3.6.1 (1). 因为

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + o(x^n).$$

所以

$$\begin{aligned} y = \frac{x^3+2x+1}{x-1} &= x^2 + x + 3 - \frac{4}{1-x} \\ &= -1 - 3x - 3x^2 - 4x^3 - 4x^4 - 4x^5 - \cdots - 4x^n + o(x^n). \end{aligned}$$

(2). 因为

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + o(x^{2n}),$$

所以

$$y = \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{4}x^2 - \frac{1}{48}x^4 + \cdots + \frac{(-1)^{n+1}}{2\cdot(2n)!}x^{2n} + o(x^{2n}).$$

3.6.2 因为 $x \to 0$ 时也有 $\sin x \to 0$，而且 $\sin x \sim x$，所以

$$\mathrm{e}^{\sin x} = 1 + \sin x + \frac{1}{2}\sin^2 x + \frac{1}{6}\sin^3 x + o(x^3),$$

我们先展开 $\sin x$，

$$\sin x = x + \frac{1}{6}x^3 + o(x^3),$$

然后考虑

$$\begin{aligned} \sin^2 x &= \left(x - \frac{1}{6}x^3 + o(x^3)\right)^2 \\ &= x^2 + \frac{1}{36}x^6 + o(x^6) - \frac{1}{3}x^4 + o(x^4) + o(x^6) \\ &= x^2 + o(x^3), \\ \sin^3 x &= \sin^2 x \cdot \sin x \\ &= \big(x^2 + o(x^3)\big) \cdot \left(x - \frac{1}{6}x^3 + o(x^3)\right) \\ &= x^3 - \frac{1}{6}x^5 + o(x^5) + o(x^4) + o(x^6) + o(x^6) \\ &= x^3 + o(x^3), \end{aligned}$$

因此

$$\begin{aligned} \mathrm{e}^{\sin x} &= 1 + \sin x + \frac{1}{2}\sin^2 x + \frac{1}{6}\sin^3 x + o(x^3) \\ &= 1 + \left(x - \frac{1}{6}x^3 + o(x^3)\right) + \frac{1}{2}\big(x^2 + o(x^3)\big) + \frac{1}{6}\big(x^3 + o(x^3)\big) \\ &= 1 + x + \frac{1}{2}x^2 + o(x^3). \end{aligned}$$

3.6.3 因为 $x \to 0$ 时 $\cos x \to 1$，我们先对 $\ln(x+1)$ 在 $x=0$ 处展开：

$$\ln(x+1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + o(x^6),$$

再展开 $\cos x - 1$，

$$\cos x - 1 = -\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + o(x^6),$$

进一步地，

$$\begin{aligned}
(\cos x - 1)^2 &= \left(-\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + o(x^6)\right)^2 \\
&= \frac{1}{4}x^4 - \frac{1}{24}x^6 + o(x^6), \\
(\cos x - 1)^3 &= \left(-\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + o(x^6)\right)^3 \\
&= -\frac{1}{8}x^6 + o(x^6),
\end{aligned}$$

对于更高次数，均是 $o(x^6)$. 因此，我们得到

$$\begin{aligned}
\ln\cos x &= \ln((\cos x - 1) + 1) \\
&= (\cos x - 1) - \frac{1}{2}(\cos x - 1)^2 + \frac{1}{3}(\cos x - 1)^3 + o(x^6) \\
&= -\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 - \frac{1}{8}x^4 + \frac{1}{48}x^6 - \frac{1}{24}x^6 + o(x^6) \\
&= -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 + o(x^6).
\end{aligned}$$

3.6.4 对 $f(x)$ 在 $x=2$ 处展开，

$$\begin{aligned}
f(x) &= f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \frac{1}{6}f'''(2)(x-2)^3 + \frac{1}{24}f''''(2)(x-2)^4 \\
&= -1 + (x-2)^2 - 2(x-2)^3 + (x-2)^4, \\
f'(x) &= 2(x-2) - 6(x-2)^2 + 4(x-2)^3, \\
f''(x) &= 2 - 12(x-2) + 12(x-2)^2,
\end{aligned}$$

因此 $f(-1) = 143$，$f'(0) = -60$，$f'''(1) = 26$.

