Solution 5

39. Pascal 分布 $f(x,\theta) = C_{x-1}^{r-1} p^r (1-p)^{x-r}$

$$= p^r \cdot \exp\{\ln(1-p)(x-r)\} \cdot C_{x-1}^{r-1}$$

其中 $\theta = p$，$C(\theta) = p^r$　$\omega_1(\theta) = \ln(1-p)$　$t_1(x) = x-r$　$h(x) = C_{x-1}^{r-1}$

令 $\eta = \ln(1-p) \Rightarrow p^r = (1-e^{\eta})^r$

自然形式：$f(x,\eta) = (1-e^{\eta})^r \cdot \exp\{\eta(x-r)\} \cdot C_{x-1}^{r-1}$

由 $p \in (0,1) \Rightarrow$ 自然参数空间 $\Theta^* = \{\mu \mid \mu \in (-\infty, 0)\}$

指数分布：$f(x,\lambda) = \lambda e^{-\lambda x} I_{\{x>0\}}$　$(\lambda > 0)$

$C(\theta) = \lambda$，$\omega_1(\theta) = -\lambda$　$t_1(x) = x$　$h(x) = I_{\{x>0\}}$

令 $-\lambda = \eta \Rightarrow$ 自然形式 $f(\eta) = -\eta \exp\{\eta x\} I_{\{x>0\}}$

自然参数空间 $\Theta^* = \{\mu \mid \mu \in (-\infty, 0)\}$. □

40. $f(x,\theta) = C(\theta) \exp\{\sum_{j=1}^{k} \theta_j T_j(x)\} h(x)$

$$\int C(\theta) \exp\{\sum_{j=1}^{k} \theta_j T_j(x)\} h(x)\,dx = 1$$

$$\frac{\partial}{\partial \theta_j} \int C(\theta) \exp\{\sum_{j=1}^{k} \theta_j T_j(x)\} h(x)\,dx = 0$$

$$\xRightarrow{\text{Thm 2.6.2}} \int \frac{\partial C(\theta)}{\partial \theta_j} \exp\{\sum_{j=1}^{k} \theta_j T_j(x)\} h(x)\,dx + \int T_j(x) C(\theta) \exp\{\sum_{j=1}^{k} \theta_j T_j(x)\} h(x)\,dx = 0$$

$$\frac{1}{C(\theta)} \frac{\partial C(\theta)}{\partial \theta_j} \int C(\theta) \exp\{\sum_{j=1}^{k} \theta_j T_j(x)\} h(x)\,dx + \int T_j(x) C(\theta) \exp\{\sum_{j=1}^{k} \theta_j T_j(x)\} h(x)\,dx = 0$$

$$\frac{1}{C(\theta)} \frac{\partial C(\theta)}{\partial \theta_j} + E_\theta[T_j(x)] = 0$$

$$\Rightarrow E_\theta[T_j(x)] = -\frac{1}{C(\theta)} \frac{\partial C(\theta)}{\partial \theta_j} = -\frac{\partial \ln C(\theta)}{\partial \theta_j}$$

转下页.

$$0 = \frac{\partial^2 0}{\partial \theta_j \partial \theta_s} \int c(\theta) \exp\left\{\sum_{j=1}^{k} \theta_j T_j(x)\right\} h(x) dx$$

$$= \frac{\partial^2 \ln c(\theta)}{\partial \theta_j \partial \theta_s} + \int T_j(x) \frac{\partial c(\theta)}{\partial \theta_s} \exp\left\{\sum_{j=1}^{k} \theta_j T_j(x)\right\} h(x) dx$$

$$+ \int T_j(x)\, c(\theta) \cdot \exp\left\{\sum_{j=1}^{k} \theta_j T_j(x)\right\} T_s(x)\, h(x)\, dx$$

$$= \frac{\partial^2 \ln c(\theta)}{\partial \theta_j \partial \theta_s} + E(T_j(x) \cdot T_s(x)) + \frac{1}{c(\theta)} \cdot \frac{\partial c(\theta)}{\partial \theta_s} \cdot \int T_j(x) c(\theta) \cdot \exp\left\{\sum_{j=1}^{k} \theta_j T_j(x)\right\} h(x) dx$$

$$= \frac{\partial^2 \ln c(\theta)}{\partial \theta_j \partial \theta_s} + E(T_j(x) \cdot T_s(x)) - E(T_s(x)) \cdot E(T_j(x))$$

$$= \frac{\partial^2 \ln c(\theta)}{\partial \theta_j \partial \theta_s} + Cov(T_j(x), T_s(x))$$

$$\Rightarrow \quad Cov(T_j(x), T_s(x)) = -\frac{\partial^2 \ln c(\theta)}{\partial \theta_j \partial \theta_s}$$

