习题 (9.3.8). 设 $z = x^2 + y^2$ ，其中 $y = y(x)$ 为由方程 $x^2 - xy + y^2 = 1$ 所确定的函数，求 $\frac{\mathrm{d}z}{\mathrm{d}x}$ 及 $\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}$.

解答: 由 $z = x^2 + y^2$ 及 $x^2 - xy + y^2 = 1$ 所确定的 $y = y(x)$，求 $\frac{\mathrm{d}z}{\mathrm{d}x}$ 和 $\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}$。

首先由隐函数方程求 $\frac{\mathrm{d}y}{\mathrm{d}x}$。对方程 $x^2 - xy + y^2 = 1$ 两边关于 x 求导：

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 - xy + y^2\right) &= 0, \\
2x - \left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0, \\
2x - y - x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0, \\
(2y - x)\frac{\mathrm{d}y}{\mathrm{d}x} &= y - 2x, \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y - 2x}{2y - x}.
\end{aligned}$$

利用链式法则求 $\frac{\mathrm{d}z}{\mathrm{d}x}$：

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \\
&= 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} \\
&= 2x + 2y \cdot \frac{y - 2x}{2y - x}.
\end{aligned}$$

合并为单分数：

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}x} &= \frac{2x(2y - x) + 2y(y - 2x)}{2y - x} \\
&= \frac{4xy - 2x^2 + 2y^2 - 4xy}{2y - x} \\
&= \frac{2(y^2 - x^2)}{2y - x}.
\end{aligned}$$

为求二阶导数，令 $u = 2(y^2 - x^2)$，$D = 2y - x$，则 $\frac{\mathrm{d}z}{\mathrm{d}x} = u/D$。于是

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{u'D - uD'}{D^2},$$

其中撇号表示对 x 求导。计算 u' 与 D'：

$$\begin{aligned}
u' &= 2\left(2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\right) = 4y\frac{\mathrm{d}y}{\mathrm{d}x} - 4x, \\
D' &= 2\frac{\mathrm{d}y}{\mathrm{d}x} - 1.
\end{aligned}$$

代入 $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-2x}{2y-x}$：

$$\begin{aligned}
u' &= 4y \cdot \frac{y - 2x}{2y - x} - 4x = \frac{4y(y - 2x)}{2y - x} - 4x, \\
D' &= 2 \cdot \frac{y - 2x}{2y - x} - 1 = \frac{2(y - 2x) - (2y - x)}{2y - x} = \frac{-3x}{2y - x}.
\end{aligned}$$

进而

$$\begin{aligned} u'D &= \Big(\frac{4y(y-2x)}{2y-x} - 4x\Big)(2y-x) = 4y(y-2x) - 4x(2y-x) \\ &= 4y^2 - 8xy - 8xy + 4x^2 = 4(x^2 - 4xy + y^2), \\ uD' &= 2(y^2 - x^2) \cdot \frac{-3x}{2y-x} = \frac{-6x(y^2-x^2)}{2y-x}. \end{aligned}$$

因此

$$\begin{aligned} \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} &= \frac{4(x^2-4xy+y^2) + \dfrac{6x(y^2-x^2)}{2y-x}}{(2y-x)^2} \\ &= \frac{4(x^2-4xy+y^2)(2y-x) + 6x(y^2-x^2)}{(2y-x)^3}. \end{aligned}$$

展开分子：

$$\begin{aligned} &4(x^2-4xy+y^2)(2y-x) + 6x(y^2-x^2) \\ &= 4\big[2y(x^2-4xy+y^2) - x(x^2-4xy+y^2)\big] + 6xy^2 - 6x^3 \\ &= 4\big[2x^2y - 8xy^2 + 2y^3 - x^3 + 4x^2y - xy^2\big] + 6xy^2 - 6x^3 \\ &= 4\big[-x^3 + 6x^2y - 9xy^2 + 2y^3\big] + 6xy^2 - 6x^3 \\ &= -4x^3 + 24x^2y - 36xy^2 + 8y^3 + 6xy^2 - 6x^3 \\ &= -10x^3 + 24x^2y - 30xy^2 + 8y^3. \end{aligned}$$

