习题 (9.2.13). (1) $z = \ln(x^2 + y^2)$;
(3) $u = \frac{s+t}{s-t}$;
(5) $z = \sin(xy)$ 在点 $(0, 0)$;

解答:
(1). 对于 $z = \ln(x^2 + y^2)$，计算偏导数：

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2},$$
$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}.$$

(3). 对于 $u = \dfrac{s+t}{s-t}$，计算偏导数：

$$\frac{\partial u}{\partial s} = -\frac{2t}{(s-t)^2},$$
$$\frac{\partial u}{\partial t} = \frac{2s}{(s-t)^2}.$$

(5). 对于 $z = \sin(xy)$，梯度为

$$\nabla z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right).$$

计算偏导数：

$$\frac{\partial z}{\partial x} = y\cos(xy),$$
$$\frac{\partial z}{\partial y} = x\cos(xy).$$

在点 $(0, 0)$ 处，

$$\left.\frac{\partial z}{\partial x}\right|_{(0,0)} = 0 \cdot \cos(0) = 0,$$
$$\left.\frac{\partial z}{\partial y}\right|_{(0,0)} = 0 \cdot \cos(0) = 0.$$

因此，$\nabla z(0, 0) = (0, 0)$。

习题 (9.2.15). 根据可微的定义证明：函数 $f(x, y) = \sqrt{|xy|}$ 在原点处不可微.

解答: 首先计算函数在原点的值：

$$f(0, 0) = \sqrt{|0 \cdot 0|} = 0.$$

其次，计算偏导数。由定义，

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{\sqrt{|h \cdot 0|} - 0}{h} = 0,$$
$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0} \frac{\sqrt{|0 \cdot k|} - 0}{k} = 0.$$

因此，若 f 在 $(0,0)$ 可微，则其全微分必为 $L(x,y)=0$。

由可微的定义，f 在 $(0,0)$ 可微当且仅当

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-f_x(0,0)h-f_y(0,0)k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}} = 0.$$

考虑路径 $k=h$（$h\to 0$），沿此路径有

$$\frac{\sqrt{|h\cdot h|}}{\sqrt{h^2+h^2}} = \frac{\sqrt{h^2}}{\sqrt{2h^2}} = \frac{|h|}{\sqrt{2}\,|h|} = \frac{1}{\sqrt{2}} \quad (h\neq 0).$$

因此

$$\lim_{h\to 0} \frac{\sqrt{|h^2|}}{\sqrt{h^2+h^2}} = \frac{1}{\sqrt{2}} \neq 0.$$

由于沿不同路径极限值不同，原极限不存在且不为零，故 f 在 $(0,0)$ 不可微。

习题 (9.2.17). 函数 $f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0\end{cases}$ 在点 $(0,0)$ 连续且偏导数存在，但偏导数在点 $(0,0)$ 不连续，而 f 在原点 $(0,0)$ 可微.

解答:

(1) 连续性: 当 $(x,y)\neq(0,0)$ 时，

$$|f(x,y)| = \left|(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}\right| \leqslant x^2+y^2.$$

由于 $\lim_{(x,y)\to(0,0)}(x^2+y^2)=0$，由夹逼定理得

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0).$$

故 f 在 $(0,0)$ 连续。

(2) 偏导数存在性: 按定义，

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{|h|}}{h} = \lim_{h\to 0} h\sin\frac{1}{|h|} = 0,$$
$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} k\sin\frac{1}{|k|} = 0.$$

所以 $f_x(0,0)$ 与 $f_y(0,0)$ 均存在且为零。

(3) 偏导数的不连续性: 当 $(x,y)\neq(0,0)$ 时，记 $r=\sqrt{x^2+y^2}$，由求导法则得

$$f_x(x,y) = 2x\sin\frac{1}{r} - \frac{x}{r}\cos\frac{1}{r},$$
$$f_y(x,y) = 2y\sin\frac{1}{r} - \frac{y}{r}\cos\frac{1}{r}.$$

考虑沿正 x 轴 $(y=0,\ x>0)$ 趋近于 $(0,0)$，此时

$$f_x(x,0) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}.$$

当 $x\to 0^+$ 时，$2x\sin(1/x)\to 0$，而 $\cos(1/x)$ 在 $[-1,1]$ 间振荡，故 $\lim_{x\to 0^+} f_x(x,0)$ 不存在。因此 f_x 在 $(0,0)$ 不连续。同理 f_y 亦不连续。

(4) 可微性: 需验证

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-f_x(0,0)h-f_y(0,0)k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} = 0.$$

令 $r=\sqrt{h^2+k^2}$，则

$$\frac{f(h,k)}{r} = \frac{r^2\sin(1/r)}{r} = r\sin\frac{1}{r}.$$

由于 $|r\sin(1/r)| \leqslant r \to 0$（当 $r\to 0^+$），故该极限为零。因此 f 在 $(0,0)$ 可微，且微分为零。

习题 (9.2.19). 求下列复合函数的偏导数或导数.

(1) 设 $u=\mathrm{e}^t+\arctan(t^2+1), t=x^y$, 求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$;

(2) 设 $u=\mathrm{e}^{xyz}, x=rs, y=\frac{r}{s}, z=r^s$, 求 $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}$;

(3) 设 $u=\ln(x^2+y^2), x=\mathrm{e}^{t+s+r}, y=4(s^2+t^2)$, 求 $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}$;

(4) 设 $u=\frac{\mathrm{e}^{ax}(y-z)}{a^2+1}, y=a\sin x, z=\cos x$, 求 $\frac{\mathrm{d}u}{\mathrm{d}x}$.

