Other Regularization Techniques

Part 1:
Bounds Constraints
Sparsity and L1 Constraint

Part 2:
Compressive Sensing
Total Variation

Sparsity and Compressive Sensing

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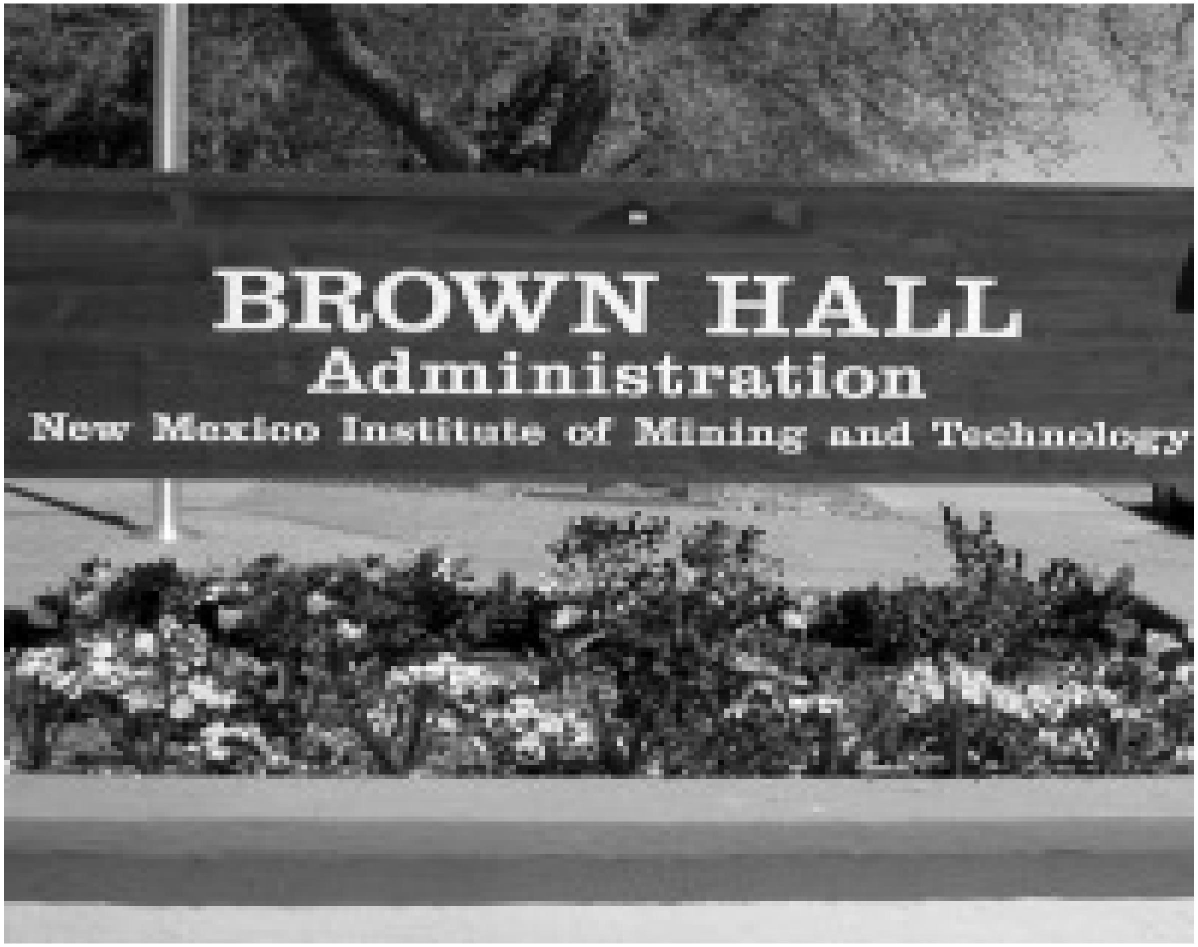
The basis vectors could be wavelets, Fourier coefficients, etc.

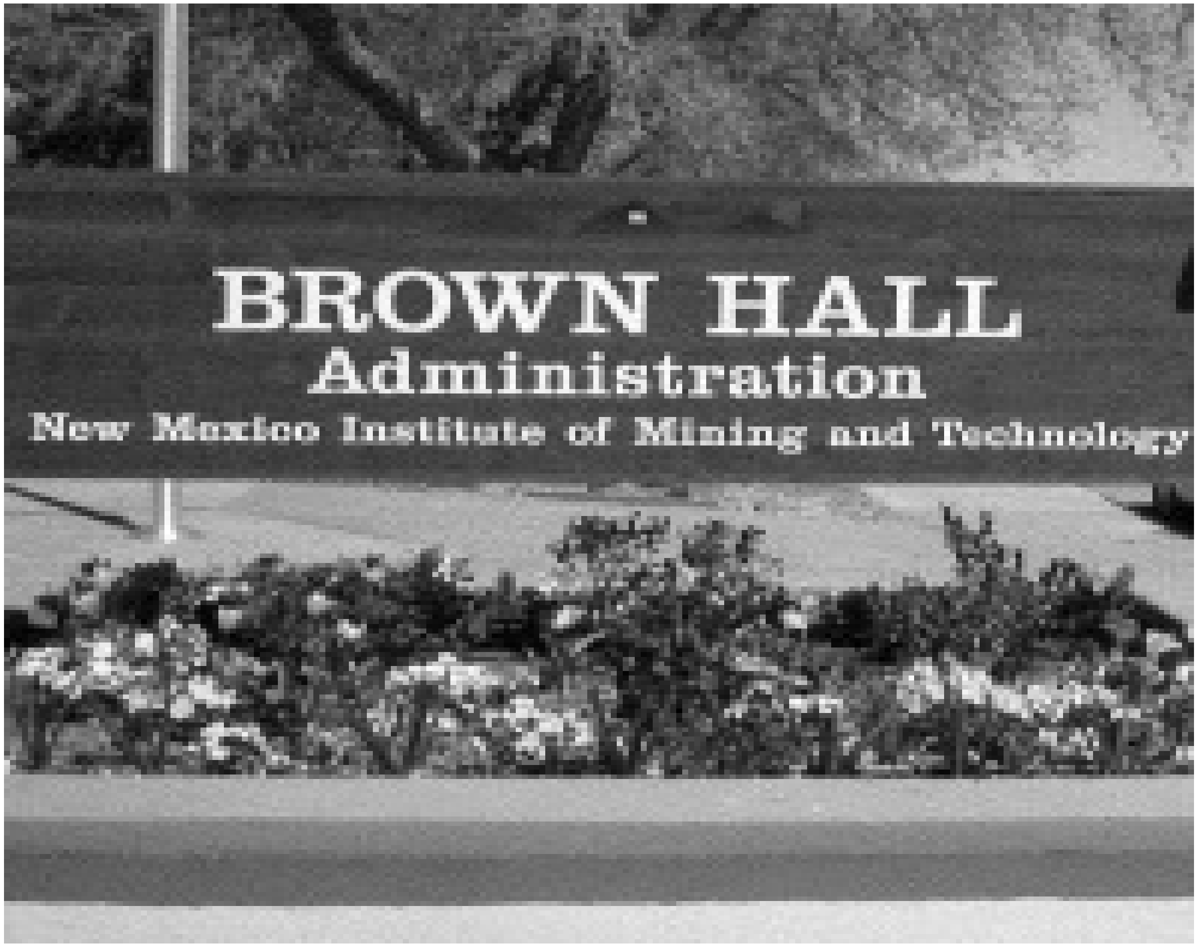
Sparse representation is very important in sound and image compression!

Apply discrete cosine transform (DCT) to an image:



The DCT has 40,000 coefficients!

Throw away (set to 0) 60% of the DCT) coefficients, and transform back to a compressed version of the image:



Only 16,000 coefficients needed to recover image beautifully!

How can this be applied in geophysics?

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Tomographic inversion using ℓ_1-norm regularization of wavelet coefficients

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SUMMARY

We propose the use of ℓ_1 regularization in a wavelet basis for the solution of linearized seismic tomography problems $\mathbf{Am} = \mathbf{d}$, allowing for the possibility of sharp discontinuities superimposed on a smoothly varying background. An iterative method is used to find a sparse solution $\mathbf{m}$ that contains no more fine-scale structure than is necessary to fit the data $\mathbf{d}$ to within its assigned errors.

Key words: inverse problem, one-norm, sparsity, tomography, wavelets.

QUESTIONS?

Compressive Sensing (CS)

The idea here is that we extract "samples" of a signal **m**(t) and from the sparse samples try to reconstruct the original signal with reasonable accuracy.