故

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{-10x^3 + 24x^2y - 30xy^2 + 8y^3}{(2y-x)^3}.$$

分子分母同乘以 -1 得

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{10x^3 - 24x^2y + 30xy^2 - 8y^3}{(x-2y)^3} = \frac{2(5x^3 - 12x^2y + 15xy^2 - 4y^3)}{(x-2y)^3}.$$

习题 (9.3.10). 设 $x = x(z)$, $y = y(z)$ 是方程组 $\begin{cases} x+y+z=0, \\ x^2+y^2+z^2=1 \end{cases}$ 所确定的隐函数组，求 $\frac{\mathrm{d}x}{\mathrm{d}z}, \frac{\mathrm{d}y}{\mathrm{d}z}$.

解答: 对给定方程组关于 z 求导，将 x 和 y 视为 z 的函数。

对第一个方程求导：

$$\frac{\mathrm{d}}{\mathrm{d}z}(x+y+z) = \frac{\mathrm{d}x}{\mathrm{d}z} + \frac{\mathrm{d}y}{\mathrm{d}z} + 1 = 0 \quad\Longrightarrow\quad \frac{\mathrm{d}x}{\mathrm{d}z} + \frac{\mathrm{d}y}{\mathrm{d}z} = -1. \tag{1}$$

对第二个方程求导：

$$\frac{\mathrm{d}}{\mathrm{d}z}(x^2+y^2+z^2) = 2x\frac{\mathrm{d}x}{\mathrm{d}z} + 2y\frac{\mathrm{d}y}{\mathrm{d}z} + 2z = 0 \quad\Longrightarrow\quad x\frac{\mathrm{d}x}{\mathrm{d}z} + y\frac{\mathrm{d}y}{\mathrm{d}z} = -z. \tag{2}$$

将 (1) 和 (2) 视为关于 $\frac{\mathrm{d}x}{\mathrm{d}z}$ 和 $\frac{\mathrm{d}y}{\mathrm{d}z}$ 的线性方程组。令 $u = \frac{\mathrm{d}x}{\mathrm{d}z}, v = \frac{\mathrm{d}y}{\mathrm{d}z}$，则

$$u + v = -1,$$
$$xu + yv = -z.$$

系数行列式为

$$\Delta = \begin{vmatrix} 1 & 1 \\ x & y \end{vmatrix} = y - x = -(x - y).$$

由 Cramer 法则得

$$u = \frac{\begin{vmatrix} -1 & 1 \\ -z & y \end{vmatrix}}{\Delta} = \frac{(-1)\cdot y - 1\cdot(-z)}{y - x} = \frac{-y + z}{y - x} = \frac{y - z}{x - y},$$
$$v = \frac{\begin{vmatrix} 1 & -1 \\ x & -z \end{vmatrix}}{\Delta} = \frac{1\cdot(-z) - (-1)\cdot x}{y - x} = \frac{-z + x}{y - x} = \frac{z - x}{x - y}.$$

因此

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{y - z}{x - y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{z - x}{x - y}.$$

（上述表达式在 $x \neq y$ 时成立，此时隐函数定理适用。）利用 $x + y + z = 0$ 可进一步简化：代入 $y = -x - z$ 得

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{(-x - z) - z}{x - (-x - z)} = \frac{-x - 2z}{2x + z}, \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{z - x}{2x + z}.$$

习题 (9.3.11). 设 $u = u(x, y), v = v(x, y)$ 是由下列方程组所确定的隐函数组，求 $\frac{\partial(u,v)}{\partial(x,y)}$.

