4.1.5 (1).

$$\begin{aligned}\int x\sin x\mathrm{d}x &= -\int x\mathrm{d}(\cos x)\\ &= -x\cos x+\int\cos x\mathrm{d}x\\ &= -x\cos x+\sin x+C.\end{aligned}$$

(3).

$$\begin{aligned}\int \cos\ln x\mathrm{d}x &= \int\cos t\mathrm{d}(\mathrm{e}^t)\quad (t=\ln x)\\ &= \mathrm{e}^t\cos t+\int\mathrm{e}^t\sin t\mathrm{d}t\\ &= \mathrm{e}^t\cos t+\int\sin t\mathrm{d}(\mathrm{e}^t)\\ &= \mathrm{e}^t\cos t+\mathrm{e}^t\sin t-\int\mathrm{e}^t\cos t\mathrm{d}t\\ &= \frac{1}{2}\mathrm{e}^t(\sin t+\cos t)+C\\ &= \frac{1}{2}x\big(\sin\ln x+\cos\ln x\big)+C.\end{aligned}$$

(5).

$$\begin{aligned}\int\sec^3 x\mathrm{d}x &= \int\sec x\sec^2 x\mathrm{d}x\\ &= \int\sec x\mathrm{d}(\tan x)\\ &= \sec x\tan x-\int\tan x\mathrm{d}(\sec x)\\ &= \sec x\tan x-\int\tan x\sec x\tan x\mathrm{d}x\\ &= \sec x\tan x-\int\sec x\tan^2 x\mathrm{d}x\\ &= \sec x\tan x-\int\sec x(\sec^2 x-1)\mathrm{d}x\\ &= \sec x\tan x-\int\sec^3 x\mathrm{d}x+\int\sec x\mathrm{d}x,\\ \Rightarrow\int\sec^3 x\mathrm{d}x &= \frac{1}{2}\sec x\tan x+\frac{1}{2}\int\sec x\mathrm{d}x\\ &= \frac{1}{2}\sec x\tan x+\frac{1}{2}\ln|\sec x+\tan x|+C.\end{aligned}$$

(7).

$$\begin{aligned}\int x\arcsin x\mathrm{d}x &= \int \sin t\cdot t\cdot\cos t\mathrm{d}t \quad (x=\sin t)\\ &= \int t\sin t\cos t\mathrm{d}t\\ &= -\frac{1}{2}t\cos^2 t+\frac{1}{2}\int\cos^2 t\mathrm{d}t\\ &= -\frac{1}{2}t\cos^2 t+\frac{1}{2}\int\frac{1+\cos 2t}{2}\mathrm{d}t\\ &= -\frac{1}{2}t\cos^2 t+\frac{1}{4}t+\frac{1}{8}\sin 2t+C\\ &= \Big(\frac{1}{2}x^2-\frac{1}{4}\Big)\arcsin x+\frac{1}{4}x\sqrt{1-x^2}+C.\end{aligned}$$

(9).

$$\begin{aligned}\int(\arcsin x)^2\mathrm{d}x &= \int t^2\cos t\mathrm{d}t \quad (x=\sin t)\\ &= \int t^2\mathrm{d}(\sin t)\\ &= t^2\sin t-\int\sin t\mathrm{d}(t^2)\\ &= t^2\sin t-\int 2t\sin t\mathrm{d}t\\ &= t^2\sin t-\Big(-2t\cos t+2\int\cos t\mathrm{d}t\Big)\\ &= t^2\sin t+2t\cos t-2\sin t+C\\ &= x(\arcsin x)^2+2\arcsin x\sqrt{1-x^2}-2x+C.\end{aligned}$$

4.1.6 (1). 对于 $n\geqslant 2$，

$$\begin{aligned}\int\sin^n x\mathrm{d}x &= -\int\sin^{n-1}x\mathrm{d}(\cos x)\\ &= -\sin^{n-1}x\cos x+\int\cos x\mathrm{d}(\sin^{n-1}x)\\ &= -\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\cos^2 x\mathrm{d}x\\ &= -\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x(1-\sin^2 x)\mathrm{d}x\\ &= -\sin^{n-1}x\cos x+(n-1)\int\sin^{n-2}x\mathrm{d}x-(n-1)\int\sin^n x\mathrm{d}x,\end{aligned}$$

$$\Rightarrow \int\sin^n x\mathrm{d}x = -\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int\sin^{n-2}x\mathrm{d}x.$$