解答:

1. 给定 $u=e^t+\arctan(t^2+1)$，$t=x^y$。由链式法则，

$$\frac{\partial u}{\partial x} = \frac{du}{dt}\cdot\frac{\partial t}{\partial x},$$
$$\frac{\partial u}{\partial y} = \frac{du}{dt}\cdot\frac{\partial t}{\partial y}.$$

计算得

$$\frac{du}{dt} = e^t + \frac{2t}{(t^2+1)^2+1},$$

$$\frac{\partial t}{\partial x} = yx^{y-1}, \quad \frac{\partial t}{\partial y} = x^y\ln x.$$

因此

$$\frac{\partial u}{\partial x} = yx^{y-1}\left(e^{x^y} + \frac{2x^y}{((x^y)^2+1)^2+1}\right),$$
$$\frac{\partial u}{\partial y} = x^y\ln x\left(e^{x^y} + \frac{2x^y}{((x^y)^2+1)^2+1}\right).$$

2. 给定 $u=e^{xyz}$，$x=rs$，$y=\frac{r}{s}$，$z=r^s$。首先注意到 $xyz=rs\cdot\frac{r}{s}\cdot r^s=r^{s+2}$，故 $u=e^{r^{s+2}}$。利用链式法则

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r},$$
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}.$$

其中

$$\frac{\partial u}{\partial x} = yze^{xyz}, \quad \frac{\partial u}{\partial y} = xze^{xyz}, \quad \frac{\partial u}{\partial z} = xye^{xyz},$$

$$\frac{\partial x}{\partial r}=s,\ \frac{\partial x}{\partial s}=r,\ \frac{\partial y}{\partial r}=\frac{1}{s},\ \frac{\partial y}{\partial s}=-\frac{r}{s^2},\ \frac{\partial z}{\partial r}=sr^{s-1},\ \frac{\partial z}{\partial s}=r^s\ln r.$$

代入并化简得

$$\frac{\partial u}{\partial r}=e^{r^{s+2}}r^{s+1}(2+s),$$
$$\frac{\partial u}{\partial s}=e^{r^{s+2}}r^{s+2}\ln r.$$

3. 给定 $u=\ln(x^2+y^2)$，$x=e^{t+s+r}$，$y=4(s^2+t^2)$。由链式法则

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r},$$
$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$
$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}.$$

计算得

$$\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2},\quad \frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2},$$
$$\frac{\partial x}{\partial r}=e^{t+s+r},\ \frac{\partial x}{\partial s}=e^{t+s+r},\ \frac{\partial x}{\partial t}=e^{t+s+r},$$
$$\frac{\partial y}{\partial r}=0,\ \frac{\partial y}{\partial s}=8s,\ \frac{\partial y}{\partial t}=8t.$$

于是

$$\frac{\partial u}{\partial r}=\frac{2x}{x^2+y^2}\cdot e^{t+s+r}=\frac{2x^2}{x^2+y^2}=\frac{2e^{2(t+s+r)}}{e^{2(t+s+r)}+16(s^2+t^2)^2},$$
$$\frac{\partial u}{\partial s}=\frac{2x}{x^2+y^2}\cdot e^{t+s+r}+\frac{2y}{x^2+y^2}\cdot 8s=\frac{2x^2+16sy}{x^2+y^2}=\frac{2e^{2(t+s+r)}+64s(s^2+t^2)}{e^{2(t+s+r)}+16(s^2+t^2)^2},$$
$$\frac{\partial u}{\partial t}=\frac{2x}{x^2+y^2}\cdot e^{t+s+r}+\frac{2y}{x^2+y^2}\cdot 8t=\frac{2x^2+16ty}{x^2+y^2}=\frac{2e^{2(t+s+r)}+64t(s^2+t^2)}{e^{2(t+s+r)}+16(s^2+t^2)^2}.$$

4. 给定 $u=\frac{e^{ax}(y-z)}{a^2+1}$，$y=a\sin x$，$z=\cos x$。代入得 $u=\frac{e^{ax}(a\sin x-\cos x)}{a^2+1}$。对 x 求导

$$\begin{aligned}\frac{du}{dx}&=\frac{1}{a^2+1}\left[ae^{ax}(a\sin x-\cos x)+e^{ax}(a\cos x+\sin x)\right]\\&=\frac{e^{ax}}{a^2+1}\left(a^2\sin x+\sin x\right)\\&=e^{ax}\sin x.\end{aligned}$$

习题 (9.2.20). 求下列复合函数的偏导数或导数，其中 f 均有连续的二阶偏导数.

(1) 设 $u=f(x,y),x=t^3,y=2t^2$，求 $\frac{\mathrm{d}u}{\mathrm{d}t}$;

(2) 设 $u=f(x,y,z),x=\sin t,y=\cos t,z=\mathrm{e}^t$，求 $\frac{\mathrm{d}u}{\mathrm{d}t}$;

(3) 设 $u=f(x^2-y^2,\mathrm{e}^{xy})$，求 $\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x\partial y}$;

(4) 设 $u=f(x+y+z,x^2+y^2+z^2)$，求 $\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x\partial y}$.

