Solution 9.

38. 由因子分解定理和 Th2.8.1 得 $\bar{X}$ 为充分完全统计量

$E[(\bar{X})^2-\frac{1}{n}]=\theta^2$. ∴ $\theta^2$ 的 UMVUE 为 $(\bar{X})^2-\frac{1}{n}$.

$f(x,\theta)=\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\theta)^2}{2}}$ 指数族

$I(\theta)=-E_\theta[\frac{\partial^2}{\partial\theta^2}\log f(x,\theta)]$

$=1$

$\Rightarrow$ 方差下界为 $\frac{(g'(\theta))^2}{nI(\theta)}=\frac{4\theta^2}{n}$

$D((\bar{X})^2-\frac{1}{n})=D((\bar{X})^2)$ 　　又 $\bar{X}\sim N(\theta,\frac{1}{n})$

$=\frac{1}{n^2}D((\sqrt{n}\bar{X})^2)$ 　　$\sqrt{n}\bar{X}\sim N(\sqrt{n}\theta,1)$. 　$(\sqrt{n}\bar{X})^2\sim\chi^2_{1,\sqrt{n}\theta}$

$=\frac{1}{n^2}(2\times1+4(\sqrt{n}\theta)^2)$

$=\frac{2}{n^2}+\frac{4\theta^2}{n}>\frac{4\theta^2}{n}$ 　　∴不是有效估计.

40. $f(x,\theta)$ 为指数族, 满足 C-R 不等式条件.

$f(x,\theta)=\frac{1}{\theta}e^{-\frac{x}{\theta}}\cdot I_{(0,+\infty)}(x)$ 　　$\frac{\partial^2}{\partial\theta^2}l(\theta)=\frac{1}{\theta^2}-\frac{2x}{\theta^3}$

$I(\theta)=-E_\theta[\frac{\partial^2}{\partial\theta^2}l(\theta)]=\frac{1}{\theta^2}$

∴ C-R 下界为 $\frac{1}{nI(\theta)}=\frac{\theta^2}{n}$

考虑 $g(x)=\bar{X}$. $E(\bar{X})=\theta$, 故 $\theta$ 的无偏估计.

$D(\bar{X})=\frac{1}{n}D(X)=\frac{\theta^2}{n}$ 达到 C-R 不等式下界.

i.e. $\bar{X}$ 为 $\theta$ 的 UMVUE.

42. (1) $N(0, \sigma^2)$

$$f(x, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

$$l(\theta) = l(\sigma^2) = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\sigma^2 - \frac{x^2}{2\sigma^2} \quad 令 \sigma^2 = \theta$$

$$= -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\theta - \frac{x^2}{2\theta}$$

指数族 $\Rightarrow$ $I(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2} l(\theta)\right]$

$$= \frac{1}{2\theta^2}$$

$\therefore$ $\sigma^2$ 的 C-R 下界为 $\frac{2\sigma^4}{n}$.

(2). $f(x_1, \cdots, x_n) = (\sqrt{2\pi}\sigma)^{-n} e^{-\frac{\sum x_i^2}{2\sigma^2}}$

由指数族 $\Rightarrow$ $\frac{1}{n}\sum_{i=1}^{n} x_i^2$ 为充分完全统计量

$$E\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) = \sigma^2$$

$\therefore$ $\frac{1}{n}\sum_{i=1}^{n} x_i^2$ 为 $\sigma^2$ 的 UMVUE.

$$Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) = \frac{1}{n} Var(x_1^2)$$

$$= \frac{\sigma^4}{n} Var\left(\left(\frac{x_1}{\sigma}\right)^2\right)$$

$$= \frac{2\sigma^4}{n}$$

$\therefore$ 效率 $e_{g}(\theta) = \frac{\frac{2\sigma^4}{n}}{\frac{2\sigma^4}{n}} = 1$.

□.