The sparse sampling is combined with sparse representation of the signal using an appropriate set of basis functions:

$$\mathbf{m} = \mathbf{W}\,\mathbf{x}$$

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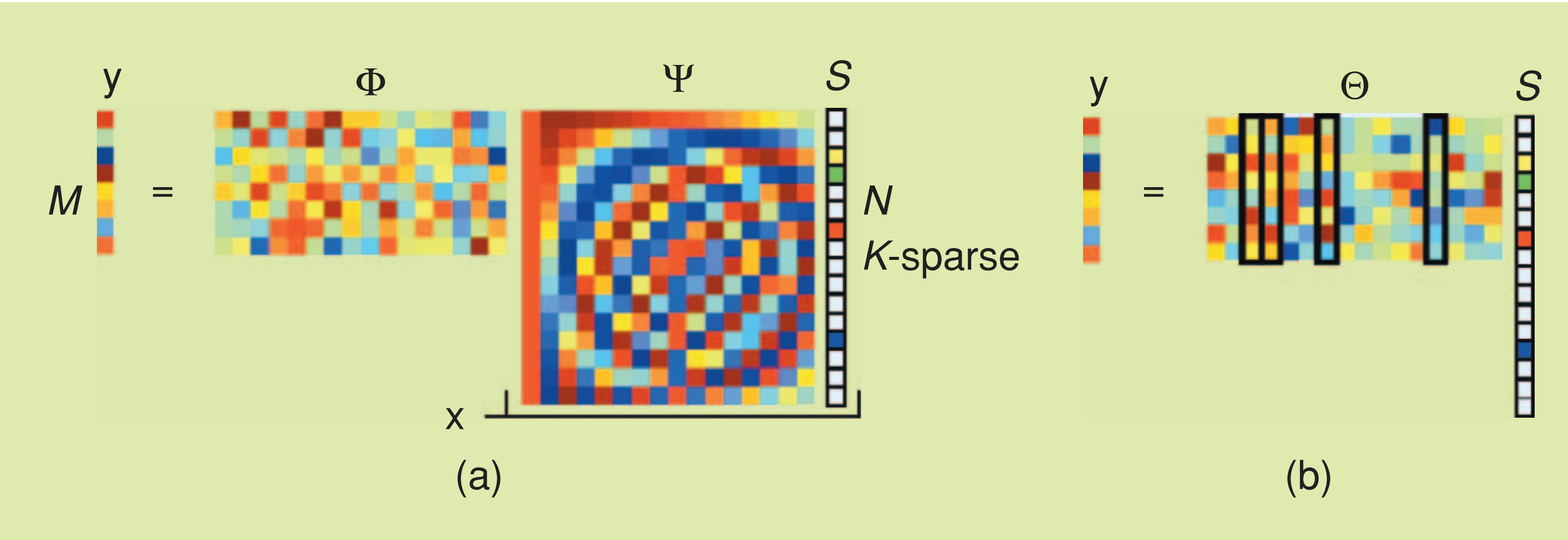
The surprising aspect of CS is that it works by evaluating linear combinations of the original signal with random vectors!

Compressive Sensing

Richard G. Baraniuk



[FIG1] (a) Compressive sensing measurement process with a random Gaussian measurement matrix Φ and discrete cosine transform (DCT) matrix Ψ. The vector of coefficients s is sparse with $K = 4$. (b) Measurement process with $\Theta = \Phi\Psi$. There are four columns that correspond to nonzero s_i coefficients; the measurement vector y is a linear combination of these columns.

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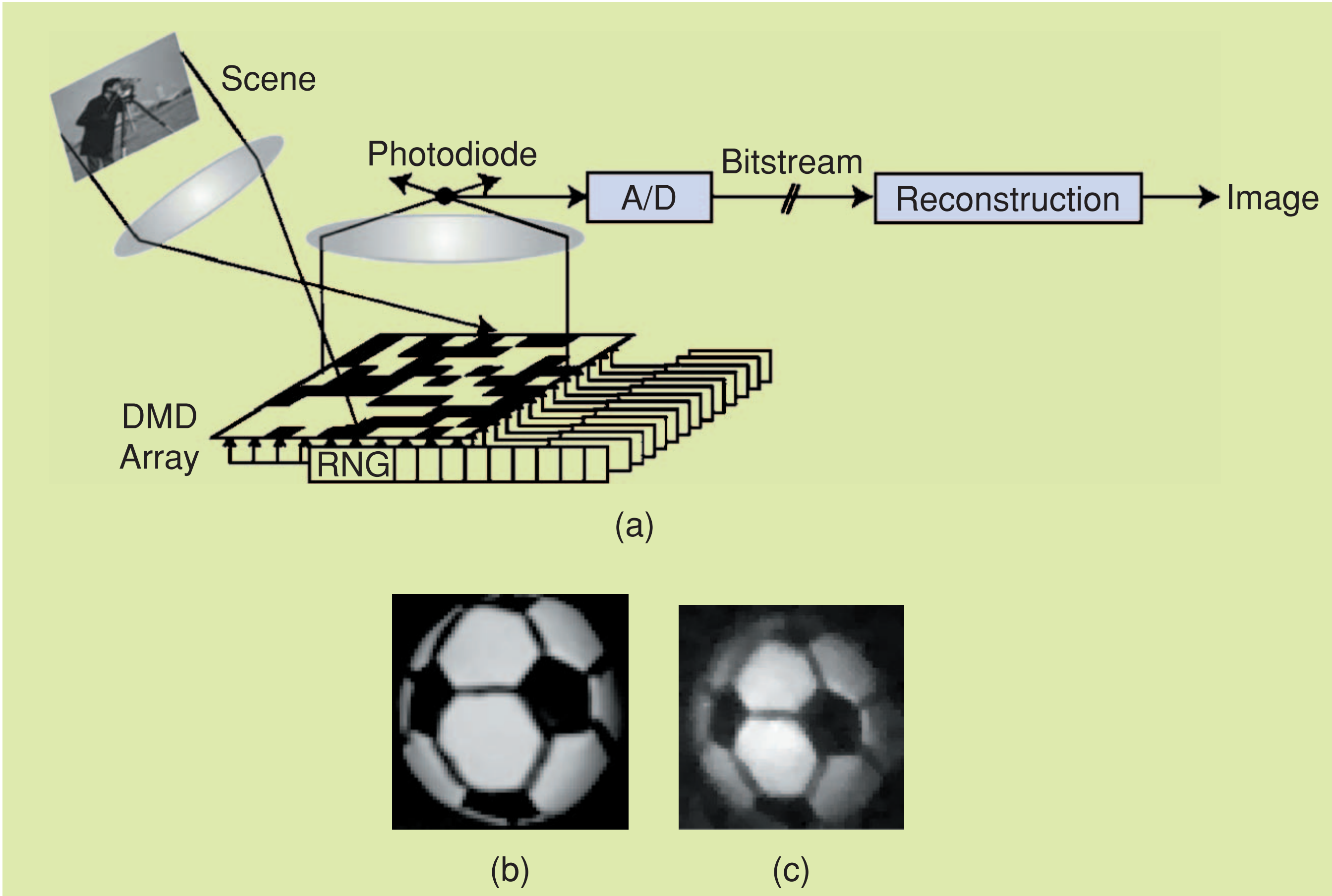
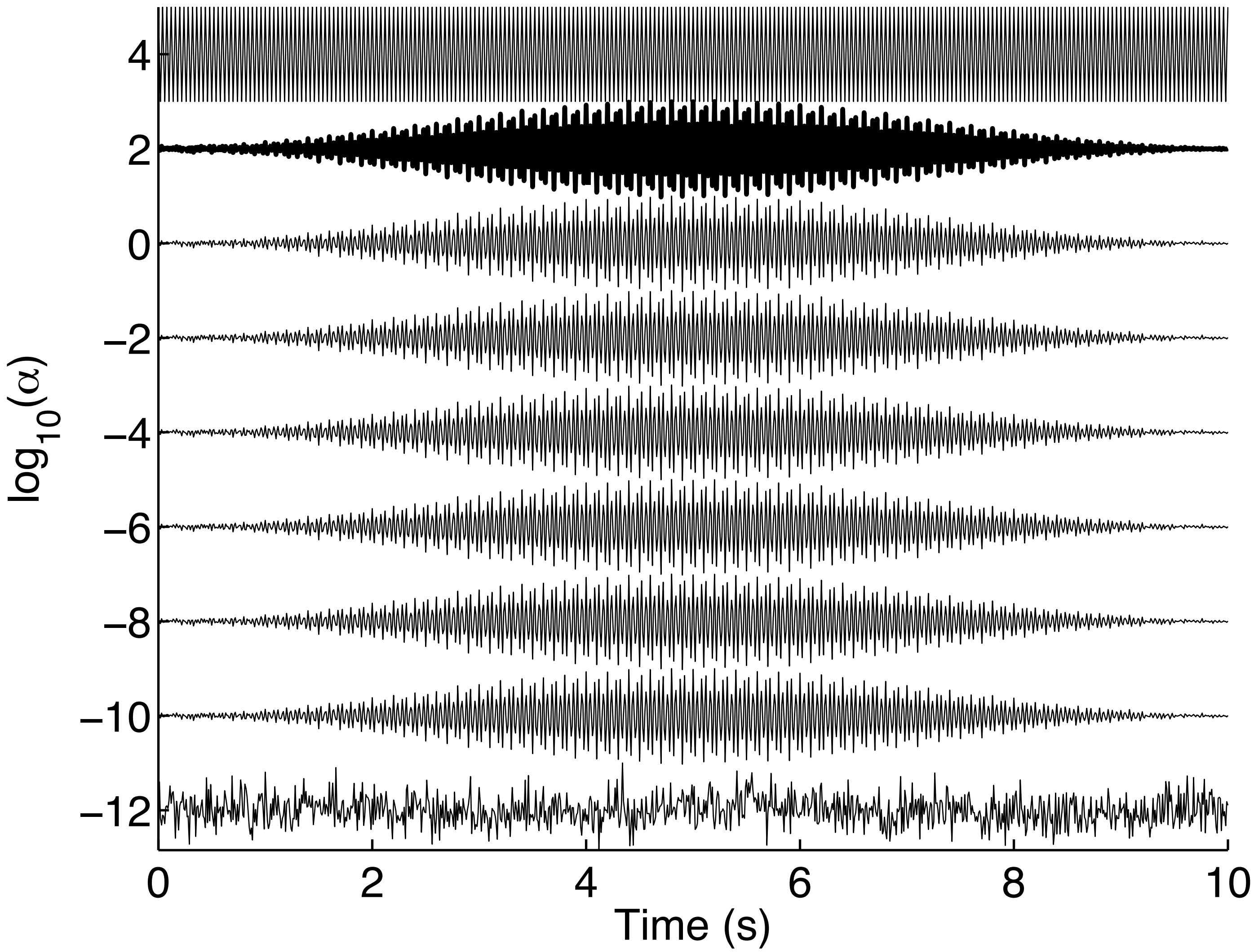