解答:

1. 由链式法则，
$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$
其中 $x=t^3$, $y=2t^2$，故 $\frac{\mathrm{d}x}{\mathrm{d}t}=3t^2$, $\frac{\mathrm{d}y}{\mathrm{d}t}=4t$。代入得
$$\frac{\mathrm{d}u}{\mathrm{d}t}=3t^2\,f_x(t^3,2t^2)+4t\,f_y(t^3,2t^2).$$

2. 由链式法则，
$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}.$$
已知 $x=\sin t$, $y=\cos t$, $z=e^t$，故 $\frac{\mathrm{d}x}{\mathrm{d}t}=\cos t$, $\frac{\mathrm{d}y}{\mathrm{d}t}=-\sin t$, $\frac{\mathrm{d}z}{\mathrm{d}t}=e^t$。代入得
$$\frac{\mathrm{d}u}{\mathrm{d}t}=\cos t\,f_x(\sin t,\cos t,e^t)-\sin t\,f_y(\sin t,\cos t,e^t)+e^t\,f_z(\sin t,\cos t,e^t).$$

3. 令 $s=x^2-y^2$, $t=e^{xy}$，则 $u=f(s,t)$。由链式法则，
$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial x}=f_s\cdot 2x+f_t\cdot ye^{xy}.$$
即
$$\frac{\partial u}{\partial x}=2x\,f_s(s,t)+ye^{xy}\,f_t(s,t).$$
接下来计算混合偏导数 $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$。首先
$$\frac{\partial}{\partial y}\big(2xf_s\big)=2x\,\frac{\partial f_s}{\partial y}.$$
而 f_s 是 s,t 的函数，故
$$\begin{aligned}\frac{\partial f_s}{\partial y}&=\frac{\partial f_s}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial f_s}{\partial t}\frac{\partial t}{\partial y}\\&=f_{ss}\cdot(-2y)+f_{st}\cdot xe^{xy}\\&=-2yf_{ss}+xe^{xy}f_{st}.\end{aligned}$$
所以
$$\frac{\partial}{\partial y}\big(2xf_s\big)=2x\big(-2yf_{ss}+xe^{xy}f_{st}\big)=-4xyf_{ss}+2x^2e^{xy}f_{st}.$$
其次，计算 $\frac{\partial}{\partial y}\big(ye^{xy}f_t\big)$，使用乘积法则：
$$\begin{aligned}\frac{\partial}{\partial y}\big(ye^{xy}f_t\big)&=\frac{\partial y}{\partial y}\,e^{xy}f_t+y\,\frac{\partial}{\partial y}\big(e^{xy}\big)f_t+ye^{xy}\,\frac{\partial f_t}{\partial y}\\&=e^{xy}f_t+y\cdot xe^{xy}f_t+ye^{xy}\left(\frac{\partial f_t}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial f_t}{\partial t}\frac{\partial t}{\partial y}\right)\\&=e^{xy}f_t+xye^{xy}f_t+ye^{xy}\big(f_{ts}\cdot(-2y)+f_{tt}\cdot xe^{xy}\big)\\&=e^{xy}f_t+xye^{xy}f_t-2y^2e^{xy}f_{ts}+xye^{2xy}f_{tt}.\end{aligned}$$

由于 f 具有连续的二阶偏导数，故 $f_{st}=f_{ts}$。将两部分相加，得

$$\begin{aligned}\frac{\partial^2 u}{\partial x\partial y}&=\left(-4xyf_{ss}+2x^2e^{xy}f_{st}\right)+\left(e^{xy}f_t+xye^{xy}f_t-2y^2e^{xy}f_{st}+xye^{2xy}f_{tt}\right)\\&=-4xyf_{ss}+2e^{xy}(x^2-y^2)f_{st}+xye^{2xy}f_{tt}+e^{xy}(1+xy)f_t.\end{aligned}$$

其中所有 f 的偏导数均在 $(s,t)=(x^2-y^2,e^{xy})$ 处取值。

4. 令 $p=x+y+z,\, q=x^2+y^2+z^2$，则 $u=f(p,q)$。由链式法则，

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=f_p\cdot 1+f_q\cdot 2x=f_p+2xf_q.$$

接下来计算二阶偏导数。首先

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(f_p+2xf_q\right).$$

注意 $f_p,\, f_q$ 均为 p,q 的函数，故

$$\begin{aligned}\frac{\partial f_p}{\partial x}&=\frac{\partial f_p}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f_p}{\partial q}\frac{\partial q}{\partial x}=f_{pp}\cdot 1+f_{pq}\cdot 2x=f_{pp}+2xf_{pq},\\\frac{\partial f_q}{\partial x}&=\frac{\partial f_q}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f_q}{\partial q}\frac{\partial q}{\partial x}=f_{qp}\cdot 1+f_{qq}\cdot 2x=f_{pq}+2xf_{qq}\quad(\text{因}f_{pq}=f_{qp}).\end{aligned}$$

于是

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\left(f_{pp}+2xf_{pq}\right)+\left(2f_q+2x(f_{pq}+2xf_{qq})\right)\\&=f_{pp}+2xf_{pq}+2f_q+2xf_{pq}+4x^2f_{qq}\\&=f_{pp}+4xf_{pq}+4x^2f_{qq}+2f_q.\end{aligned}$$