(2). 对于 $n \geqslant 1$，

$$\begin{aligned}\int x^n \mathrm{e}^x \mathrm{d}x &= \int x^n \mathrm{d}(\mathrm{e}^x)\\ &= x^n \mathrm{e}^x - n\int \mathrm{e}^x x^{n-1}\mathrm{d}x.\end{aligned}$$

4.1.7 (2).

$$\begin{aligned}\int \frac{x^2-1}{x^4+x^2+1}\mathrm{d}x &= \int \frac{x^2-1}{(x^2+x+1)(x^2-x+1)}\mathrm{d}x\\ &= \int \left[\frac{-x-\frac{1}{2}}{x^2+x+1}+\frac{x-\frac{1}{2}}{x^2-x+1}\right]\mathrm{d}x\\ &= -\frac{1}{2}\int \frac{2x+1}{x^2+x+1}\mathrm{d}x+\frac{1}{2}\int \frac{2x-1}{x^2-x+1}\mathrm{d}x\\ &= -\frac{1}{2}\ln\big(x^2+x+1\big)+\frac{1}{2}\ln\big(x^2-x+1\big)+C.\end{aligned}$$

(4).

$$\begin{aligned}\int x\sqrt{x-2}\mathrm{d}x &= \int (t+2)\sqrt{t}\mathrm{d}t \qquad (t=x-2)\\ &= \int \big(t^{3/2}+2t^{1/2}\big)\mathrm{d}t\\ &= \frac{2}{5}t^{5/2}+\frac{4}{3}t^{3/2}+C\\ &= \frac{2}{5}(x-2)^{5/2}+\frac{4}{3}(x-2)^{3/2}+C.\end{aligned}$$

(6).

$$\begin{aligned}\int \frac{x\mathrm{e}^x}{\sqrt{\mathrm{e}^x-2}}\mathrm{d}x &= \int 2\ln(\mathrm{e}^x)\mathrm{d}\Big(\sqrt{\mathrm{e}^x-2}\Big) \qquad \Big(t=\sqrt{\mathrm{e}^x-2}\Big)\\ &= 2\int \ln(t^2+2)\mathrm{d}t\\ &= 2\Big[t\ln(t^2+2)-2\int \frac{t^2}{t^2+2}\mathrm{d}t\Big]\\ &= 2\Big[t\ln(t^2+2)-2\int\Big(1-\frac{2}{t^2+2}\Big)\mathrm{d}t\Big]\\ &= 2t\ln(t^2+2)-4t+4\sqrt{2}\arctan\frac{t}{\sqrt{2}}+C\\ &= 2x\sqrt{\mathrm{e}^x-2}-4\sqrt{\mathrm{e}^x-2}+4\sqrt{2}\arctan\frac{\sqrt{\mathrm{e}^x-2}}{\sqrt{2}}+C.\end{aligned}$$

(8).

$$\begin{aligned}\int \frac{1}{(1+\tan x)\sin^2 x}\mathrm{d}x &= \int \frac{1}{(1+t)t^2}\mathrm{d}t \qquad (t=\tan x)\\ &= \int \left(-\frac{1}{t}+\frac{1}{t^2}+\frac{1}{1+t}\right)\mathrm{d}t\\ &= -\ln|t|-\frac{1}{t}+\ln|1+t|+C\\ &= \ln\left|\frac{1+\tan x}{\tan x}\right|-\cot x+C.\end{aligned}$$

(10).

$$\begin{aligned}\int \sqrt{\frac{x-1}{x+1}}\frac{1}{x^2}\mathrm{d}x &= \int \frac{2\sqrt{u}}{(1+u)^2}\mathrm{d}u \qquad \left(x=\frac{1+u}{1-u}\right)\\ &= \int \frac{4t^2}{(1+t^2)^2}\mathrm{d}t \qquad (u=t^2)\\ &= \int \left(\frac{4}{1+t^2}-\frac{4}{(1+t^2)^2}\right)\mathrm{d}t\\ &= 4\arctan t-2\arctan t-\frac{2t}{1+t^2}+C\\ &= 2\arctan t-\frac{2t}{1+t^2}+C\\ &= 2\arctan\sqrt{\frac{x-1}{x+1}}-\frac{\sqrt{x^2-1}}{x}+C.\end{aligned}$$

(12).

$$\begin{aligned}\int \frac{x}{1+\sin x}\mathrm{d}x &= \int x\Big(\sec^2 x-\tan x\sec x\Big)\mathrm{d}x\\ &= x(\tan x-\sec x)-\int(\tan x-\sec x)\mathrm{d}x\\ &= x(\tan x-\sec x)+\ln|\cos x|+\ln|\sec x+\tan x|+C.\end{aligned}$$

