6(r, w) \subset Y^6(r, v_1) \cap Y^6(r, v_2)$

New question

- If there is a rational vector v perpendicular to w, then

$$X^{2n}(r,w) \subset Y^{2n}(r,v) \Rightarrow c(X^{2n}(r,v)) \leq r\|v\|$$

In a somewhat opposite direction, here is another question.

Question (private communications with Xue, 2024)

Suppose $D^{n-1}_{\mathrm{perp}}(r)$ contains no rational vectors, does there still exist symplectic embedding from $P^{2n}(r)$ to $X^{2n}(1,w)$ for any $r > 0$?

Theorem (F.-Zhang, 2024)

Let w be a unit (irrational) vector in $\mathbb{R}^3$ admitting a ***biased approximation****, then there exist a symplectic embedding from $P^6(r)$ to $X^6(1,w)$ for any $r > 0$.*

- If $v = (v_1, v_2, v_3)$ is $\mathbb{Q}$-linear dependent, e.g. $v = (1, \sqrt{2}, \sqrt{2} - 1)$, then v does not admit a biased approximation by Theorem before.

Biased approximation

Definition

A non-zero irrational vector $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ admits a biased approximation if there exist a sequence $\{Q_i\}_{i\in\mathbb{N}}$ and a sequence of coprime triples $\{(p_{i,1}, p_{i,2}, p_{i,3})\}_{i=1}^{\infty}$ such that

$$\left|\frac{p_{i,1}}{p_{i,3}} - \frac{v_1}{v_3}\right| < \frac{1}{|p_{i,3}|Q_i} \text{ and } \left|\frac{p_{i,2}}{p_{i,3}} - \frac{v_2}{v_3}\right| < \frac{1}{|p_{i,3}|Q_i}$$

with $p_{i,3} \to \infty$ and $Q_i \to \infty$ as $i \to \infty$, and there exist a constant $C > 0$ (independent of coprime triples above) such that

$$|p_{i,1}v_2 - p_{i,2}v_1| < \min\left\{\frac{C}{Q_i}|p_{i,2}v_3 - p_{i,3}v_2|, \frac{C}{Q_i}|p_{i,1}v_3 - p_{i,3}v_1|\right\}$$

- If $1, \alpha_1, \alpha_2$ is a basis of a real algebraic number field of degree 3, e.g. $(\alpha_1, \alpha_2) = (\sqrt[3]{2}, \sqrt[3]{4})$, then $v = (1, \alpha_1, \alpha_2)$ admits a biased approximation.
- The set of vectors admit biased approximation is generic in $\mathbb{R}^3$.

Proof (for $n = 2$)

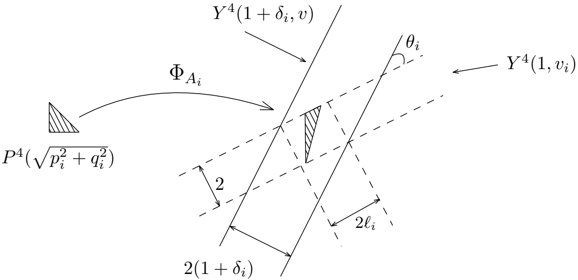
- For any irrational vector v, we will construct the following data:

(1) an appropriate sequence $\{A_i \in \mathrm{SL}_2(\mathbb{Z})\}_{i\in\mathbb{N}}$ such that their shrinking directions v_i approximates v;

(2) a sequence of positive scalars $r_i \to \infty$ such that for the linear map Ψ_{A_i} as above, we have $\Phi_{A_i} : P^4(r_i) \hookrightarrow Y^4(1, v_i)$,

so that $\Phi_{A_i}(P^4(r_i)) \subset Y^4(1 + \delta_i, v)$ where δ_i is **uniformly bounded**.

- A schematic picture is given below.



Proof (for $n = 2$)

Here is the recipe to cook up vectors v_i , matrices A_i and scalars r_i. Up to a rescaling, assume $v = (1, \kappa)$ for an irrational κ.

(1) Due to **Dirichlet's approximation theorem**, there exists a sequence of coprime pairs $\{(p_i, q_i)\}_{i\in\mathbb{N}}$ satisfying

$$\left|\frac{p_i}{q_i} - \kappa\right| < \frac{1}{q_i^2} \quad \text{and} \quad \lim_{i\to\infty} q_i = \infty.$$

Then the approximating v_i is a rescaling of vector (p_i, q_i).

(2) By **Bézout's identity**, there exist $a_i, c_i \in \mathbb{Z}$ such that $a_i p_i + c_i q_i = 1$ and $|a_i| \leq |q_i|, |c_i| \leq |p_i|$. Take

$$A = \begin{pmatrix} a_i & -q_i \\ c_i & p_i \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(3) Take $r_i := \sqrt{p_i^2 + q_i^2}$ (hence, $r_i \to \infty$ as $i \to \infty$).

Then one can verify that $\delta_i \leq \frac{p_i^2+q_i^2}{q_i^2} \leq C\kappa^2$ for some C.

THANK YOU!