其次，计算混合偏导数

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\left(f_p+2xf_q\right).$$

类似地，

$$\begin{aligned}\frac{\partial f_p}{\partial y}&=\frac{\partial f_p}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f_p}{\partial q}\frac{\partial q}{\partial y}=f_{pp}\cdot 1+f_{pq}\cdot 2y=f_{pp}+2yf_{pq},\\\frac{\partial f_q}{\partial y}&=\frac{\partial f_q}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f_q}{\partial q}\frac{\partial q}{\partial y}=f_{pq}\cdot 1+f_{qq}\cdot 2y=f_{pq}+2yf_{qq}.\end{aligned}$$

因此

$$\begin{aligned}\frac{\partial^2 u}{\partial x\partial y}&=\left(f_{pp}+2yf_{pq}\right)+2x\left(f_{pq}+2yf_{qq}\right)\\&=f_{pp}+2yf_{pq}+2xf_{pq}+4xyf_{qq}\\&=f_{pp}+2(x+y)f_{pq}+4xyf_{qq}.\end{aligned}$$

以上所有 f 的偏导数均在 $(p,q)=(x+y+z,\, x^2+y^2+z^2)$ 处取值。

习题 (9.2.22). 试求函数 $z=\arctan\frac{y}{x}$ 在圆 $x^2+y^2-2x=0$ 上一点 $P\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ 处沿该圆周逆时针方向上的方向导数.

解答: 函数 $z=\arctan\frac{y}{x}$ 的梯度为

$$\nabla z=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right).$$

计算偏导数：

$$\frac{\partial z}{\partial x}=-\frac{y}{x^2+y^2},\qquad \frac{\partial z}{\partial y}=\frac{x}{x^2+y^2}.$$

在点 $P\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ 处，有 $x^2+y^2=1$，故

$$\left.\frac{\partial z}{\partial x}\right|_P=-\frac{\sqrt{3}/2}{1}=-\frac{\sqrt{3}}{2},\qquad \left.\frac{\partial z}{\partial y}\right|_P=\frac{1/2}{1}=\frac{1}{2}.$$

因此

$$\nabla z(P)=\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right).$$

圆 $x^2+y^2-2x=0$ 可写为 $(x-1)^2+y^2=1$。为求点 P 处沿逆时针方向的单位切向量，将圆参数化为

$$x=1+\cos\theta,\quad y=\sin\theta,$$

其中 θ 增加的方向即为逆时针方向。在点 P 处，$\cos\theta=-\frac{1}{2}$，$\sin\theta=\frac{\sqrt{3}}{2}$，故 $\theta=\frac{2\pi}{3}$。速度向量为

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta},\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)=(-\sin\theta,\cos\theta),$$

其模长为 1。于是逆时针方向的单位切向量为

$$\boldsymbol{u}=(-\sin\theta,\cos\theta)\big|_{\theta=2\pi/3}=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right).$$

方向导数为梯度与单位切向量的点积：

$$\begin{aligned}D_{\boldsymbol{u}}z(P)&=\nabla z(P)\cdot\boldsymbol{u}\\&=\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)\cdot\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)\\&=\left(-\frac{\sqrt{3}}{2}\right)\times\left(-\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}\right)\times\left(-\frac{1}{2}\right)\\&=\frac{3}{4}-\frac{1}{4}=\frac{2}{4}=\frac{1}{2}.\end{aligned}$$

习题 (9.2.24). 设 $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$，$r=|\boldsymbol{r}|$，试求（1）$\mathrm{grad}\,\frac{1}{r^2}$；（2）$\mathrm{grad}\ln r$。

解答: 设 $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$，$r=|\boldsymbol{r}|=\sqrt{x^2+y^2+z^2}$。梯度算子 $\nabla=\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}$。

(1). 计算 $\nabla\left(\frac{1}{r^2}\right)$。令 $f(x,y,z)=\frac{1}{r^2}=\frac{1}{x^2+y^2+z^2}$，则

$$\begin{aligned}\frac{\partial f}{\partial x}&=-\frac{2x}{(x^2+y^2+z^2)^2}=-\frac{2x}{r^4},\\ \frac{\partial f}{\partial y}&=-\frac{2y}{r^4},\\ \frac{\partial f}{\partial z}&=-\frac{2z}{r^4}.\end{aligned}$$

因此

$$\nabla\left(\frac{1}{r^2}\right)=-\frac{2x}{r^4}\boldsymbol{i}-\frac{2y}{r^4}\boldsymbol{j}-\frac{2z}{r^4}\boldsymbol{k}=-\frac{2\boldsymbol{r}}{r^4}=-\frac{2\hat{\boldsymbol{r}}}{r^3},$$

其中 $\hat{\boldsymbol{r}}=\boldsymbol{r}/r$ 为径向单位向量。

(2). 计算 $\nabla(\ln r)$。令 $g(x,y,z)=\ln r=\ln\sqrt{x^2+y^2+z^2}=\frac{1}{2}\ln(x^2+y^2+z^2)$，则

$$\begin{aligned}\frac{\partial g}{\partial x}&=\frac{1}{2}\cdot\frac{2x}{x^2+y^2+z^2}=\frac{x}{r^2},\\ \frac{\partial g}{\partial y}&=\frac{y}{r^2},\\ \frac{\partial g}{\partial z}&=\frac{z}{r^2}.\end{aligned}$$

因此

$$\nabla(\ln r)=\frac{x}{r^2}\boldsymbol{i}+\frac{y}{r^2}\boldsymbol{j}+\frac{z}{r^2}\boldsymbol{k}=\frac{\boldsymbol{r}}{r^2}=\frac{\hat{\boldsymbol{r}}}{r}.$$