(14).

$$\begin{aligned}\int \frac{x+\sin x}{1+\cos x}\mathrm{d}x &= \int \frac{x}{1+\cos x}\mathrm{d}x+\int\frac{\sin x}{1+\cos x}\mathrm{d}x\\ &= \int x\Big(\csc^2 x-\cot x\csc x\Big)\mathrm{d}x+\int(\csc x-\cot x)\mathrm{d}x\\ &= x(\csc x-\cot x)-\int(\csc x-\cot x)\mathrm{d}x+\int(\csc x-\cot x)\mathrm{d}x\\ &= x(\csc x-\cot x)+C.\end{aligned}$$

(16).

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}}\mathrm{d}x &= \frac{1}{2}\int \frac{x^2\cdot 2x}{\sqrt{1+x^2}}\mathrm{d}x\\ &= \frac{1}{2}\int \frac{t-1}{\sqrt{t}}\mathrm{d}t \qquad (t=1+x^2)\\ &= \frac{1}{2}\int \left(t^{1/2}-t^{-1/2}\right)\mathrm{d}t\\ &= \frac{1}{3}t^{3/2}-t^{1/2}+C\\ &= \frac{1}{3}(1+x^2)^{3/2}-\sqrt{1+x^2}+C.\end{aligned}$$

(18).

$$\begin{aligned}\int \frac{\arctan \mathrm{e}^x}{\mathrm{e}^x}\mathrm{d}x &= \int \frac{\arctan t}{t^2}\mathrm{d}t \qquad (t=\mathrm{e}^x)\\ &= uv-\int v\mathrm{d}u \quad \begin{cases} u=\arctan t, \quad \mathrm{d}u=\dfrac{\mathrm{d}t}{1+t^2},\\ v=-\dfrac{1}{t}\end{cases} \quad = -\frac{\arctan t}{t}+\int \frac{1}{t(1+t^2)}\mathrm{d}t\\ &= -\frac{\arctan t}{t}+\int \left(\frac{1}{t}-\frac{t}{1+t^2}\right)\mathrm{d}t\\ &= -\frac{\arctan t}{t}+\ln|t|-\frac{1}{2}\ln(1+t^2)+C\\ &= -\frac{\arctan(\mathrm{e}^x)}{\mathrm{e}^x}+x-\frac{1}{2}\ln\left(1+\mathrm{e}^{2x}\right)+C.\end{aligned}$$

(20).

$$\begin{aligned}\int \frac{x^2}{(x\sin x+\cos x)^2}\mathrm{d}x &= \int \frac{x^2}{x\cos x}\mathrm{d}\left(\frac{1}{x\sin x+\cos x}\right)\\ &= \int \frac{x}{\cos x}\mathrm{d}\left(\frac{1}{x\sin x+\cos x}\right)\\ &= \frac{x}{\cos x}\cdot\frac{1}{x\sin x+\cos x}-\int \frac{1}{x\sin x+\cos x}\mathrm{d}\left(\frac{x}{\cos x}\right)\\ &= \frac{x}{\cos x}\cdot\frac{1}{x\sin x+\cos x}-\int \frac{1}{\cos^2 x}\mathrm{d}x\\ &= \frac{x}{\cos x}\cdot\frac{1}{x\sin x+\cos x}-\tan x+C.\end{aligned}$$

(22).

$$\begin{aligned}\int \frac{1}{\sqrt{x-1+\sqrt{x+1}}}\mathrm{d}x &= \int \frac{2t}{\sqrt{t^2+t-2}}\mathrm{d}t \qquad (t=\sqrt{x+1}) \\ &= \int \frac{2t+1}{\sqrt{t^2+t-2}}\mathrm{d}t - \int \frac{1}{\sqrt{t^2+t-2}}\mathrm{d}t \\ &= 2\sqrt{t^2+t-2} - \int \frac{1}{\sqrt{t^2+t-2}}\mathrm{d}t \\ &= 2\sqrt{t^2+t-2} - \ln\left|t+\frac{1}{2}+\sqrt{t^2+t-2}\right| + C \\ &= 2\sqrt{x-1+\sqrt{x+1}} - \ln\left(\sqrt{x+1}+\frac{1}{2}+\sqrt{x-1+\sqrt{x+1}}\right) + C.\end{aligned}$$

(24).