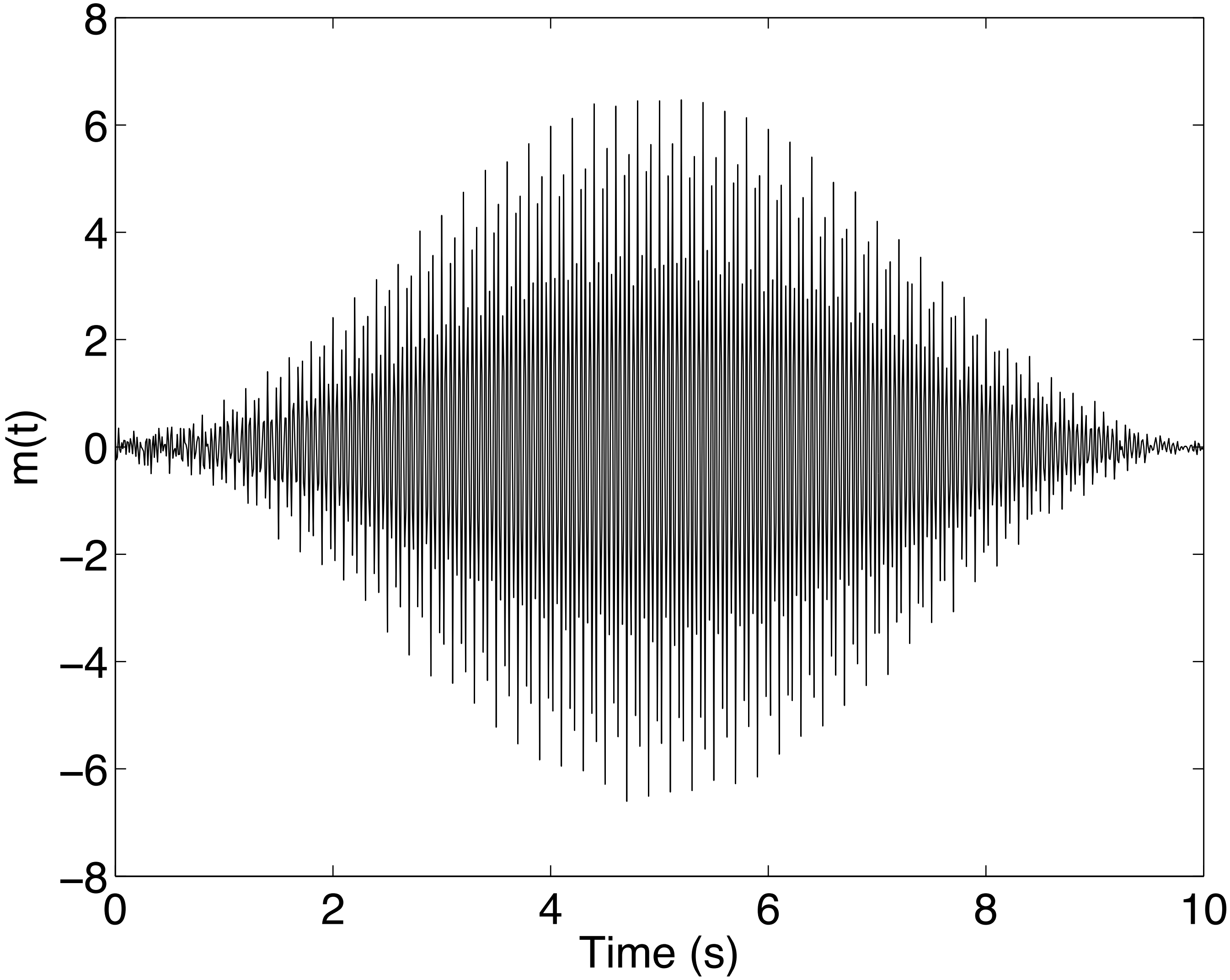
[FIG3] (a) Single-pixel, compressive sensing camera. (b) Conventional digital camera image of a soccer ball. (c) 64 $\times$ 64 black-and-white image $\widehat{x}$ of the same ball ($N = 4{,}096$ pixels) recovered from $M = 1{,}600$ random measurements taken by the camera in (a). The images in (b) and (c) are not meant to be aligned.

MATLAB
Example

Compressive sensing solutions for different regularizations



Preferred compressive sensing solution



QUESTIONS?

Compressive Sensing Example from the Literature

Tomographic SAR Inversion by L_1-Norm Regularization—The Compressive Sensing Approach

Xiao Xiang Zhu, *Student Member, IEEE*, and Richard Bamler, *Fellow, IEEE*

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Compressive Sensing Example from the Literature

Abstract—Synthetic aperture radar (SAR) tomography (TomoSAR) extends the synthetic aperture principle into the elevation direction for 3-D imaging. The resolution in the elevation direction depends on the size of the elevation aperture, i.e., on the spread of orbit tracks. Since the orbits of modern meter-resolution spaceborne SAR systems, like TerraSAR-X, are tightly controlled, the tomographic elevation resolution is at least an order of magnitude lower than in range and azimuth. Hence, super-resolution reconstruction algorithms are desired. The high anisotropy of the 3-D tomographic resolution element renders the signals *sparse* in the elevation direction; only a few pointlike reflections are expected per azimuth–range cell. This property suggests using compressive sensing (CS) methods for tomographic reconstruction. This paper presents the theory of 4-D (differential, i.e., space–time) CS TomoSAR and compares it with parametric (nonlinear least squares) and nonparametric (singular value decomposition) reconstruction methods. Super-resolution properties and point localization accuracies are demonstrated using simulations and real data. A CS reconstruction of a building complex from TerraSAR-X spotlight data is presented.

IV. TomoSAR via CS

A. CS

CS is a new and popular approach for sparse signal reconstruction. A signal of interest $\mathbf{x}$ with a length of L is said to be K-sparse in an orthogonal basis $\mathbf{\Psi}$ if the projection coefficient vector $\mathbf{s} = \mathbf{\Psi}\mathbf{x}$ has only K nonzero or significant elements. $\mathbf{x}$ is represented by $\mathbf{\Psi}^{\mathbf{H}}\mathbf{s}$. N measurements $\mathbf{y}$ can be obtained by projecting the signal onto N random basis functions $\mathbf{\Phi}$ (the sensing matrix)

$$\mathbf{y} = \mathbf{\Phi}\mathbf{x}. \tag{5}$$

The measurement vector can be rewritten as

$$\mathbf{y} = \mathbf{\Phi}\mathbf{\Psi}^{\mathbf{H}}\mathbf{s} = \mathbf{\Theta}\mathbf{s}. \tag{6}$$

III. TomoSAR Imaging Model

For a single SAR acquisition, the focused complex-valued measurement $g_n(x_0, r_0)$ of an azimuth–range pixel (x_0, r_0) for the nth acquisition at aperture position b_n and at time t_n is the integral (tomographic projection) of the reflected signal along the elevation direction [20], as shown in Fig. 1 (the deformation term is ignored here for simplicity) [9]

$$g_n = \int_{\Delta s} \gamma(s) \exp(-j2\pi\xi_n s)\, ds, \qquad n = 1, \ldots, N \tag{1}$$

where $\gamma(s)$ represents the reflectivity function along elevation s. $\xi_n = -2b_n/(\lambda r)$ is the spatial (elevation) frequency.

continuous-space system model of (1) can be approximated by discretizing the continuous-reflectivity function along s (ignoring an inconsequential constant)

$$\mathbf{g} = \mathbf{R}\boldsymbol{\gamma} \tag{2}$$

where $\mathbf{g}$ is the measurement vector with N elements g_n, $\mathbf{R}$ is an $N \times L$ mapping matrix with $R_{nl} = \exp(-j2\pi\xi_n s_l)$, and $\boldsymbol{\gamma}$ is the discrete reflectivity vector with L elements $\gamma_l = \gamma(s_l)$. s_l $(l = 1, \ldots, L)$ denotes the discrete elevation positions. Equation (1) is an irregularly sampled discrete Fourier transform of the elevation profile $\gamma(s)$. The objective of TomoSAR is to retrieve the reflectivity profile for each azimuth–range pixel.