习题 (9.2.31). 试证：方程 $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}\cos x-\frac{\partial^2 u}{\partial y^2}\sin^2 x-\frac{\partial u}{\partial y}\sin x=0$ 经变换 $\xi=x-\sin x+y$, $\eta=x+\sin x-y$ 后变成 $\frac{\partial^2 u}{\partial\xi\partial\eta}=0$。（其中二阶偏导数均连续。）

解答: 令变换

$$\xi=x-\sin x+y,\qquad \eta=x+\sin x-y.$$

计算一阶偏导数

$$\xi_x=1-\cos x,\quad \xi_y=1,\quad \eta_x=1+\cos x,\quad \eta_y=-1.$$

由链式法则，

$$\begin{aligned}u_x&=u_\xi\xi_x+u_\eta\eta_x=u_\xi(1-\cos x)+u_\eta(1+\cos x),\\ u_y&=u_\xi\xi_y+u_\eta\eta_y=u_\xi-u_\eta.\end{aligned}$$

再计算二阶偏导数。首先，

$$\begin{aligned}\frac{\partial u_\xi}{\partial x}&=u_{\xi\xi}\xi_x+u_{\xi\eta}\eta_x=u_{\xi\xi}(1-\cos x)+u_{\xi\eta}(1+\cos x),\\ \frac{\partial u_\xi}{\partial y}&=u_{\xi\xi}\xi_y+u_{\xi\eta}\eta_y=u_{\xi\xi}-u_{\xi\eta},\\ \frac{\partial u_\eta}{\partial x}&=u_{\eta\xi}\xi_x+u_{\eta\eta}\eta_x=u_{\xi\eta}(1-\cos x)+u_{\eta\eta}(1+\cos x),\\ \frac{\partial u_\eta}{\partial y}&=u_{\eta\xi}\xi_y+u_{\eta\eta}\eta_y=u_{\xi\eta}-u_{\eta\eta}.\end{aligned}$$

于是

$$\begin{aligned}
u_{xx} &= \frac{\partial}{\partial x}(u_x) \\
&= \big[u_{\xi\xi}(1-\cos x) + u_{\xi\eta}(1+\cos x)\big](1-\cos x) + u_\xi \sin x \\
&\quad + \big[u_{\xi\eta}(1-\cos x) + u_{\eta\eta}(1+\cos x)\big](1+\cos x) - u_\eta \sin x \\
&= u_{\xi\xi}(1-\cos x)^2 + u_{\xi\eta}(1+\cos x)(1-\cos x) + u_\xi \sin x \\
&\quad + u_{\xi\eta}(1-\cos x)(1+\cos x) + u_{\eta\eta}(1+\cos x)^2 - u_\eta \sin x \\
&= u_{\xi\xi}(1-\cos x)^2 + 2u_{\xi\eta}\sin^2 x + u_{\eta\eta}(1+\cos x)^2 + \sin x(u_\xi - u_\eta).
\end{aligned}$$

其次，

$$\begin{aligned}
u_{xy} &= \frac{\partial}{\partial y}(u_x) \\
&= (u_{\xi\xi} - u_{\xi\eta})(1-\cos x) + (u_{\xi\eta} - u_{\eta\eta})(1+\cos x) \\
&= u_{\xi\xi}(1-\cos x) - u_{\xi\eta}(1-\cos x) + u_{\xi\eta}(1+\cos x) - u_{\eta\eta}(1+\cos x) \\
&= u_{\xi\xi}(1-\cos x) + 2u_{\xi\eta}\cos x - u_{\eta\eta}(1+\cos x).
\end{aligned}$$

最后，

$$\begin{aligned}
u_{yy} = \frac{\partial}{\partial y}(u_y) &= \frac{\partial u_\xi}{\partial y} - \frac{\partial u_\eta}{\partial y} \\
&= (u_{\xi\xi} - u_{\xi\eta}) - (u_{\xi\eta} - u_{\eta\eta}) = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.
\end{aligned}$$

将以上表达式代入原方程

$$u_{xx} + 2\cos x\, u_{xy} - \sin^2 x\, u_{yy} - \sin x\, u_y = 0.$$

计算各项：

$$\begin{aligned}
u_{xx} &= u_{\xi\xi}(1-\cos x)^2 + 2u_{\xi\eta}\sin^2 x + u_{\eta\eta}(1+\cos x)^2 + \sin x(u_\xi - u_\eta), \\
2\cos x\, u_{xy} &= 2\cos x\big[u_{\xi\xi}(1-\cos x) + 2u_{\xi\eta}\cos x - u_{\eta\eta}(1+\cos x)\big], \\
-\sin^2 x\, u_{yy} &= -\sin^2 x(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}), \\
-\sin x\, u_y &= -\sin x(u_\xi - u_\eta).
\end{aligned}$$