(1) $\begin{cases} u^2 + v^2 + x^2 + y^2 = 1, \\ u + v + x + y = 0; \end{cases}$

(2) $\begin{cases} xu - yv = 0, \\ yu + xv = 1; \end{cases}$

(3) $\begin{cases} u = f(ux, v + y), \\ v = g(u - x, v^2 y). \end{cases}$

解答:

1. 令

$$F = u^2 + v^2 + x^2 + y^2 - 1, \qquad G = u + v + x + y.$$

计算

$$\frac{\partial(F, G)}{\partial(u, v)} = \begin{vmatrix} 2u & 2v \\ 1 & 1 \end{vmatrix} = 2u - 2v = 2(u - v),$$
$$\frac{\partial(F, G)}{\partial(x, y)} = \begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix} = 2x - 2y = 2(x - y).$$

于是

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(F, G)/\partial(x, y)}{\partial(F, G)/\partial(u, v)} = \frac{2(x - y)}{2(u - v)} = \frac{x - y}{u - v}.$$

2. 令

$$F = xu - yv, \qquad G = yu + xv - 1.$$

计算

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2,$$

$$\frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} u & -v \\ v & u \end{vmatrix} = u^2 + v^2.$$

于是

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{u^2+v^2}{x^2+y^2}.$$

3. 将方程组改写为

$$F = u - f(ux,\, v+y) = 0, \qquad G = v - g(u-x,\, v^2 y) = 0.$$

记偏导数：

$$f_a = \frac{\partial f}{\partial a}\Big|_{(ux,\, v+y)}, \quad f_b = \frac{\partial f}{\partial b}\Big|_{(ux,\, v+y)}, \quad g_c = \frac{\partial g}{\partial c}\Big|_{(u-x,\, v^2y)}, \quad g_d = \frac{\partial g}{\partial d}\Big|_{(u-x,\, v^2y)}.$$

计算 F, G 的偏导数：

$$\begin{aligned} F_u &= 1 - x f_a, & F_v &= -f_b, \\ F_x &= -u f_a, & F_y &= -f_b, \\ G_u &= -g_c, & G_v &= 1 - 2vy g_d, \\ G_x &= g_c, & G_y &= -v^2 g_d. \end{aligned}$$

于是

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} 1 - x f_a & -f_b \\ -g_c & 1 - 2vy g_d \end{vmatrix} = (1 - x f_a)(1 - 2vy g_d) - f_b g_c,$$

$$\frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} -u f_a & -f_b \\ g_c & -v^2 g_d \end{vmatrix} = (-u f_a)(-v^2 g_d) - (-f_b)(g_c) = u v^2 f_a g_d + f_b g_c.$$

因此

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{u v^2 f_a g_d + f_b g_c}{(1 - x f_a)(1 - 2vy g_d) - f_b g_c}.$$

习题 (9.3.12). 求下列函数组所确定的反函数组的偏导数 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

(1) $\begin{cases} x = f(u,v), \\ y = g(u,v); \end{cases}$

(2) $\begin{cases} x = \mathrm{e}^u + u \sin v, \\ y = \mathrm{e}^u - u \cos v. \end{cases}$

解答:

1. 设 $a=\frac{\partial f}{\partial u}$, $b=\frac{\partial f}{\partial v}$, $c=\frac{\partial g}{\partial u}$, $d=\frac{\partial g}{\partial v}$。从 (u,v) 到 (x,y) 的映射的 Jacobi 矩阵为

$$J=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det J = ad-bc.$$

根据反函数定理，当 $\det J \neq 0$ 时，存在局部反函数 $u=u(x,y)$, $v=v(x,y)$，且其偏导数矩阵为 J 的逆矩阵：

$$J^{-1}=\frac{1}{\det J}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}.$$

因此

$$\frac{\partial u}{\partial x}=\frac{d}{\Delta}, \qquad \frac{\partial u}{\partial y}=-\frac{b}{\Delta},$$
$$\frac{\partial v}{\partial x}=-\frac{c}{\Delta}, \qquad \frac{\partial v}{\partial y}=\frac{a}{\Delta},$$