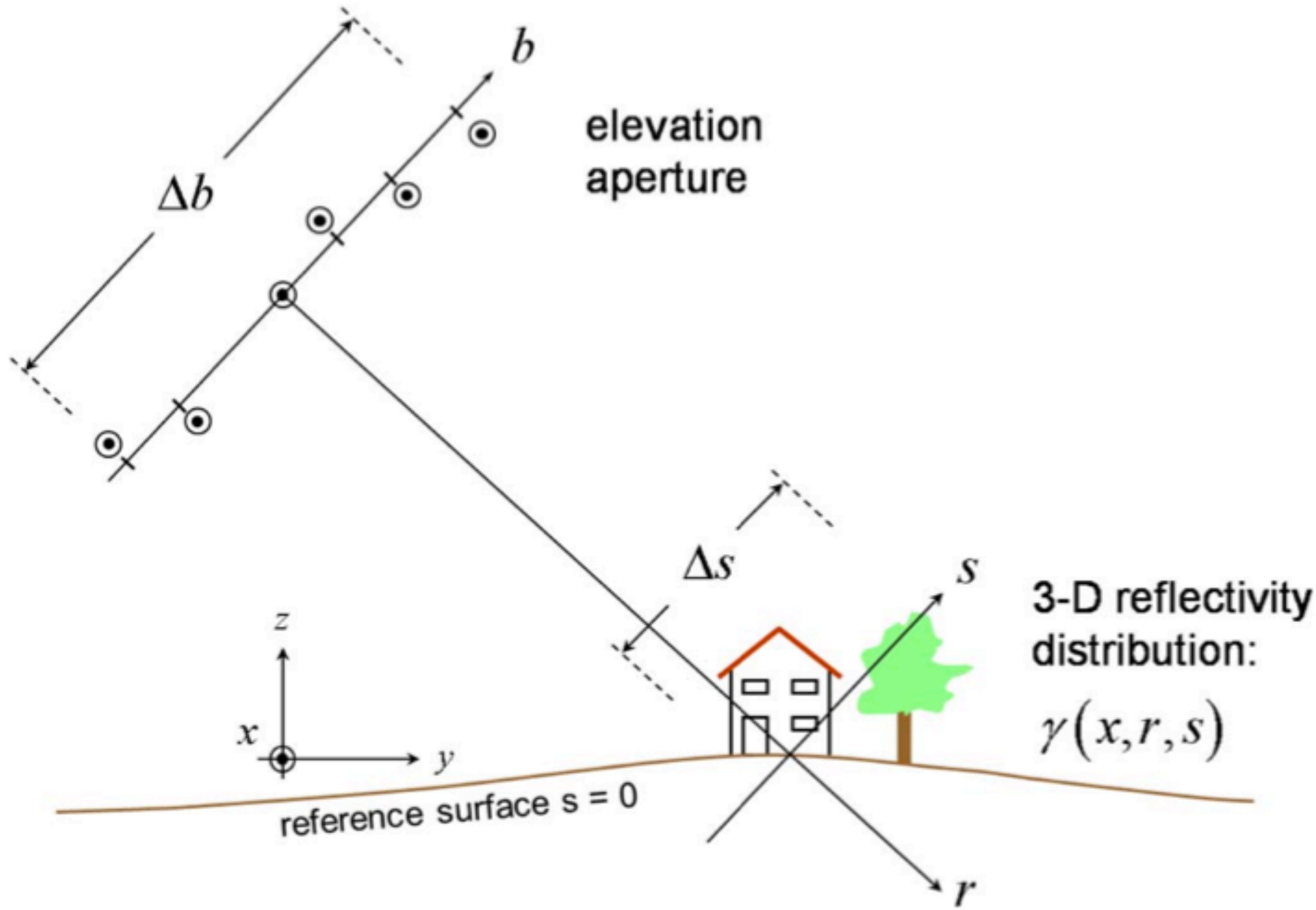


Fig. 1. TomoSAR imaging geometry. The coordinate s is referred to as elevation, and b (parallel to s) is regarded as aperture position.

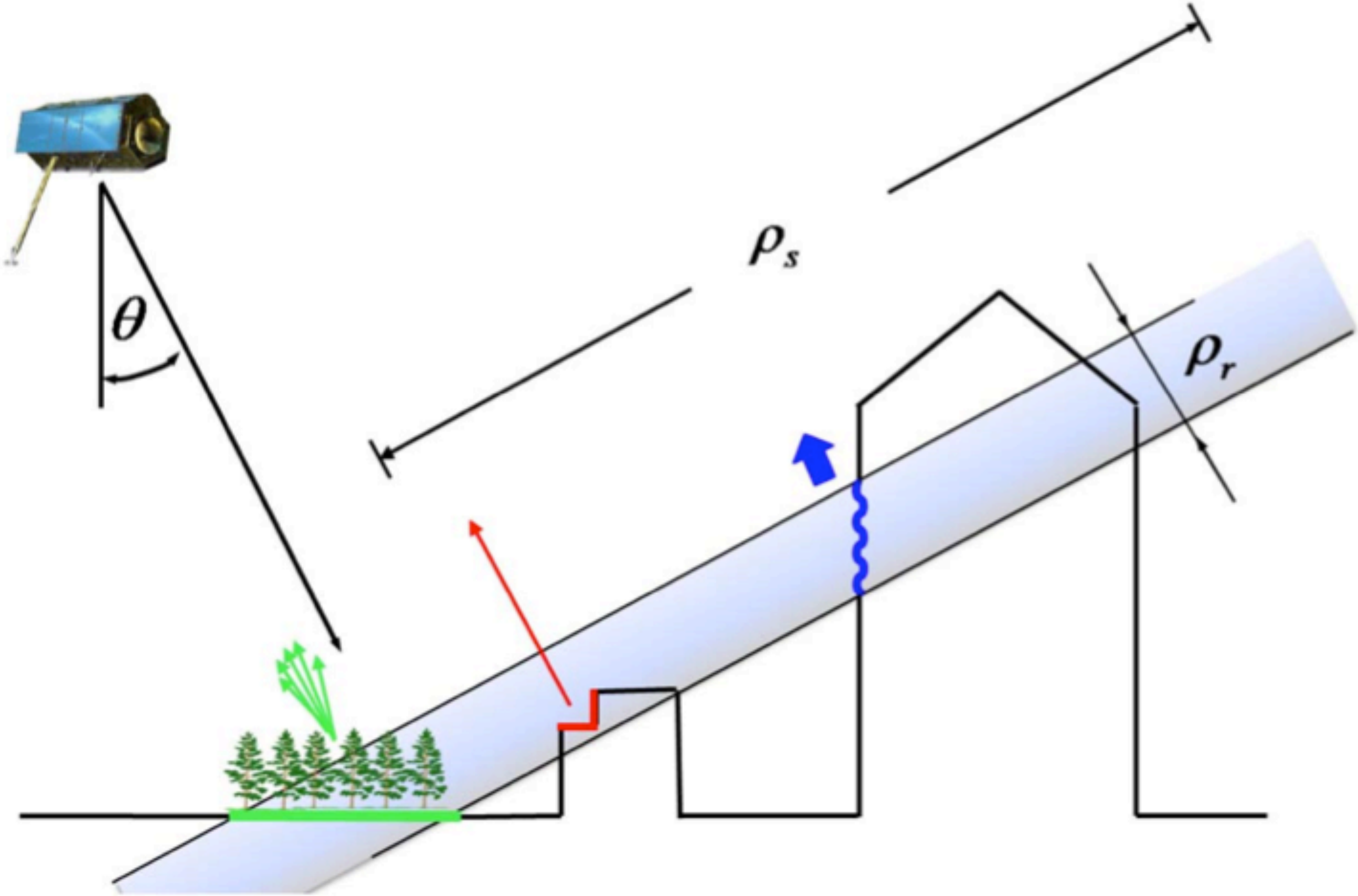


Fig. 2. Possible signal contributions in a single SAR image azimuth–range pixel. ρ_r and ρ_s: range and elevation resolutions, respectively (size of resolution cells not to scale).