注意到 $\sin x(u_\xi - u_\eta)$ 与 $-\sin x(u_\xi - u_\eta)$ 相消。合并 $u_{\xi\xi}$、$u_{\xi\eta}$、$u_{\eta\eta}$ 的系数：

$$\begin{aligned}
&u_{\xi\xi}\ \text{的系数：}(1-\cos x)^2 + 2\cos x(1-\cos x) - \sin^2 x \\
&= (1 - 2\cos x + \cos^2 x) + (2\cos x - 2\cos^2 x) - \sin^2 x \\
&= 1 - \cos^2 x - \sin^2 x = 0, \\
&u_{\eta\eta}\ \text{的系数：}(1+\cos x)^2 - 2\cos x(1+\cos x) - \sin^2 x \\
&= (1 + 2\cos x + \cos^2 x) + (-2\cos x - 2\cos^2 x) - \sin^2 x \\
&= 1 - \cos^2 x - \sin^2 x = 0, \\
&u_{\xi\eta}\ \text{的系数：}2\sin^2 x + 4\cos^2 x + 2\sin^2 x = 4\sin^2 x + 4\cos^2 x = 4.
\end{aligned}$$

因此原方程化为

$$4u_{\xi\eta} = 0 \quad \Longrightarrow \quad u_{\xi\eta} = 0.$$

即

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

习题 (9.2.32). 设变换 $\begin{cases} u = x - 2y, \\ v = x + ay \end{cases}$ 可把方程 $6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 0$ 简化为 $\frac{\partial^2 z}{\partial u \partial v} = 0$, 求常数 a. (其中二阶偏导数均连续.)

解答: 设变换 $u = x - 2y$, $v = x + ay$ 将方程 $6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 0$ 化为 $\frac{\partial^2 z}{\partial u \partial v} = 0$。求常数 a。

利用链式法则计算一阶偏导数：

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = z_u + z_v,$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -2z_u + az_v.$$

再计算二阶偏导数：

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(z_u + z_v) = (z_{uu} + z_{uv}) + (z_{uv} + z_{vv}) = z_{uu} + 2z_{uv} + z_{vv},$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}(z_u + z_v) = (-2z_{uu} + az_{uv}) + (-2z_{uv} + az_{vv}) = -2z_{uu} + (a-2)z_{uv} + az_{vv},$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(-2z_u + az_v) = 4z_{uu} - 4az_{uv} + a^2 z_{vv}.$$

代入原方程 $6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = 0$ 得：

$$6(z_{uu} + 2z_{uv} + z_{vv}) + \left(-2z_{uu} + (a-2)z_{uv} + az_{vv}\right) - \left(4z_{uu} - 4az_{uv} + a^2 z_{vv}\right) = 0.$$

合并同类项：

z_{uu} 的系数：$6 - 2 - 4 = 0,$

z_{uv} 的系数：$12 + (a-2) + 4a = 10 + 5a,$

z_{vv} 的系数：$6 + a - a^2.$

于是变换后的方程为

$$(10 + 5a)z_{uv} + (6 + a - a^2)z_{vv} = 0.$$

要使其化为 $\frac{\partial^2 z}{\partial u \partial v} = 0$，须有 z_{vv} 的系数为零且 z_{uv} 的系数非零，即

$$6 + a - a^2 = 0, \quad 10 + 5a \neq 0.$$

解 $6 + a - a^2 = 0$ 得 $a^2 - a - 6 = 0$，即 $(a-3)(a+2) = 0$，故 $a = 3$ 或 $a = -2$。当 $a = 3$ 时，$10 + 5 \cdot 3 = 25 \neq 0$；当 $a = -2$ 时，$10 + 5 \cdot (-2) = 0$，不满足非零条件。因此常数 $a = 3$。

习题 (9.2.34). 设 $u = u(x, y)$ ，当 $y = x^2$ 时有 $u = 1, \frac{\partial u}{\partial x} = x$ ，求当 $y = x^2$ 时的 $\frac{\partial u}{\partial y}$.

解答: 由于在曲线 $y=x^2$ 上 $u(x,x^2)=1$ 恒成立，对该恒等式关于 x 求导得

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x,x^2)=\frac{\mathrm{d}}{\mathrm{d}x}1=0.$$

由链式法则，

$$\frac{\partial u}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}x}+\frac{\partial u}{\partial y}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x^2)=\frac{\partial u}{\partial x}+2x\frac{\partial u}{\partial y}=0.$$

已知在曲线上 $\frac{\partial u}{\partial x}=x$，代入得

$$x+2x\frac{\partial u}{\partial y}=0.$$

当 $x\neq 0$ 时，两边除以 x 得

$$1+2\frac{\partial u}{\partial y}=0 \quad\Longrightarrow\quad \frac{\partial u}{\partial y}=-\frac{1}{2}.$$

当 $x=0$ 时，方程化为 $0=0$，不限制 $\frac{\partial u}{\partial y}$ 的值。若假设 u 连续可微（C^1），则 $\frac{\partial u}{\partial y}$ 沿曲线连续，由极限 $x\to 0$ 得 $\frac{\partial u}{\partial y}(0,0)=-\frac{1}{2}$。因此，在 $y=x^2$ 上恒有 $\frac{\partial u}{\partial y}=-\frac{1}{2}$。

习题 (9.2.35). 设 $u=u(x,y)$ 满足方程 $\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$ 以及条件 $u(x,2x)=x, u'_x(x,2x)=x^2$，求 $u''_{xx}(x,2x), u''_{xy}(x,2x), u''_{yy}(x,2x)$.（其中二阶偏导数均连续.）

解答: 设 $a=u_{xx}(x,2x)$, $b=u_{xy}(x,2x)$, $d=u_{yy}(x,2x)$，由二阶偏导连续知 $u_{xy}=u_{yx}$。由方程 $\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}=0$ 得

$$a-d=0 \quad\Longrightarrow\quad a=d. \tag{1}$$

对条件 $u(x,2x)=x$ 关于 x 求全导数：

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x,2x)=u_x(x,2x)+2\,u_y(x,2x)=1.$$