其中 $\Delta = ad-bc = \dfrac{\partial f}{\partial u}\dfrac{\partial g}{\partial v}-\dfrac{\partial f}{\partial v}\dfrac{\partial g}{\partial u}$。

2. 给定 $x=e^u+u\sin v$, $y=e^u-u\cos v$。首先计算 x, y 关于 u, v 的偏导数：

$$\frac{\partial x}{\partial u}=e^u+\sin v, \qquad \frac{\partial x}{\partial v}=u\cos v,$$
$$\frac{\partial y}{\partial u}=e^u-\cos v, \qquad \frac{\partial y}{\partial v}=u\sin v.$$

于是 Jacobi 行列式为

$$\begin{aligned}\Delta &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= (e^u+\sin v)(u\sin v)-(u\cos v)(e^u-\cos v) \\ &= ue^u\sin v+u\sin^2 v-ue^u\cos v+u\cos^2 v \\ &= u\big[e^u(\sin v-\cos v)+(\sin^2 v+\cos^2 v)\big] \\ &= u\big[1+e^u(\sin v-\cos v)\big].\end{aligned}$$

代入 (1) 中公式得

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{u\sin v}{\Delta}=\frac{\sin v}{1+e^u(\sin v-\cos v)}, \\ \frac{\partial u}{\partial y} &= -\frac{u\cos v}{\Delta}=-\frac{\cos v}{1+e^u(\sin v-\cos v)}, \\ \frac{\partial v}{\partial x} &= -\frac{e^u-\cos v}{\Delta}=-\frac{e^u-\cos v}{u\big[1+e^u(\sin v-\cos v)\big]}, \\ \frac{\partial v}{\partial y} &= \frac{e^u+\sin v}{\Delta}=\frac{e^u+\sin v}{u\big[1+e^u(\sin v-\cos v)\big]}.\end{aligned}$$

习题 (9.3.13). 设 $u=f(x,y,z),\varphi(x^2,\mathrm{e}^y,z)=0,y=\sin x$ ，其中 f,φ 都具有一阶连续偏导数，且 $\frac{\partial\varphi}{\partial z}\neq 0$. 求 $\frac{\mathrm{d}u}{\mathrm{d}x}$.

解答: 由链式法则，$u=f(x,y,z)$ 且 $y=\sin x$，$\varphi(x^2,\mathrm{e}^y,z)=0$，得

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}.$$

因为 $y=\sin x$，所以 $\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$。

为求 $\frac{\mathrm{d}z}{\mathrm{d}x}$，将约束 $\varphi(x^2,\mathrm{e}^y,z)=0$ 对 x 求导：

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x^2,\mathrm{e}^y,z)=\frac{\partial\varphi}{\partial(x^2)}\cdot\frac{\mathrm{d}(x^2)}{\mathrm{d}x}+\frac{\partial\varphi}{\partial(\mathrm{e}^y)}\cdot\frac{\mathrm{d}(\mathrm{e}^y)}{\mathrm{d}x}+\frac{\partial\varphi}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}x}=0.$$

计算得 $\frac{\mathrm{d}(x^2)}{\mathrm{d}x}=2x$，$\frac{\mathrm{d}(\mathrm{e}^y)}{\mathrm{d}x}=\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^y\cos x$，代入得

$$2x\frac{\partial\varphi}{\partial(x^2)}+\mathrm{e}^y\cos x\frac{\partial\varphi}{\partial(\mathrm{e}^y)}+\frac{\partial\varphi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}=0.$$

已知 $\frac{\partial\varphi}{\partial z}\neq 0$，解得

$$\frac{\mathrm{d}z}{\mathrm{d}x}=-\frac{2x\frac{\partial\varphi}{\partial(x^2)}+\mathrm{e}^y\cos x\frac{\partial\varphi}{\partial(\mathrm{e}^y)}}{\frac{\partial\varphi}{\partial z}}.$$