3.6.5 (1). 先求各阶导数：

$$y = \tan x,\ y' = \frac{1}{\cos^2 x},\ y'' = \frac{2\sin x}{\cos^3 x}, y''' = \frac{2 + 4\sin^2 x}{\cos^4 x}$$

因此 $y(0) = 0$，$y'(0) = 1$，$y''(0) = 0$，

$$y = \tan x = x + \frac{x^3}{3} \cdot \frac{1 + 2\sin^2\theta x}{\cos^4\theta x},\ \theta \in (0,1).$$

(2). 归纳法易证

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{1}{x}\right) = (-1)^k \frac{x^{k+1}}{k!},$$

因此

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{1}{x}\right)\bigg|_{x=-1} = -\frac{1}{k!}.$$

从而

$$y = \frac{1}{x} = -\sum_{k=0}^{n}(x+1)^k + \frac{(-1)^{n+1}}{(\theta x)^{n+2}}\cdot(x+1)^{n+1},\ \theta\in(0,1).$$

3.6.6 (1). 分母上 $\sin^4 x \sim x^4$ 是 4 阶，所以需要把分子展开到 4 阶以上：

$$\begin{aligned}\cos x - \mathrm{e}^{-\frac{1}{2}x^2} &= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)\right) - \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + o(x^4)\right)\\ &= -\frac{1}{12}x^4 + o(x^4),\end{aligned}$$

因此原极限 $= -\frac{1}{12}$.

(2). 分母上 x^2 是 2 阶，所以需要把分子展开到 2 阶以上：

$$\begin{aligned}(1+x^2)^{\frac{1}{4}} - (1-x^2)^{\frac{1}{4}} &= \left(1 + \frac{1}{4}x^2 + o(x^2)\right) - \left(1 - \frac{1}{4}x^2 + o(x^2)\right)\\ &= \frac{1}{2}x^2 + o(x^2),\end{aligned}$$

所以原极限 $= \frac{1}{2}$.

(3). 换元 $x = \frac{1}{t}$，那么

$$\lim_{x\to\infty}\left[x - x^2\ln\left(1+\frac{1}{x}\right)\right] = \lim_{t\to 0}\frac{t - \ln(1+t)}{t^2},$$

把分子展开到 2 阶以上：

$$t - \ln(1+t) = t - \left(t - \frac{1}{2}t^2 + o(t^2)\right) = \frac{1}{2}t^2 + o(t^2),$$

所以原极限 $= \frac{1}{2}$.

(4). 分母上 $\sin^4 x \sim x^4$ 是 4 阶，所以需要把分子展开到 4 阶以上。难点在于 $\cos\sin x$

的展开，与本节第 2、3 题的方法差不多。

$$\begin{aligned}
\cos x &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4), \\
\sin x &= x - \frac{x^3}{6} + o(x^4), \\
\sin^2 x &= x^2 - \frac{x^4}{3} + o(x^4), \\
\sin^4 x &= x^4 + o(x^4), \\
\cos\sin x &= 1 - \frac{1}{2}\sin^2 x + \frac{1}{24}\sin^4 x + o(x^4) \\
&= 1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 + \frac{1}{24}x^4 + o(x^4) \\
&= 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + o(x^4), \\
\cos\sin x - \cos x &= \frac{1}{6}x^4 + o(x^4),
\end{aligned}$$

从而原极限 $= \frac{1}{6}$.