C. Real Data

a) CS TomoSAR: The Las Vegas Convention Center is a very interesting test building for 3-D focusing for two reasons. First, it is very big and has a regular shape. Therefore, we are able to check the plausibility of the results. Second, it has a height of about 20 m, the critical distinguishable distance between two scatterers (one from the ground and the other from the building) by using SVD-Wiener for our elevation aperture size. The presence of two scatterers within azimuth–range pixels is expected in layover areas and has been validated by using SVD-Wiener in [22]. Thus, we are able to compare the performance of CS at the layover areas to that of the SVD-Wiener method. The left image in Fig. 8 shows the convention center visualized in Google Earth. The right image is the TerraSAR-X intensity map of the area. We choose a reference pixel according to Adam *et al.* [28], which has most likely only a single scatterer inside. The bright blue line shows the position of the analysis slice, and the area marked by the red block is a layover area. From the Google Earth image, we can see that there is a small triangular-shaped plaza on the ground made of the same material as the building. Thereby, multiple scatterers are expected.

Fig. 8. (Left) Las Vegas Convention Center (Google Earth). (Right) TerraSAR-X intensity map.

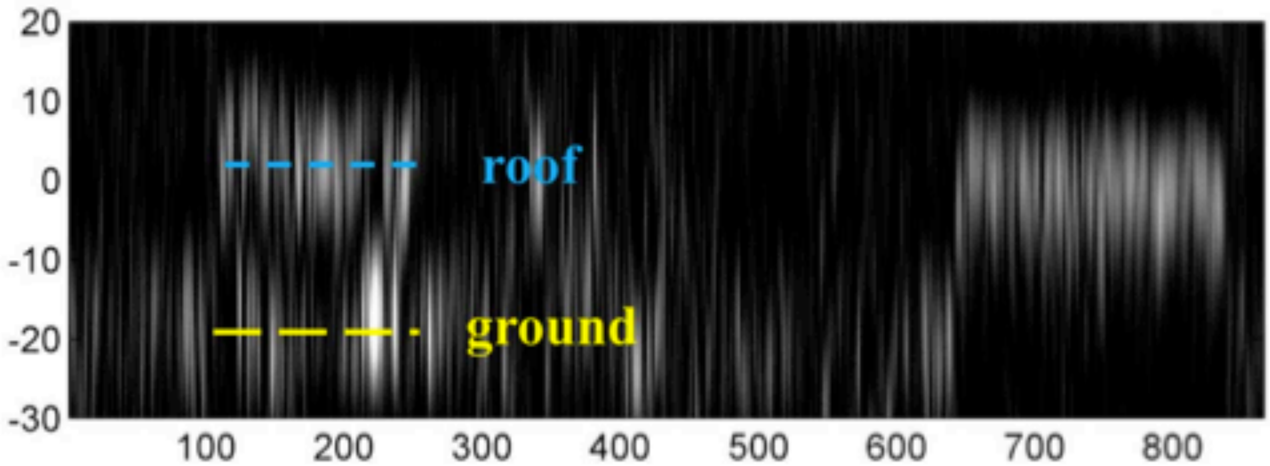
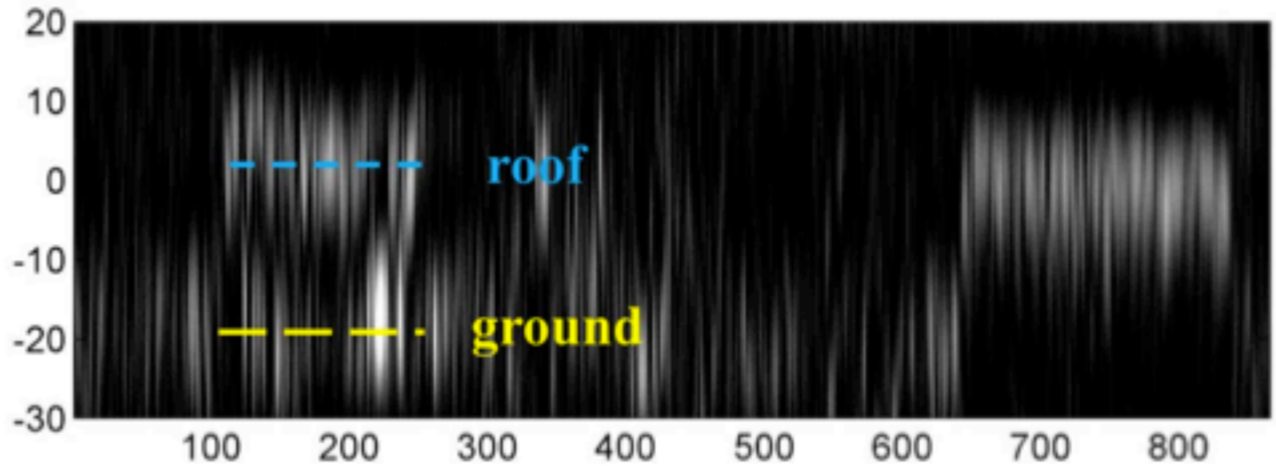


Fig. 9. Estimated reflectivity with SVD-Wiener shown in the azimuth–elevation plane [horizontal: azimuth; vertical: elevation, converted to height (in meters)].



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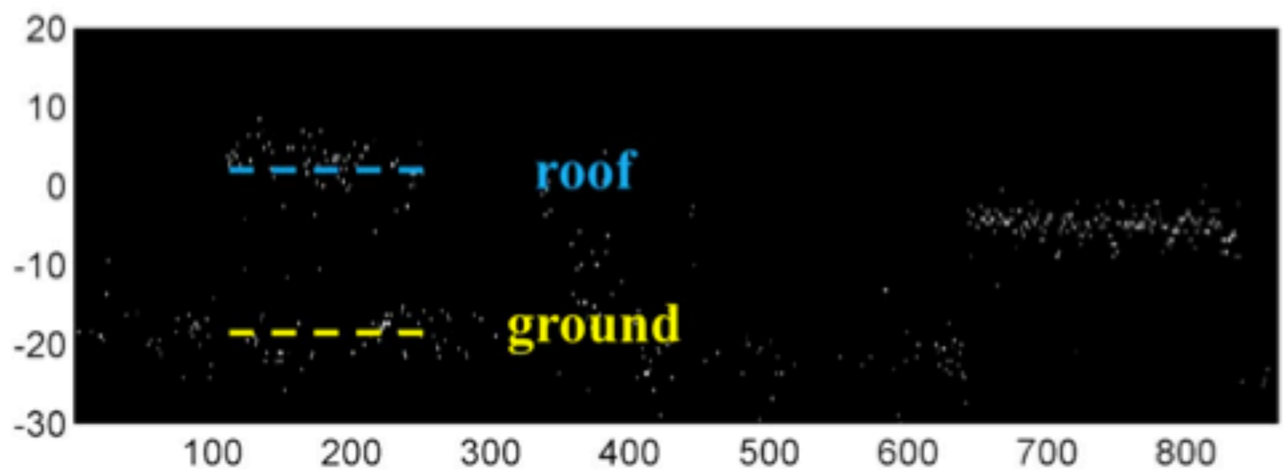

Fig. 10. Same slice as Fig. 9 but estimated by CS.

QUESTIONS?

Total Variation

For some models, we might expect there to be discontinuities. The methods we have discussed so far all would penalize sharp model changes quite heavily! So we need another approach if we believe sharp changes/discontinuities may be present.