代入 $\frac{\mathrm{d}u}{\mathrm{d}x}$ 的表达式：

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cos x-\frac{\partial f}{\partial z}\cdot\frac{2x\frac{\partial\varphi}{\partial(x^2)}+\mathrm{e}^y\cos x\frac{\partial\varphi}{\partial(\mathrm{e}^y)}}{\frac{\partial\varphi}{\partial z}}.$$

亦可写为

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\partial f}{\partial x}-\frac{2x\frac{\partial\varphi}{\partial(x^2)}\frac{\partial f}{\partial z}}{\frac{\partial\varphi}{\partial z}}+\cos x\left(\frac{\partial f}{\partial y}-\frac{\mathrm{e}^y\frac{\partial\varphi}{\partial(\mathrm{e}^y)}\frac{\partial f}{\partial z}}{\frac{\partial\varphi}{\partial z}}\right),$$

其中所有偏导数均在相应点取值：$\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}$ 在 $(x,\sin x,z)$ 处，$\frac{\partial\varphi}{\partial(x^2)},\frac{\partial\varphi}{\partial(\mathrm{e}^y)},\frac{\partial\varphi}{\partial z}$ 在 $(x^2,\mathrm{e}^{\sin x},z)$ 处，而 z 由 $\varphi(x^2,\mathrm{e}^{\sin x},z)=0$ 隐式确定。

习题 (9.3.14). 设 $y=y(x),z=z(x)$ 是由方程 $z=xf(x+y)$ 和 $F(x,y,z)=0$ 所确定的函数，其中 f 和 F 分别具有一阶连续导数和一阶连续偏导数. 求 $\frac{\mathrm{d}z}{\mathrm{d}x}$.

解答: 设 $y=y(x)$, $z=z(x)$ 由方程 $z=xf(x+y)$ 与 $F(x,y,z)=0$ 确定。对两式关于 x 求导：

$$\frac{\mathrm{d}z}{\mathrm{d}x}=f(x+y)+xf'(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)=f+xf'+xf'\frac{\mathrm{d}y}{\mathrm{d}x},$$

$$F_x+F_y\frac{\mathrm{d}y}{\mathrm{d}x}+F_z\frac{\mathrm{d}z}{\mathrm{d}x}=0.$$

消去 $\frac{\mathrm{d}y}{\mathrm{d}x}$。由第一式得

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}z}{\mathrm{d}x}-f-xf'}{xf'}.$$

由第二式得

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x + F_z\frac{\mathrm{d}z}{\mathrm{d}x}}{F_y}.$$

令两式相等：

$$\frac{\frac{\mathrm{d}z}{\mathrm{d}x} - f - xf'}{xf'} = -\frac{F_x + F_z\frac{\mathrm{d}z}{\mathrm{d}x}}{F_y}.$$

交叉相乘并整理：

$$\begin{aligned}
F_y\left(\frac{\mathrm{d}z}{\mathrm{d}x} - f - xf'\right) &= -xf'\left(F_x + F_z\frac{\mathrm{d}z}{\mathrm{d}x}\right),\\
F_y\frac{\mathrm{d}z}{\mathrm{d}x} - F_y(f + xf') &= -xf'F_x - xf'F_z\frac{\mathrm{d}z}{\mathrm{d}x},\\
F_y\frac{\mathrm{d}z}{\mathrm{d}x} + xf'F_z\frac{\mathrm{d}z}{\mathrm{d}x} &= F_y(f + xf') - xf'F_x,\\
\frac{\mathrm{d}z}{\mathrm{d}x}(F_y + xf'F_z) &= F_yf + xf'(F_y - F_x).
\end{aligned}$$

因此

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{F_yf(x+y) + xf'(x+y)(F_y - F_x)}{F_y + xf'(x+y)F_z}.$$

习题 (9.3.15). 设 $u = u(x,y), v = v(x,y)$ 是由方程 $F(x,y,u,v) = 0$ 和 $G(x,y,u,v) = 0$ 所确定的隐函数，其中 F 和 G 都具有一阶连续偏导数。求 $\mathrm{d}u, \mathrm{d}v$.