$$\begin{aligned}\int \sqrt{\frac{x}{1-x\sqrt{x}}}\mathrm{d}x &= \int \frac{2t^2}{\sqrt{1-t^3}}\mathrm{d}t \qquad (t=\sqrt{x}) \\ &= -\frac{2}{3}\int (1-t^3)^{-1/2}\mathrm{d}(1-t^3) \\ &= -\frac{4}{3}\sqrt{1-t^3}+C \\ &= -\frac{4}{3}\sqrt{1-x\sqrt{x}}+C.\end{aligned}$$

(26).

$$\begin{aligned}\int \frac{x\mathrm{e}^x}{(1+x)^2}\mathrm{d}x &= -\int x\mathrm{e}^x \mathrm{d}\left(\frac{1}{x+1}\right) \\ &= -\frac{x\mathrm{e}^x}{x+1} + \int \frac{1}{x+1}\mathrm{d}(x\mathrm{e}^x) \\ &= -\frac{x\mathrm{e}^x}{x+1} + \int \mathrm{e}^x\mathrm{d}x \\ &= -\frac{x\mathrm{e}^x}{x+1} + \mathrm{e}^x + C \\ &= \frac{\mathrm{e}^x}{x+1} + C.\end{aligned}$$

4.2.1 (1).

$$\begin{aligned}\int \frac{1}{x^2+x-2}\mathrm{d}x &= \int \frac{1}{(x-1)(x+2)}\mathrm{d}x \\ &= \int \left(\frac{1/3}{x-1} - \frac{1/3}{x+2}\right)\mathrm{d}x \\ &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C.\end{aligned}$$

(3).

$$\begin{aligned}\int\frac{x^3+1}{x^3-x}\mathrm{d}x&=\int\frac{(x^3-x)+(x+1)}{x^3-x}\mathrm{d}x\\&=\int\left(1+\frac{x+1}{x(x^2-1)}\right)\mathrm{d}x\\&=\int\left(1+\frac{1}{x-1}-\frac{1}{x}\right)\mathrm{d}x\\&=x+\ln|x-1|-\ln|x|+C.\end{aligned}$$

(5).

$$\begin{aligned}\int\frac{x}{(x+1)^2(x^2+x+1)}\mathrm{d}x&=\int\left(\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+x+1}\right)\mathrm{d}x\\&=\int\left(-\frac{1}{(x+1)^2}+\frac{1}{x^2+x+1}\right)\mathrm{d}x\qquad(\text{解得}A=C=0,B=-1,D=1)\\&=\int-\frac{1}{(x+1)^2}\mathrm{d}x+\int\frac{1}{x^2+x+1}\mathrm{d}x\\&=\frac{1}{x+1}+\int\frac{1}{(x+\frac12)^2+\left(\frac{\sqrt3}{2}\right)^2}\mathrm{d}x\\&=\frac{1}{x+1}+\frac{2}{\sqrt3}\arctan\frac{2x+1}{\sqrt3}+C.\end{aligned}$$

(7).

$$\begin{aligned}\int\frac{x^5-x}{x^8+1}\mathrm{d}x&=\int\frac{x(x^4-1)}{x^8+1}\mathrm{d}x=\frac12\int\frac{x^4-1}{x^8+1}\mathrm{d}(x^2)\quad(t=x^2)\\&=\frac12\int\frac{t^2-1}{t^4+1}\mathrm{d}t\\&=\frac12\int\left(\frac{-\frac{\sqrt2}{2}t-\frac12}{t^2+\sqrt2t+1}+\frac{\frac{\sqrt2}{2}t-\frac12}{t^2-\sqrt2t+1}\right)\mathrm{d}t\\&=\frac12\left(-\frac{\sqrt2}{4}\int\frac{2t+\sqrt2}{t^2+\sqrt2t+1}\mathrm{d}t+\frac{\sqrt2}{4}\int\frac{2t-\sqrt2}{t^2-\sqrt2t+1}\mathrm{d}t\right)\\&=\frac{\sqrt2}{8}\left[\ln\bigl(t^2-\sqrt2t+1\bigr)-\ln\bigl(t^2+\sqrt2t+1\bigr)\right]+C\\&=\frac{\sqrt2}{8}\ln\frac{x^4-\sqrt2x^2+1}{x^4+\sqrt2x^2+1}+C.\end{aligned}$$

4.2.2 令 $t=\tan x$，则

$$\mathrm{d}t=\frac{1}{\cos^2x}\mathrm{d}x=(1+\tan^2x)\mathrm{d}x=(1+t^2)\mathrm{d}x,$$