Total Variation

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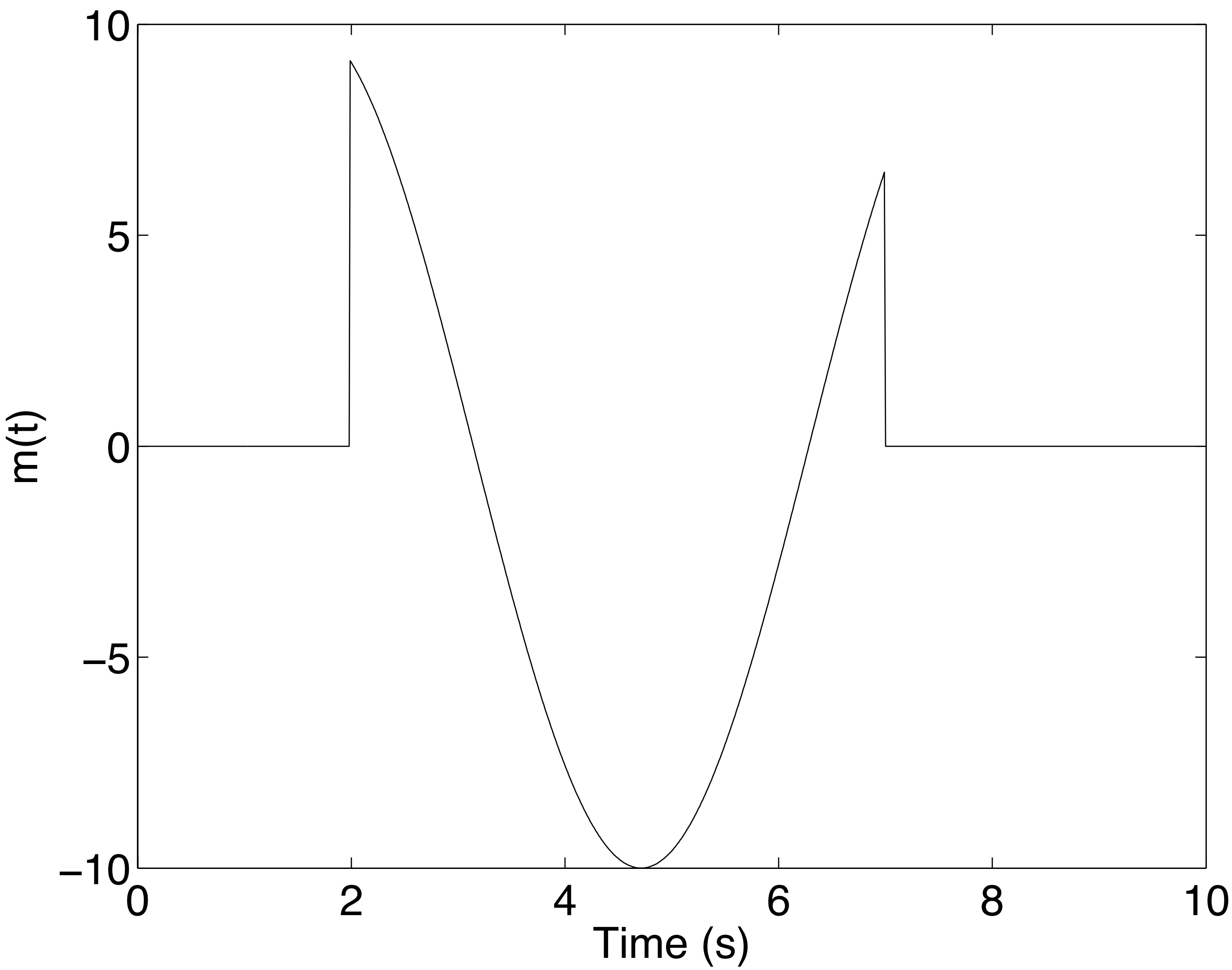
Total Variation (TV) is very effective in such cases. In a 1D case, the penalty function is

$$\begin{aligned} TV(\mathbf{m}) &= \sum_{i=1}^{n-1} |m_{i+1} - m_i| && (7.26) \\ &= \|\mathbf{Lm}\|_1 && (7.27) \end{aligned}$$