4.1.1 (1).

$$\begin{aligned}
\int x(x-1)^3 \mathrm{d}x &= \int (x-1)^4 \mathrm{d}x + \int (x-1)^3 \mathrm{d}x \\
&= \frac{1}{5}(x-1)^5 + \frac{1}{4}(x-1)^4 + C \\
&= \frac{(4x+1)(x-1)^4}{20} + C.
\end{aligned}$$

(2).

$$\begin{aligned}
\int \frac{\mathrm{e}^{3x}+1}{\mathrm{e}^x+1} \mathrm{d}x &= \int (\mathrm{e}^{2x} - \mathrm{e}^x + 1) \mathrm{d}x \\
&= \frac{1}{2}\mathrm{e}^{2x} - \mathrm{e}^x + x + C.
\end{aligned}$$

(3).

$$\begin{aligned}
\int (2^x + 3^x)^2 \mathrm{d}x &= \int (4^x + 2\cdot 6^x + 9^x) \mathrm{d}x \\
&= \frac{4^x}{\ln 4} + \frac{2\cdot 6^x}{\ln 6} + \frac{9^x}{\ln 9}.
\end{aligned}$$

(4).

$$\begin{aligned}
\int \tan^2 x \mathrm{d}x &= \int \left(\frac{1}{\cos^2 x} - 1\right) \mathrm{d}x \\
&= \tan x - x + C.
\end{aligned}$$

(5).

$$\begin{aligned}\int \frac{x^2}{1+x^2}\mathrm{d}x &= \int\left(1-\frac{1}{1+x^2}\right)\mathrm{d}x\\&= x-\arctan x + C.\end{aligned}$$

(6).

$$\begin{aligned}\int \frac{1+\cos^2 x}{1+\cos 2x}\mathrm{d}x &= \int \frac{1+\cos^2 x}{2\cos^2 x}\mathrm{d}x\\&= \int\left(\frac{1}{2\cos^2 x}+\frac{1}{2}\right)\mathrm{d}x\\&= \frac{\tan x + x}{2}+C.\end{aligned}$$

4.1.2 (1).

$$\begin{aligned}\int (2x-1)^{100}\mathrm{d}x &= \frac{1}{2}\int (2x-1)^{100}\mathrm{d}(2x-1)\\&= \frac{1}{202}(2x-1)^{101}+C.\end{aligned}$$

(2).

$$\begin{aligned}\int \frac{1}{x^2}\sin\frac{1}{x}\mathrm{d}x &= -\int \sin\frac{1}{x}\mathrm{d}\frac{1}{x}\\&= \cos\frac{1}{x}+C.\end{aligned}$$

(3).

$$\begin{aligned}\int \frac{\cos x-\sin x}{1+\sin x+\cos x}\mathrm{d}x &= \int \frac{1}{1+\sin x+\cos x}(\cos x-\sin x)\mathrm{d}x\\&= \int \frac{1}{1+\sin x+\cos x}\mathrm{d}(1+\sin x+\cos x)\\&= \ln|1+\sin x+\cos x|+C.\end{aligned}$$

(4).

$$\begin{aligned}\int \frac{\arctan x}{1+x^2}\mathrm{d}x &= \int \arctan x\cdot\frac{1}{1+x^2}\mathrm{d}x\\&= \int \arctan x\mathrm{d}(\arctan x)\\&= \frac{1}{2}\big(\arctan x\big)^2+C.\end{aligned}$$

(5).

$$\begin{aligned}\int x\sqrt{1-x^2}\mathrm{d}x &= -\frac{1}{2}\int\sqrt{1-x^2}\mathrm{d}(1-x^2)\\&= -\frac{1}{2}\cdot\frac{2}{3}(1-x^2)^{\frac{3}{2}}+C\\&= -\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C.\end{aligned}$$

(6).

$$\begin{aligned}\int \frac{1}{\sqrt{x}(1+x)}\mathrm{d}x &= \int \frac{1}{\sqrt{x}(1+x)} \cdot 2\sqrt{x}\mathrm{d}(\sqrt{x}) \\ &= 2\int \frac{1}{1+(\sqrt{x})^2}\mathrm{d}(\sqrt{x}) \\ &= 2\arctan\sqrt{x} + C.\end{aligned}$$