代入 $u_x(x,2x)=x^2$ 得

$$x^2+2\,u_y(x,2x)=1 \quad\Longrightarrow\quad u_y(x,2x)=\frac{1-x^2}{2}. \tag{2}$$

再对 $u_x(x,2x)=x^2$ 关于 x 求全导数：

$$\frac{\mathrm{d}}{\mathrm{d}x}u_x(x,2x)=u_{xx}(x,2x)+2\,u_{xy}(x,2x)=2x,$$

即

$$a+2b=2x. \tag{3}$$

对 (2) 式关于 x 求全导数：

$$\frac{\mathrm{d}}{\mathrm{d}x}u_y(x,2x)=u_{yx}(x,2x)+2\,u_{yy}(x,2x)=-x,$$

利用 $u_{yx}=u_{xy}$ 得

$$b+2d=-x. \tag{4}$$

联立 (1)、(3)、(4) 求解。将 $a=d$ 代入 (3) 得 $d+2b=2x$，即

$$b=x-\frac{d}{2}.$$

代入 (4)：

$$\begin{aligned}\left(x-\frac{d}{2}\right)+2d&=-x,\\ x+\frac{3}{2}d&=-x,\\ \frac{3}{2}d&=-2x,\\ d&=-\frac{4}{3}x.\end{aligned}$$

于是

$$a=d=-\frac{4}{3}x, \qquad b=x-\frac{1}{2}\left(-\frac{4}{3}x\right)=x+\frac{2}{3}x=\frac{5}{3}x.$$

习题 (9.2.36). 求下列复合函数的微分 $\mathrm{d}u$: (1) $u=f(t),t=x+y$; (3) $u=f(x,y,z),x=t,y=t^2,z=t^3$; (5) $u=f(\xi,\eta,\zeta),\xi=x^2+y^2,\eta=x^2-y^2,\zeta=2xy$.

解答:

1. 给定 $u=f(t)$, $t=x+y$. 由链式法则,

$$\mathrm{d}u=f'(t)\,\mathrm{d}t=f'(t)(\mathrm{d}x+\mathrm{d}y),$$

其中 $\mathrm{d}t=\mathrm{d}x+\mathrm{d}y$.

2. 给定 $u=f(x,y,z)$, $x=t$, $y=t^2$, $z=t^3$. 由链式法则,

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}=f_x\cdot 1+f_y\cdot 2t+f_z\cdot 3t^2.$$

因此,

$$\mathrm{d}u=\left(f_x+2tf_y+3t^2f_z\right)\mathrm{d}t,$$

其中偏导数在 $(x,y,z)=(t,t^2,t^3)$ 处取值.

3. 给定 $u=f(\xi,\eta,\zeta)$, $\xi=x^2+y^2$, $\eta=x^2-y^2$, $\zeta=2xy$. 由链式法则,

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}+\frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\\ &=f_\xi\cdot 2x+f_\eta\cdot 2x+f_\zeta\cdot 2y=2x(f_\xi+f_\eta)+2yf_\zeta,\\ \frac{\partial u}{\partial y}&=\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}+\frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\\ &=f_\xi\cdot 2y+f_\eta\cdot(-2y)+f_\zeta\cdot 2x=2y(f_\xi-f_\eta)+2xf_\zeta.\end{aligned}$$

于是全微分为

$$\mathrm{d}u = \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y = \big[2x(f_\xi + f_\eta) + 2yf_\zeta\big]\mathrm{d}x + \big[2y(f_\xi - f_\eta) + 2xf_\zeta\big]\mathrm{d}y.$$

亦可写为 $\mathrm{d}u = f_\xi\,\mathrm{d}\xi + f_\eta\,\mathrm{d}\eta + f_\zeta\,\mathrm{d}\zeta$, 其中

$$\mathrm{d}\xi = 2x\,\mathrm{d}x + 2y\,\mathrm{d}y, \quad \mathrm{d}\eta = 2x\,\mathrm{d}x - 2y\,\mathrm{d}y, \quad \mathrm{d}\zeta = 2y\,\mathrm{d}x + 2x\,\mathrm{d}y.$$

习题 (9.3.1). 证明下列方程在指定点的邻域内对 y 有唯一解，并求出 y 对 x 在该点处的一阶和二阶导数. (1) $x^2 + xy + y^2 = 7$，在 $(2,1)$ 处；(2) $x\cos xy = 0$，在 $\left(1, \frac{\pi}{2}\right)$ 处.