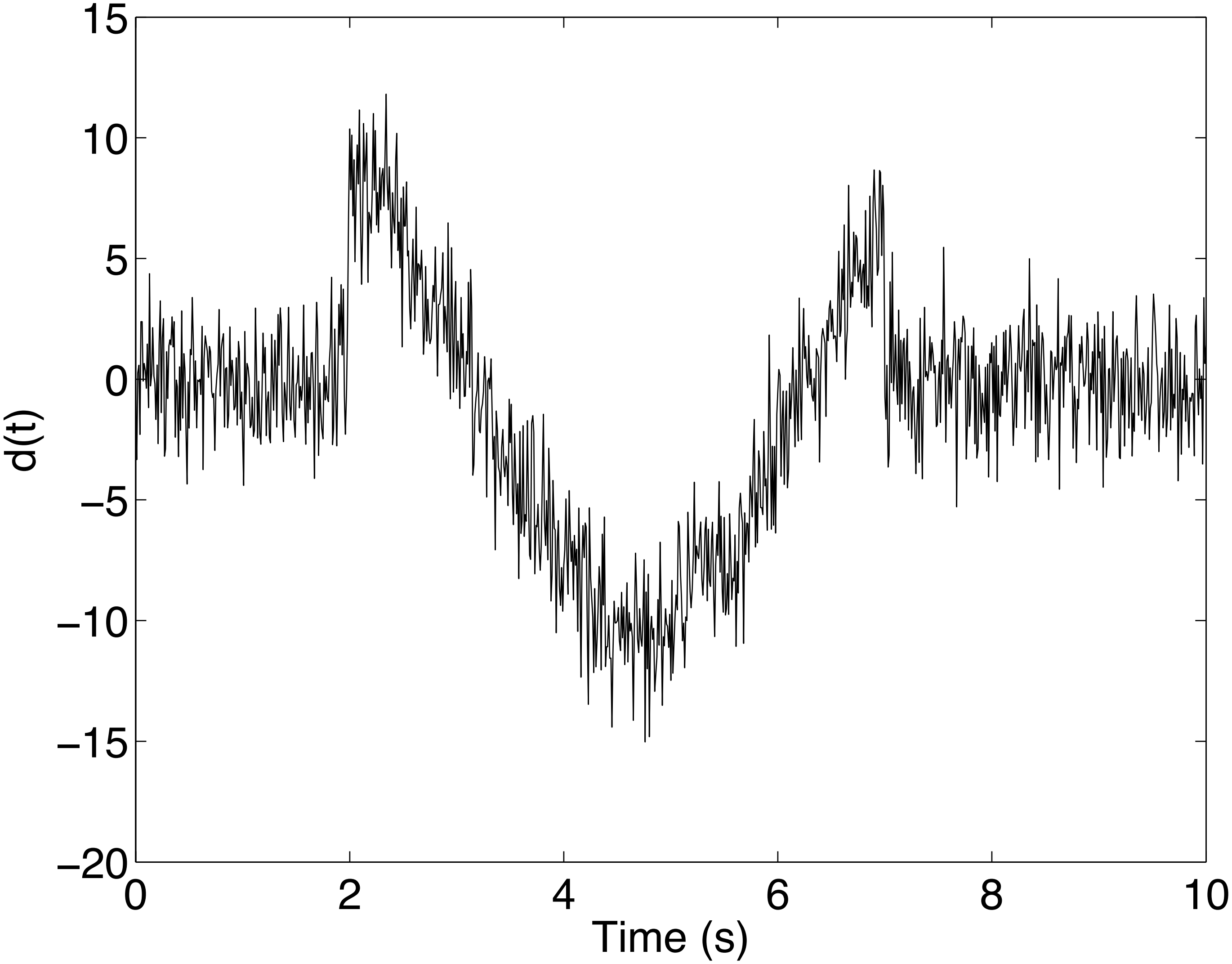
and the equation to be solved is

$$\min \|\mathbf{Gm} - \mathbf{d}\|_2^2 + \alpha \|\mathbf{Lm}\|_1 \tag{7.28}$$

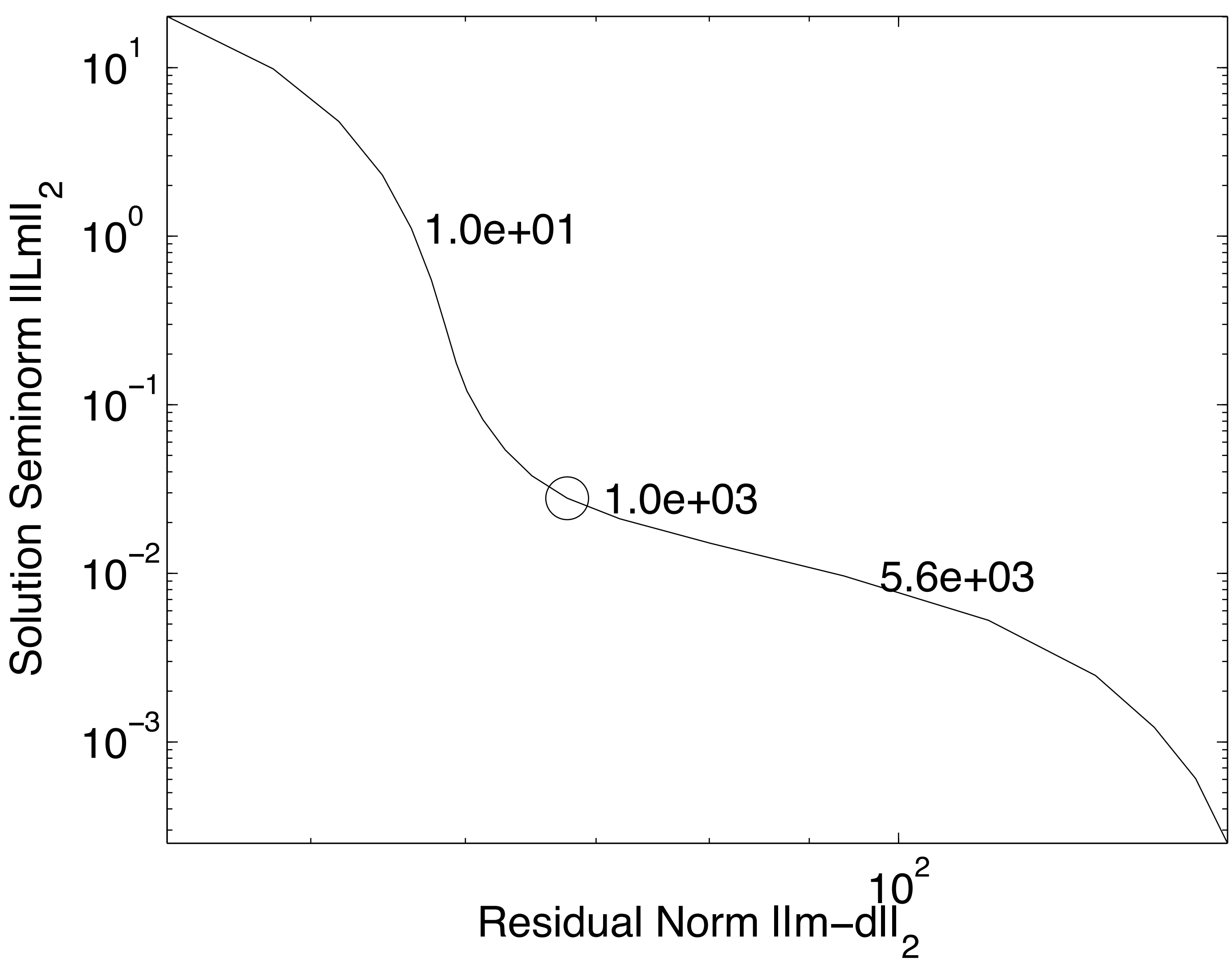
Noise-Free Model



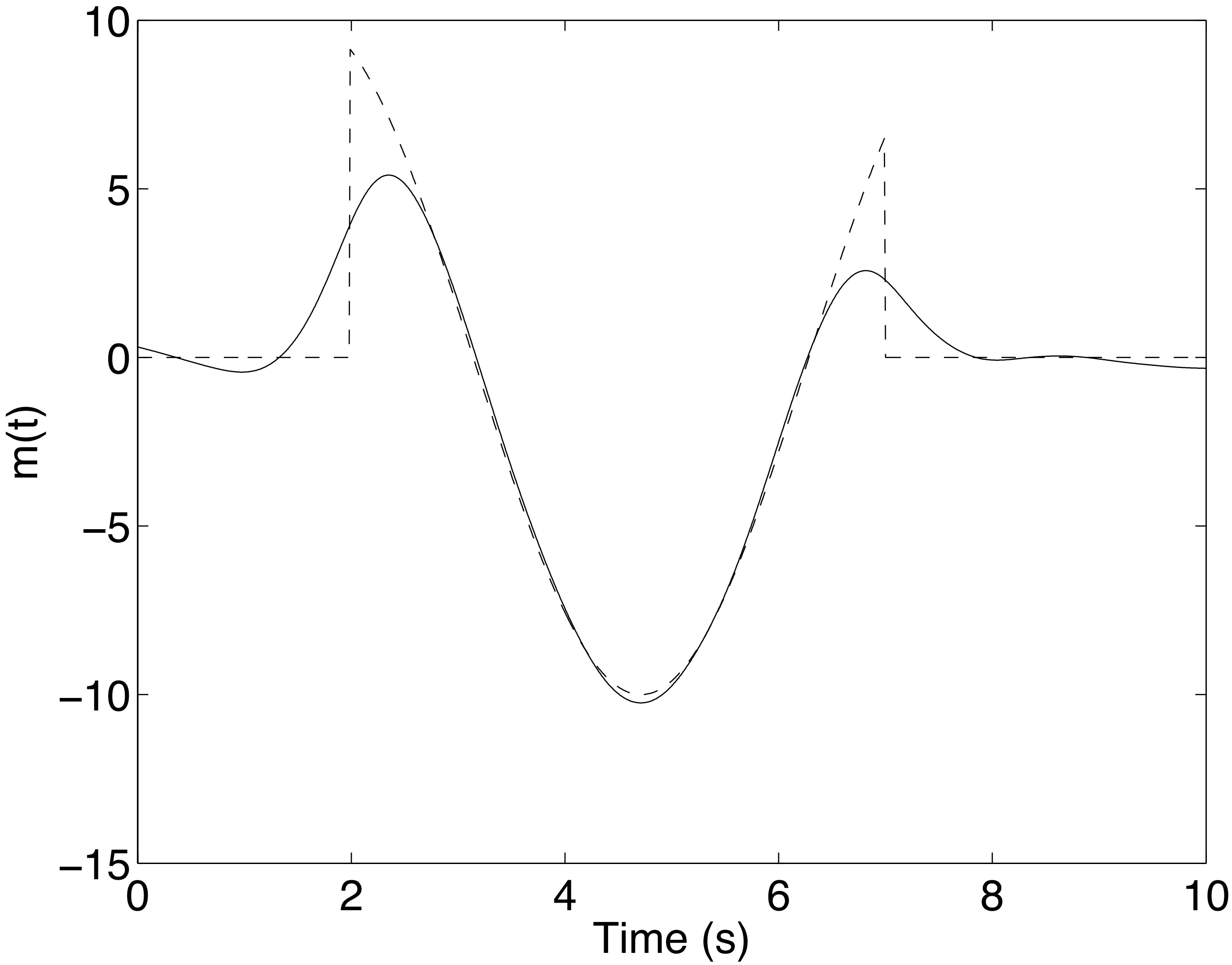
Model with Noise



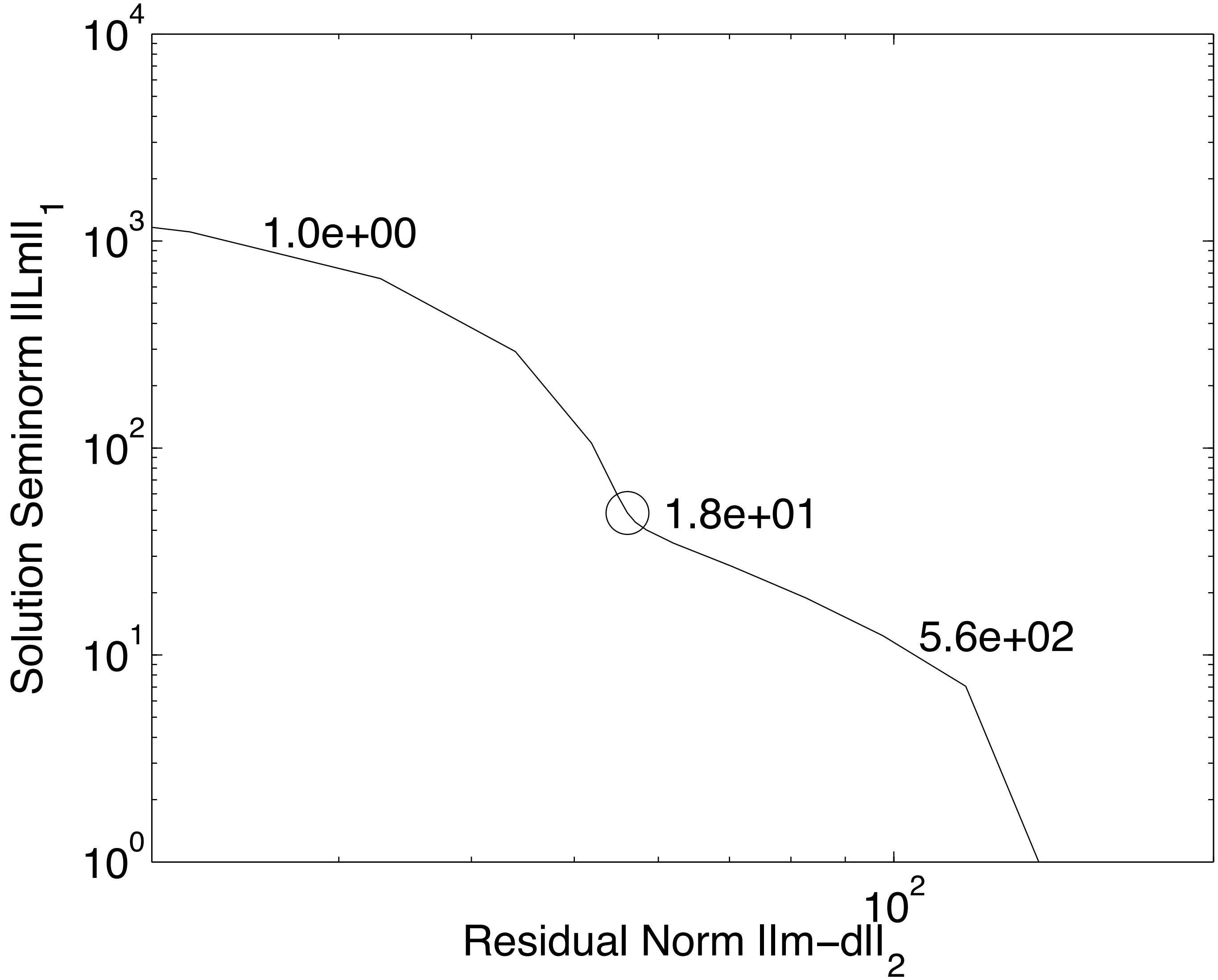
Second-Order Tikhonov L-Curve