解答: 设 F 和 G 均具有一阶连续偏导数，且隐函数定理成立，即雅可比行列式

$$J = \frac{\partial(F,G)}{\partial(u,v)} = F_uG_v - F_vG_u \neq 0.$$

对隐函数方程 $F(x,y,u,v) = 0$ 和 $G(x,y,u,v) = 0$ 求全微分，得

$$\begin{aligned}
\mathrm{d}F &= F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_u\,\mathrm{d}u + F_v\,\mathrm{d}v = 0,\\
\mathrm{d}G &= G_x\,\mathrm{d}x + G_y\,\mathrm{d}y + G_u\,\mathrm{d}u + G_v\,\mathrm{d}v = 0.
\end{aligned}$$

整理后得到关于 $\mathrm{d}u$ 和 $\mathrm{d}v$ 的线性方程组

$$\begin{cases}
F_u\,\mathrm{d}u + F_v\,\mathrm{d}v = -F_x\,\mathrm{d}x - F_y\,\mathrm{d}y,\\
G_u\,\mathrm{d}u + G_v\,\mathrm{d}v = -G_x\,\mathrm{d}x - G_y\,\mathrm{d}y.
\end{cases}$$

利用克拉默法则求解。记系数行列式 $J = F_uG_v - F_vG_u$，则

$$\begin{aligned}
\mathrm{d}u &= \frac{1}{J}\begin{vmatrix} -F_x\,\mathrm{d}x - F_y\,\mathrm{d}y & F_v \\ -G_x\,\mathrm{d}x - G_y\,\mathrm{d}y & G_v \end{vmatrix} \\
&= \frac{1}{J}\big[(-F_x\,\mathrm{d}x - F_y\,\mathrm{d}y)G_v - F_v(-G_x\,\mathrm{d}x - G_y\,\mathrm{d}y)\big] \\
&= \frac{1}{J}\big[(F_vG_x - F_xG_v)\,\mathrm{d}x + (F_vG_y - F_yG_v)\,\mathrm{d}y\big], \\
\mathrm{d}v &= \frac{1}{J}\begin{vmatrix} F_u & -F_x\,\mathrm{d}x - F_y\,\mathrm{d}y \\ G_u & -G_x\,\mathrm{d}x - G_y\,\mathrm{d}y \end{vmatrix} \\
&= \frac{1}{J}\big[F_u(-G_x\,\mathrm{d}x - G_y\,\mathrm{d}y) - (-F_x\,\mathrm{d}x - F_y\,\mathrm{d}y)G_u\big] \\
&= \frac{1}{J}\big[(F_xG_u - F_uG_x)\,\mathrm{d}x + (F_yG_u - F_uG_y)\,\mathrm{d}y\big].
\end{aligned}$$

因此，所求微分为

$$\mathrm{d}u = \frac{F_vG_x - F_xG_v}{F_uG_v - F_vG_u}\,\mathrm{d}x + \frac{F_vG_y - F_yG_v}{F_uG_v - F_vG_u}\,\mathrm{d}y,$$
$$\mathrm{d}v = \frac{F_xG_u - F_uG_x}{F_uG_v - F_vG_u}\,\mathrm{d}x + \frac{F_yG_u - F_uG_y}{F_uG_v - F_vG_u}\,\mathrm{d}y.$$

等价地，用雅可比行列式表示为

$$\mathrm{d}u = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}\,\mathrm{d}x - \frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}\,\mathrm{d}y,$$
$$\mathrm{d}v = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}\,\mathrm{d}x - \frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}}\,\mathrm{d}y.$$

习题 (9.4.2). 设 $\boldsymbol{r}(t)$ 是单位向量，试证明 $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\perp\boldsymbol{r}$，并说明它的几何意义.