□

41. 设指数族自然形式为

$$f(x,\theta) = c(\theta) \exp\left\{\sum_{i=1}^{k} \theta_i T_i(x)\right\} h(x)$$

$$\Theta^* = \left\{\theta : \int_{\mathcal{X}} \exp\left\{\sum_{i=1}^{k} \theta_i T_i(x)\right\} h(x) dx < +\infty\right\}$$

只需证 $\forall \theta^{(1)} = (\theta_1^{(1)}, \cdots, \theta_k^{(1)}),\ \theta^{(2)} = (\theta_1^{(2)}, \cdots, \theta_k^{(2)}) \in \Theta^*,\ \forall \alpha \in (0,1)$

有 $\alpha\theta^{(1)} + (1-\alpha)\theta^{(2)} \in \Theta^*$

事实上：

$$\int_{\mathcal{X}} e^{\sum_{i=1}^{k} (\alpha\theta_i^{(1)} + (1-\alpha)\theta_i^{(2)}) T_i(x)} \cdot h(x)\, dx$$

$$= \int_{\mathcal{X}} \left[e^{\sum_{i=1}^{k} \theta_i^{(1)} T_i(x)} h(x)\right]^{\alpha} \left[e^{\sum_{i=1}^{k} \theta_i^{(2)} T_i(x)} h(x)\right]^{1-\alpha} dx$$

$$\overset{\text{Hölder}}{\leq} \left[\int_{\mathcal{X}} e^{\sum_{i=1}^{k} \theta_i^{(1)} T_i(x)} h(x) dx\right]^{\alpha} \cdot \left[\int_{\mathcal{X}} e^{\sum_{i=1}^{k} \theta_i^{(2)} T_i(x)} h(x) dx\right]^{1-\alpha}$$

$< +\infty$

□

45. $f(x,\lambda) = \lambda e^{-\lambda x} I_{\{x>0\}}$

$f(x_1,x_2,\cdots,x_n,\lambda) = \lambda^n \cdot e^{-n\lambda\cdot\bar{x}} \cdot I_{\{x_{(1)}>0\}}$

$\therefore\ g(\bar{x},\lambda) = \lambda^n e^{-n\lambda\cdot\bar{x}} \qquad h(x) = I_{\{x_{(1)}>0\}}$

由因子分解定理 $\bar{x}$ 为充分统计量

□

47. 证完全性. $(T_1, T_2, T_3) = (\bar{X}, \bar{Y}, S^2)$

$f(\vec{x},\vec{y},\vec{\theta}) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{m+n} e^{-\frac{1}{2\sigma^2}(ma^2+nb^2)} \cdot \exp\{-\frac{1}{2\sigma^2}[(n-m-2)T_3+mT_1^2+nT_2^2]+\frac{a}{\sigma^2}mT_1+\frac{b}{\sigma^2}nT_2\}$

取 $\eta_1 = -\frac{1}{2\sigma^2}$, $\eta_2 = \frac{a}{\sigma^2}$, $\eta_3 = \frac{b}{\sigma^2}$

$\Theta^* = \{(\eta_1,\eta_2,\eta_3) \mid (\eta_1,\eta_2,\eta_3) \in (-\infty,0)\times R\times R\}$ 作为 R^3 子集有内点

$\Rightarrow ((n+m-2)T_3+mT_1^2+nT_2^2, mT_1, nT_2)$ 为完全统计量, 记为 (T_1', T_2', T_3')

显然存在一个可测函数 φ s.t. $\varphi(T_1',T_2',T_3') = (T_1,T_2,T_3)$

$(T_1',T_2',T_3') \longmapsto \left(\frac{T_2'}{m}, \frac{T_3'}{n}, \frac{T_1'-\frac{1}{m}T_2'^2-\frac{1}{n}T_3'^2}{n+m-2}\right)$

$\therefore\ (T_1,T_2,T_3)$ 为完全统计量

□.