即

$$\mathrm{d}x=\frac{1}{1+t^2}\mathrm{d}t.$$

(2).

$$\begin{aligned}
\int \frac{\sin^5 x}{\cos x}\mathrm{d}x &= \int \frac{\sin^4 x \tan x}{1}\mathrm{d}x \\
&= \int \frac{\sin^4 x \tan x}{(\sin^2 x + \cos^2 x)^2}\mathrm{d}x \\
&= \int \frac{\tan^5 x}{(\tan^2 x + 1)^2}\mathrm{d}x \\
&= \int \frac{t^5}{(t^2+1)^3}\mathrm{d}t \\
&= \frac{1}{2}\int \frac{(u-1)^2}{u^3}\mathrm{d}u \qquad (u = t^2+1) \\
&= \frac{1}{2}\int \left(u^{-1} - 2u^{-2} + u^{-3}\right)\mathrm{d}u \\
&= \frac{1}{2}\ln u + u^{-1} - \frac{1}{4}u^{-2} + C \\
&= -\ln|\cos x| + \cos^2 x - \frac{1}{4}\cos^4 x + C.
\end{aligned}$$

(4).

$$\begin{aligned}
\int \frac{\sin^2 x \cos x}{\sin x + \cos x}\mathrm{d}x &= \int \frac{t^2}{(t+1)(1+t^2)^2}\mathrm{d}t \\
&\text{设}\frac{t^2}{(t+1)(1+t^2)^2} = \frac{A}{t+1} + \frac{Bt+C}{1+t^2} + \frac{Dt+E}{(1+t^2)^2}, \\
&\text{解得}A = \tfrac{1}{4},\ B = -\tfrac{1}{4},\ C = \tfrac{1}{4},\ D = \tfrac{1}{2},\ E = -\tfrac{1}{2}, \\
\Rightarrow \int \frac{\sin^2 x \cos x}{\sin x + \cos x}\mathrm{d}x &= \int \left(\frac{1/4}{t+1} + \frac{-t/4+1/4}{1+t^2} + \frac{t/2-1/2}{(1+t^2)^2}\right)\mathrm{d}t \\
&= \frac{1}{4}\ln|t+1| - \frac{1}{8}\ln(1+t^2) - \frac{t+1}{4(1+t^2)} + C \\
&= \frac{1}{4}\ln|\sin x + \cos x| - \frac{1}{4}(\sin x + \cos x)\cos x + C.
\end{aligned}$$

(6).

$$\begin{aligned}
\int \frac{\sin^2 x}{1+\sin^2 x}\mathrm{d}x &= \int \frac{t^2}{(1+t^2)(1+2t^2)}\mathrm{d}t \\
&= \int \left(\frac{1}{1+t^2} - \frac{1}{1+2t^2}\right)\mathrm{d}t \\
&= \arctan t - \frac{1}{\sqrt{2}}\arctan(\sqrt{2}, t) + C \\
&= x - \frac{1}{\sqrt{2}}\arctan!\left(\sqrt{2}, \tan x\right) + C.
\end{aligned}$$

(8).

$$\begin{aligned}
\int \frac{1}{\sin^4 x \cos^4 x} \mathrm{d}x &= \int \frac{(1+t^2)^4}{t^4} \cdot \frac{1}{1+t^2} \mathrm{d}t \\
&= \int \frac{(1+t^2)^3}{t^4} \mathrm{d}t \\
&= \int \left(t^{-4} + 3t^{-2} + 3 + t^2\right) \mathrm{d}t \\
&= -\frac{1}{3t^3} - \frac{3}{t} + 3t + \frac{1}{3} t^3 + C \\
&= -\frac{1}{3} \cot^3 x - 3 \cot x + 3 \tan x + \frac{1}{3} \tan^3 x + C.
\end{aligned}$$

(10). 注意

$$\cos x = \frac{a}{a^2+b^2}(a \cos x - b \sin x) + \frac{b}{a^2+b^2}(a \sin x + b \cos x),$$

于是

$$\begin{aligned}
\int \frac{\cos x}{a \sin x + b \cos x} \mathrm{d}x &= \frac{a}{a^2+b^2} \int \frac{a \cos x - b \sin x}{a \sin x + b \cos x} \mathrm{d}x + \frac{b}{a^2+b^2} \int \mathrm{d}x \\
&= \frac{a}{a^2+b^2} \ln |a \sin x + b \cos x| + \frac{b}{a^2+b^2} x + C.
\end{aligned}$$