解答: 由于 $\boldsymbol{r}(t)$ 是单位向量，故对任意 t 有 $\boldsymbol{r}(t)\cdot\boldsymbol{r}(t) = 1$。两边对 t 求导得

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}\cdot\boldsymbol{r}) = 0.$$

利用点积的乘积法则 $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}\cdot\boldsymbol{v}) = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$，可得

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\boldsymbol{r} + \boldsymbol{r}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = 2\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\boldsymbol{r} = 0.$$

因此 $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\boldsymbol{r} = 0$，即 $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\perp\boldsymbol{r}$。

几何意义：$\boldsymbol{r}(t)$ 的末端始终位于以原点为中心的单位球面上。导数 $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$ 表示该末端点的瞬时速度向量。正交性表明速度向量总是与球面相切，这意味着运动被约束在球面上，没有径向分量。因此，导数代表了纯粹的方向变化，保持了向量的单位长度。

习题 (9.4.4). 设 $r = \left(\frac{t}{1+t}, \frac{1+t}{t}, t^2\right)(t>0)$，判断它是不是简单曲线，是不是光滑曲线，并求出它在 $t=1$ 时的切线方程和法平面方程.

解答:

(1). 是简单曲线。因为参数域 $t>0$ 上，第三个分量 t^2 严格单调递增，故映射 $t \mapsto \boldsymbol{r}(t)$ 是单射，曲线无自交。

(2). 是光滑曲线。求导得

$$\boldsymbol{r}'(t) = \left(\frac{1}{(1+t)^2},\ -\frac{1}{t^2},\ 2t\right),$$

各分量在 $t>0$ 上连续且 $\boldsymbol{r}'(t) \neq \boldsymbol{0}$（三个分量均不为零）。

(3). 切线方程（$t=1$）：点 $\boldsymbol{r}(1) = \left(\frac{1}{2},\ 2,\ 1\right)$，切向量 $\boldsymbol{r}'(1) = \left(\frac{1}{4},\ -1,\ 2\right)$。参数式：

$$(x,\ y,\ z) = \left(\frac{1}{2},\ 2,\ 1\right) + s\left(\frac{1}{4},\ -1,\ 2\right), \quad s \in \mathbb{R}.$$

对称式：

$$\frac{x-1/2}{1/4} = \frac{y-2}{-1} = \frac{z-1}{2},$$

或等价地

$$\frac{x-1/2}{1} = \frac{y-2}{-4} = \frac{z-1}{8}.$$

(4). 法平面方程（$t=1$）：法向量为 $\boldsymbol{r}'(1) = \left(\frac{1}{4},\ -1,\ 2\right)$，过点 $\left(\frac{1}{2},\ 2,\ 1\right)$。方程为

$$\frac{1}{4}\left(x-\frac{1}{2}\right) - 1\cdot(y-2) + 2(z-1) = 0.$$

化简得

$$2x - 8y + 16z = 1.$$

习题 (9.4.6). 求下列曲面在所给点处的切平面与法线方程.

(1) $x = u\cos v, y = u\sin v, z = av$, 在 (u_0, v_0);

(2) $x = a\sin\theta\cos\varphi, y = b\sin\theta\sin\varphi, z = c\cos\theta$, 在 (θ_0, φ_0).

解答:

1. 对于曲面 $\boldsymbol{r}(u,v) = (u\cos v,\ u\sin v,\ av)$，在参数点 (u_0, v_0) 处：

$$\boldsymbol{r}_u = (\cos v,\ \sin v,\ 0),$$
$$\boldsymbol{r}_v = (-u\sin v,\ u\cos v,\ a).$$

在 (u_0, v_0) 处：

$$\boldsymbol{r}_u(u_0,v_0) = (\cos v_0,\ \sin v_0,\ 0),$$
$$\boldsymbol{r}_v(u_0,v_0) = (-u_0\sin v_0,\ u_0\cos v_0,\ a).$$