(7). 注意

$$\arctan\frac{1}{x} = \frac{\pi}{2} - \arctan x,$$

所以

$$\mathrm{d}\arctan\frac{1}{x} = -\frac{1}{1+x^2}\mathrm{d}x,$$

于是

$$\begin{aligned}\int \frac{\arctan\frac{1}{x}}{1+x^2}\mathrm{d}x &= -\int \arctan\frac{1}{x}\mathrm{d}\left(\arctan\frac{1}{x}\right) \\ &= -\frac{1}{2}\left(\arctan\frac{1}{x}\right)^2 + C.\end{aligned}$$

(8).

$$\begin{aligned}\int \frac{1+\ln x}{1+x\ln x}\mathrm{d}x &= \int \frac{1}{1+x\ln x}(1+\ln x)\mathrm{d}x \\ &= \int \frac{1}{1+x\ln x}\mathrm{d}(1+x\ln x) \\ &= \ln|1+x\ln x| + C.\end{aligned}$$

(9).

$$\begin{aligned}\int \sin^2 x\mathrm{d}x &= \frac{1}{2}\int(1-\cos 2x)\mathrm{d}x \\ &= \frac{1}{2}\int \mathrm{d}x - \frac{1}{2}\int \cos 2x\mathrm{d}x \\ &= \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\int \cos 2x\mathrm{d}(2x) \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C.\end{aligned}$$

(10).

$$\begin{aligned}\int \sin^5 x\cos x\mathrm{d}x &= \int \sin^5 x\mathrm{d}(\sin x) \\ &= \frac{1}{6}\sin^6 x + C.\end{aligned}$$

4.1.3 (1).

$$令t=\sqrt{e^x-2},\ t^2=e^x-2,\ \mathrm{d}t=\frac{e^x}{2\sqrt{e^x-2}}\mathrm{d}x,\ 则\mathrm{d}x=\frac{2t}{t^2+2}\mathrm{d}t,$$

$$\begin{aligned}\int\sqrt{e^x-2}\mathrm{d}x&=\int t\cdot\frac{2t}{t^2+2}\mathrm{d}t\\&=2\int\frac{t^2}{t^2+2}\mathrm{d}t\\&=2\int\left(1-\frac{2}{t^2+2}\right)\mathrm{d}t\\&=2t-4\int\frac{1}{t^2+2}\mathrm{d}t\\&=2t-\frac{4}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C\\&=2\sqrt{e^x-2}-2\sqrt{2}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}+C.\end{aligned}$$

(3).

$$\begin{aligned}&令x=a\sec t,\ \mathrm{d}x=a\sec t\tan t\mathrm{d}t,\\&\Rightarrow x^2-a^2=a^2\tan^2t,\ (x^2-a^2)^{3/2}=a^3\tan^3t,\\\int\frac{1}{(x^2-a^2)^{3/2}}\mathrm{d}x&=\int\frac{a\sec t\tan t}{a^3\tan^3t}\mathrm{d}t\\&=\frac{1}{a^2}\int\frac{\sec t}{\tan^2t}\mathrm{d}t\\&=\frac{1}{a^2}\int\frac{\cos t}{\sin^2t}\mathrm{d}t\\&=\frac{1}{a^2}\int\frac{1}{\sin^2t}\mathrm{d}(\sin t)\\&=-\frac{1}{a^2}\frac{1}{\sin t}+C\\&=-\frac{1}{a^2}\cdot\frac{x}{\sqrt{x^2-a^2}}+C.\end{aligned}$$

(5).

$$\begin{aligned}&令t=\sqrt{x+1},\ x=t^2-1,\ \mathrm{d}x=2t\mathrm{d}t,\\\int\frac{1}{1+\sqrt{x+1}}\mathrm{d}x&=\int\frac{1}{1+t}\cdot2t\mathrm{d}t\\&=2\int\frac{t}{1+t}\mathrm{d}t\\&=2\int\left(1-\frac{1}{1+t}\right)\mathrm{d}t\\&=2t-2\ln(1+t)+C\\&=2\sqrt{x+1}-2\ln\left(1+\sqrt{x+1}\right)+C.\end{aligned}$$

(7).