Second-Order Tikhonov Solution

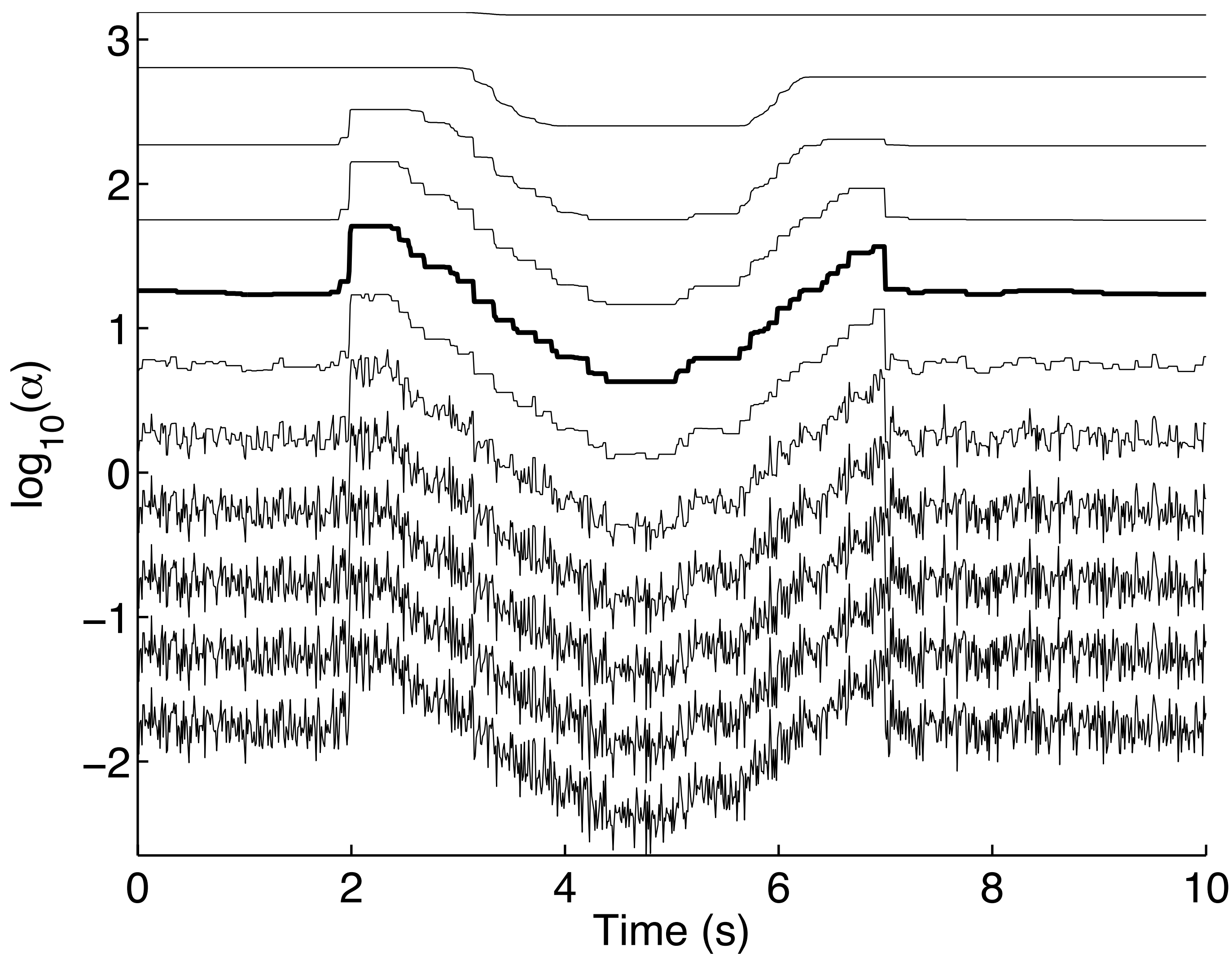


TV L-Curve

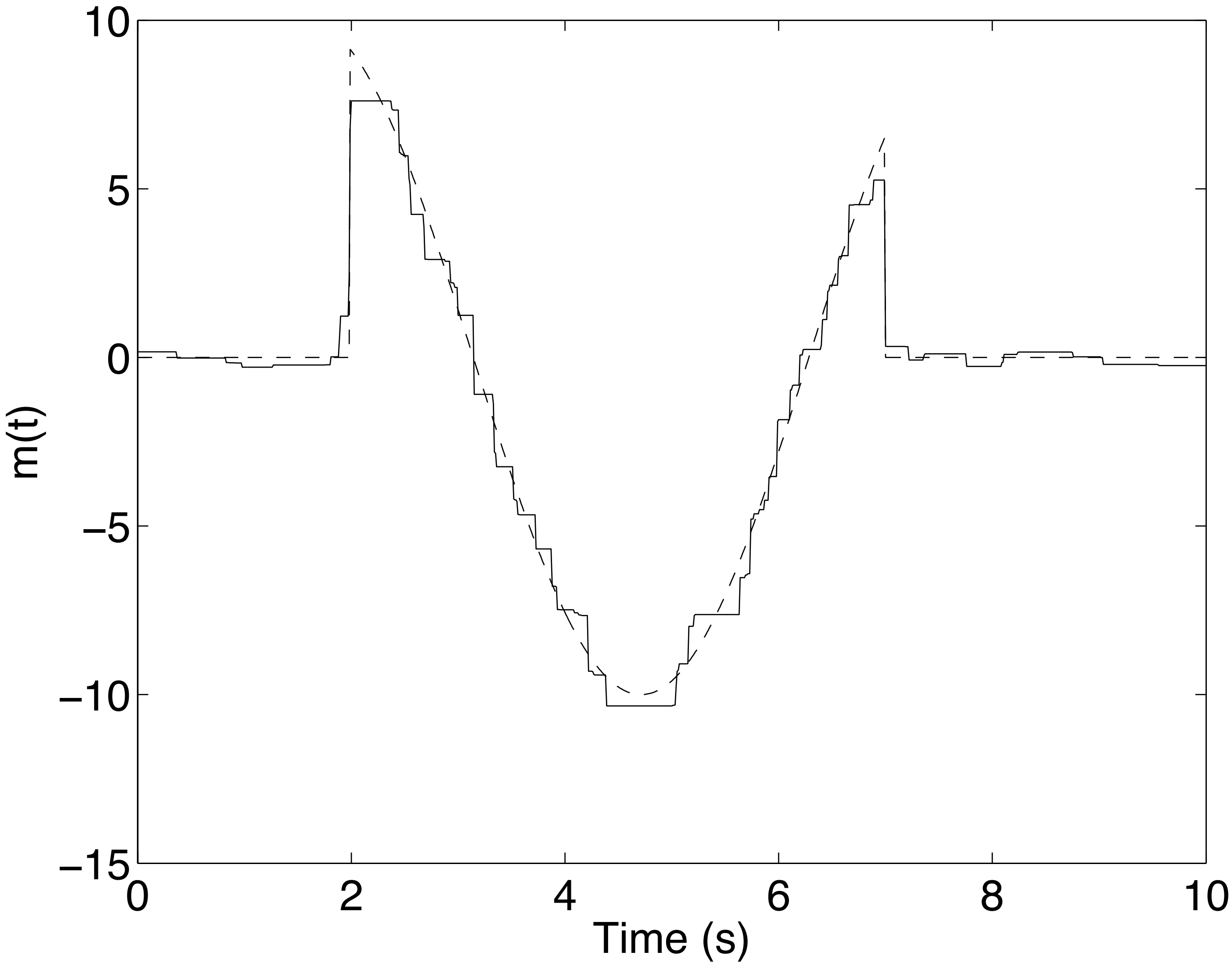


10^4
10^3
10^2
10^1
10^0
Solution Seminorm $||Lm||_1$
1.0e+00
1.8e+01
5.6e+02
10^2
Residual Norm $||m-d||_2$

Set of TV Solutions



Selected TV Solution



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