解答:

1. 定义 $F(x,y) = x^2 + xy + y^2 - 7$。则 $F(2,1) = 0$，且

$$F_y(x,y) = x + 2y, \quad F_y(2,1) = 4 \neq 0.$$

由隐函数定理，在 $x = 2$ 的某邻域内存在唯一的 C^2 函数 $y = y(x)$ 使得 $y(2) = 1$ 且 $F(x, y(x)) = 0$。
对方程 $F(x,y) = 0$ 隐式求导：

$$F_x + F_y y' = 0, \quad F_x = 2x + y, \quad F_y = x + 2y,$$

故

$$y' = -\frac{2x+y}{x+2y}.$$

在点 $(2,1)$ 处，

$$y'(2) = -\frac{2\cdot 2+1}{2+2\cdot 1} = -\frac{5}{4}.$$

再次求导：

$$\frac{\mathrm{d}}{\mathrm{d}x}(F_x + F_y y') = 0 \Longrightarrow (2+y') + (1+2y')y' + (x+2y)y'' = 0.$$

代入 $x = 2$，$y = 1$，$y' = -5/4$：

$$\begin{aligned}
2 - \frac{5}{4} + \left(1 + 2\left(-\frac{5}{4}\right)\right)\left(-\frac{5}{4}\right) + (2+2)y'' &= 0, \\
\frac{3}{4} + \left(-\frac{3}{2}\right)\left(-\frac{5}{4}\right) + 4y'' &= 0, \\
\frac{3}{4} + \frac{15}{8} + 4y'' = 0 \Longrightarrow \frac{21}{8} + 4y'' &= 0,
\end{aligned}$$

所以

$$y''(2) = -\frac{21}{32}.$$

2. 定义 $G(x,y)=x\cos(xy)$。则 $G\left(1,\frac{\pi}{2}\right)=0$，且

$$G_y(x,y)=-x^2\sin(xy),\quad G_y\left(1,\frac{\pi}{2}\right)=-1\neq 0.$$

由隐函数定理，在 $x=1$ 的某邻域内存在唯一的 C^2 函数 $y=y(x)$ 使得 $y(1)=\frac{\pi}{2}$ 且 $G(x,y(x))=0$。
对方程 $G(x,y)=0$ 隐式求导：

$$G_x+G_y y'=0,\quad G_x=\cos(xy)-xy\sin(xy),\quad G_y=-x^2\sin(xy),$$

故

$$y'=-\frac{G_x}{G_y}=\frac{\cos(xy)-xy\sin(xy)}{x^2\sin(xy)}.$$

在点 $\left(1,\frac{\pi}{2}\right)$ 处，$\cos\left(\frac{\pi}{2}\right)=0$，$\sin\left(\frac{\pi}{2}\right)=1$，于是

$$y'(1)=\frac{0-\frac{\pi}{2}\cdot 1}{1\cdot 1}=-\frac{\pi}{2}.$$

再次隐式求导。令 $u=xy$，由 $G_x+G_y y'=0$ 得

$$\cos u-u\sin u-x^2\sin u\cdot y'=0.$$

对 x 求导：

$$-\sin u\cdot u'-u'(\sin u+u\cos u)-\left[2x\sin u\cdot y'+x^2\cos u\cdot u'\cdot y'+x^2\sin u\cdot y''\right]=0.$$

在 $x=1$，$y=\frac{\pi}{2}$ 处，$u=\frac{\pi}{2}$，$\sin u=1$，$\cos u=0$，$y'=-\frac{\pi}{2}$，$u'=y+xy'=\frac{\pi}{2}+\left(-\frac{\pi}{2}\right)=0$。代入得

$$0-0-\left[2\cdot 1\cdot 1\cdot\left(-\frac{\pi}{2}\right)+0+1\cdot 1\cdot y''\right]=0\Longrightarrow \pi-y''=0,$$

所以

$$y''(1)=\pi.$$

习题 (9.3.2). 求由下列方程所确定的隐函数的导数. (2) $\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$, 求 $\frac{\mathrm{d}y}{\mathrm{d}x}$ 和 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$;

解答: 将方程 $\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$ 化简为

$$\frac{1}{2}\ln(x^2+y^2)=\arctan\frac{y}{x}.$$

对两边关于 x 求导（视 y 为 x 的函数），记 $y'=\frac{\mathrm{d}y}{\mathrm{d}x}$。左端求导：

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln(x^2+y^2)\right)&=\frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot(2x+2yy')\\&=\frac{x+yy'}{x^2+y^2}.\end{aligned}$$

右端求导：

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{y'x-y}{x^2} = \frac{x^2}{x^2+y^2}\cdot\frac{xy'-y}{x^2} = \frac{xy'-y}{x^2+y^2}.$$

令两式相等：

$$\frac{x+yy'}{x^2+y^2} = \frac{xy'-y}{x^2+y^2}.$$

由于 $x^2+y^2>0$，比较分子得

$$x+yy' = xy'-y.$$

解出 y'：

$$x+y = xy'-yy' = y'(x-y),$$
$$y' = \frac{x+y}{x-y}, \quad x\neq y.$$

接下来求二阶导数 $y''=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$。对 $y'=\frac{x+y}{x-y}$ 关于 x 求导，使用商法则：

$$y'' = \frac{(1+y')(x-y)-(x+y)(1-y')}{(x-y)^2}.$$

化简分子：

$$(1+y')(x-y)-(x+y)(1-y') = (x-y)+y'(x-y)-(x+y)+y'(x+y) = -2y+2xy' = 2(xy'-y).$$

于是

$$y'' = \frac{2(xy'-y)}{(x-y)^2}.$$

代入 $y'=\frac{x+y}{x-y}$：

$$xy'-y = x\cdot\frac{x+y}{x-y}-y = \frac{x^2+xy-y(x-y)}{x-y} = \frac{x^2+xy-xy+y^2}{x-y} = \frac{x^2+y^2}{x-y}.$$

因此

$$y'' = \frac{2\cdot\frac{x^2+y^2}{x-y}}{(x-y)^2} = \frac{2(x^2+y^2)}{(x-y)^3}.$$