$$
\begin{aligned}
&\text{令} t = \ln x,\ x = e^t,\ \mathrm{d}x = e^t \mathrm{d}t,\\
\int \frac{1-\ln x}{(x-\ln x)^2}\mathrm{d}x &= \int \frac{1-t}{(e^t-t)^2} e^t \mathrm{d}t\\
&= \int \frac{e^t(1-t)}{(e^t-t)^2}\mathrm{d}t\\
&= \int \mathrm{d}!\left(\frac{t}{e^t-t}\right)\\
&= \frac{t}{e^t-t} + C\\
&= \frac{\ln x}{x-\ln x} + C.
\end{aligned}
$$

(9).

$$
\begin{aligned}
&\text{令} t = \sqrt[3]{2x+1},\ 2x+1 = t^3,\ x = \frac{t^3-1}{2},\ \mathrm{d}x = \frac{3}{2}t^2\mathrm{d}t,\\
&x+2 = \frac{t^3-1}{2} + 2 = \frac{t^3+3}{2},\\
\int \frac{x+2}{\sqrt[3]{2x+1}}\mathrm{d}x &= \int \frac{t^3+3}{2t}\cdot\frac{3}{2}t^2\mathrm{d}t\\
&= \frac{3}{4}\int (t^3+3)t\mathrm{d}t\\
&= \frac{3}{4}\int (t^4+3t)\mathrm{d}t\\
&= \frac{3}{4}\left(\frac{t^5}{5}+\frac{3t^2}{2}\right) + C\\
&= \frac{3}{20}t^5 + \frac{9}{8}t^2 + C\\
&= \frac{3}{20}(2x+1)^{5/3} + \frac{9}{8}(2x+1)^{2/3} + C.
\end{aligned}
$$

(11).

$$\begin{aligned}
&\text{令} t = \frac{1}{x},\ \mathrm{d}t = -\frac{1}{x^2}\mathrm{d}x,\ \mathrm{d}x = -x^2\mathrm{d}t,\\
\int \frac{x-1}{x^2\sqrt{x^2-1}}\mathrm{d}x &= \int \frac{x-1}{x^2\sqrt{x^2-1}} \cdot (-x^2)\mathrm{d}t\\
&= -\int \frac{x-1}{\sqrt{x^2-1}}\mathrm{d}t\\
&= -\int \frac{\frac{1}{t}-1}{\sqrt{\frac{1}{t^2}-1}}\mathrm{d}t\\
&= -\int \frac{1-t}{\sqrt{1-t^2}}\mathrm{d}t\\
&= \int \left(\frac{t}{\sqrt{1-t^2}} - \frac{1}{\sqrt{1-t^2}}\right)\mathrm{d}t\\
&= \int \frac{t}{\sqrt{1-t^2}}\mathrm{d}t - \int \frac{1}{\sqrt{1-t^2}}\mathrm{d}t\\
&= -\sqrt{1-t^2} - \arcsin t + C\\
&= -\sqrt{1-\frac{1}{x^2}} - \arcsin\frac{1}{x} + C.
\end{aligned}$$

4.1.4 (1).

$$|x| = \begin{cases} -x & ,x < 0. \\ x & ,x \geqslant 0. \end{cases}$$

所以

$$\int |x|\mathrm{d}x = \begin{cases} -\frac{1}{2}x^2 + C_1 & ,x < 0. \\ \frac{1}{2}x^2 + C_2 & ,x \geqslant 0. \end{cases}$$

为确保可微性，应当 $C_1 = C_2$.

(2).

$$\max\{1, x^2\} = \begin{cases} x^2 & ,x \leqslant -1. \\ 1 & ,x \in (-1,1). \\ x^2 & ,x \geqslant 1. \end{cases}$$

所以

$$\int \max\{1, x^2\}\mathrm{d}x = \begin{cases} \frac{1}{3}x^3 + C_1 & ,x \leqslant -1. \\ x + C_2 & ,x \in (-1,1). \\ \frac{1}{3}x^3 + C_3 & ,x \geqslant 1. \end{cases}$$

为确保可微性，应当 $C_1 = C_2 = C_3$.