法向量 $\boldsymbol{n} = \boldsymbol{r}_u \times \boldsymbol{r}_v$ 为

$$\boldsymbol{n} = (a\sin v_0,\ -a\cos v_0,\ u_0).$$

对应曲面上的点为 $(x_0, y_0, z_0) = (u_0 \cos v_0,\ u_0 \sin v_0,\ a v_0)$。切平面方程为

$$a \sin v_0\,(x - u_0 \cos v_0) - a \cos v_0\,(y - u_0 \sin v_0) + u_0\,(z - a v_0) = 0,$$

化简得

$$a \sin v_0\, x - a \cos v_0\, y + u_0 z = a u_0 v_0.$$

法线方程：参数形式

$$x = u_0 \cos v_0 + a t \sin v_0, \quad y = u_0 \sin v_0 - a t \cos v_0, \quad z = a v_0 + u_0 t.$$

对称形式（当分母非零时）

$$\frac{x - u_0 \cos v_0}{a \sin v_0} = \frac{y - u_0 \sin v_0}{-a \cos v_0} = \frac{z - a v_0}{u_0}.$$

2. 对于曲面 $\boldsymbol{r}(\theta, \varphi) = (a \sin\theta \cos\varphi,\ b \sin\theta \sin\varphi,\ c \cos\theta)$，在参数点 (θ_0, φ_0) 处：

$$\begin{aligned}\boldsymbol{r}_\theta &= (a \cos\theta \cos\varphi,\ b \cos\theta \sin\varphi,\ -c \sin\theta),\\ \boldsymbol{r}_\varphi &= (-a \sin\theta \sin\varphi,\ b \sin\theta \cos\varphi,\ 0).\end{aligned}$$

在 (θ_0, φ_0) 处：

$$\begin{aligned}\boldsymbol{r}_\theta(\theta_0, \varphi_0) &= (a \cos\theta_0 \cos\varphi_0,\ b \cos\theta_0 \sin\varphi_0,\ -c \sin\theta_0),\\ \boldsymbol{r}_\varphi(\theta_0, \varphi_0) &= (-a \sin\theta_0 \sin\varphi_0,\ b \sin\theta_0 \cos\varphi_0,\ 0).\end{aligned}$$

法向量 $\boldsymbol{n} = \boldsymbol{r}_\theta \times \boldsymbol{r}_\varphi$ 为

$$\boldsymbol{n} = (bc \sin^2\theta_0 \cos\varphi_0,\ ac \sin^2\theta_0 \sin\varphi_0,\ ab \sin\theta_0 \cos\theta_0).$$

对应曲面上的点为 $(x_0, y_0, z_0) = (a \sin\theta_0 \cos\varphi_0,\ b \sin\theta_0 \sin\varphi_0,\ c \cos\theta_0)$。切平面方程为

$$bc \sin^2\theta_0 \cos\varphi_0\,(x - x_0) + ac \sin^2\theta_0 \sin\varphi_0\,(y - y_0) + ab \sin\theta_0 \cos\theta_0\,(z - z_0) = 0.$$

利用 $\sin^2\theta_0 + \cos^2\theta_0 = 1$ 化简得

$$bc \sin\theta_0 \cos\varphi_0\, x + ac \sin\theta_0 \sin\varphi_0\, y + ab \cos\theta_0\, z = abc.$$

由于曲面满足 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$，切平面亦可写为

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1.$$

法线方程：参数形式（取梯度方向）

$$x = x_0 + \frac{x_0}{a^2} t, \quad y = y_0 + \frac{y_0}{b^2} t, \quad z = z_0 + \frac{z_0}{c^2} t.$$

对称形式

$$\frac{a^2(x - x_0)}{x_0} = \frac{b^2(y - y_0)}{y_0} = \frac{c^2(z - z_0)}{z_0}.$$

或直接用法向量（当各分量非零时）

$$\frac{x - x_0}{bc \sin^2\theta_0 \cos\varphi_0} = \frac{y - y_0}{ac \sin^2\theta_0 \sin\varphi_0} = \frac{z - z_0}{ab \sin\theta_0 \cos\theta_0}.$$