48. ① $f(x_1,\cdots,x_n,\theta) = (2\theta)^{-n} \cdot \exp\{-\theta^{-1}(\sum_{i=1}^n |x_i|)\} \qquad \theta>0$

$c(\theta) = (2\theta)^{-n} \qquad \theta_1 = -\theta^{-1} \qquad T_1(x) = \sum_{i=1}^n |x_i| \qquad h(x)=1$

由因子分解定理: $T(x)$ 为充分统计量

② $\eta = -\frac{1}{\theta}$, $f(x_1,\cdots,x_n,\eta) = 2^{-n}\cdot(-\varphi)^n \cdot \exp\{\varphi T(x)\}$

自然参数空间 $\Theta^* = \{\varphi \mid \varphi<0\}$ 作为 R^1 的子集有内点

$\therefore\ T = \sum_{i=1}^n |X_i|$ 是 θ 的充分完全统计量.

□

49. ① $f(x,\theta) = \exp\{-(x-\theta)\} I_{(\theta,+\infty)}^{(x)}$

$$f(x_1,\cdots,x_n;\theta) = e^{n\theta}\cdot \exp\{-\sum_{i=1}^{n} x_i\}\cdot I_{(\theta,+\infty)}^{(X_{(1)})}$$

$$= g(T(\vec{x});\theta)\cdot h(\vec{x})$$

其中 $g(T(\vec{x});\theta) = e^{n\theta}\cdot I_{(\theta,+\infty)}^{(X_{(1)})}$ $\quad h(\vec{x}) = \exp\{-\sum_{i=1}^{n} x_i\}$

由因子分解定理 $X_{(1)}$ 充分

② 设 $\varphi(t)$ 为 R 上 $\forall$ 可测函数，满足 $E_\theta\varphi(T)=0$

$X_{(1)}$ 的 pdf 为 $f(t,\theta) = n(1-F(t))^{n-1} f(t)$

$$= n\cdot \exp^{-n(t-\theta)} I_{(\theta,+\infty)}^{(t)}$$

$$\cancel{= g(T(\vec{x});\theta)\cdot h(\vec{x})}$$

~~其中~~ $\cancel{g(T(\vec{x});\theta) = e^{n\theta}\cdot I_{(\theta,+\infty)}^{(X_{(1)})}}$ $\quad \cancel{h(\vec{x}) = \exp\{-\sum_{i=1}^{n} x_i\}}$

对 R 上 $\forall$ 可测函数 φ，满足 $E_\theta\varphi(T)=0$，则

$$\int_\theta^{+\infty} \varphi(t)\cdot n\cdot e^{-n(t-\theta)} I_{(\theta,+\infty)}^{(t)}\, dt = 0$$

$$\Longleftrightarrow \int_\theta^{+\infty} \varphi(t)\cdot e^{-nt}\, dt = 0$$

对 θ 求导 $\quad \varphi(\theta)\cdot e^{-n\theta} = 0 \quad a.s. P_\theta$

$$\Longleftrightarrow \varphi(\theta) = 0 \quad a.s. P_\theta$$

由定义 $T(\vec{x}) = X_{(1)}$ 为完全统计量.

综上 $T(\vec{x}) = X_{(1)}$ 充分完全. $\square$

50. ① $f(x_1,\cdots,x_n;\theta) = \frac{1}{\theta^n}\cdot\prod_{i=1}^{n} I_{(-\frac{\theta}{2},\frac{\theta}{2})}^{(x_i)} \quad (\theta>0)$

$= \frac{1}{\theta^n} I_{(-\frac{\theta}{2},X_{(n)})}^{(X_{(1)})} \cdot I_{(-\frac{\theta}{2},\frac{\theta}{2})}^{(X_{(n)})}$

$= g(T(\vec{x}),\theta)\cdot h(\vec{x})$

其中 $g(T(\vec{x}),\theta) = \frac{1}{\theta^n} I_{(-\frac{\theta}{2},X_{(n)})}^{(X_{(1)})} \cdot I_{(-\frac{\theta}{2},\frac{\theta}{2})}^{(X_{(n)})}$ $\quad h(\vec{x})=1$

$\therefore T(\vec{x}) = (X_{(1)}, X_{(n)})$ 充分.

② 考虑 $Y_i = \frac{X_i}{\theta}$，则 $Y_1,\cdots,Y_n$ i.i.d. $\sim U(-\frac{1}{2},\frac{1}{2})$

若 $X_{(1)}=X_{(n)}$ 同时为0，则 $X_1=\cdots=X_n=Y_1=\cdots=Y_n=0$ 为一个样本点.

在要求 $\varphi(T)=0$ a.s. P_θ 情况下可忽略.

$X_{(1)}, X_{(n)}$ 不同时为0时，不妨假设 $X_{(1)}\neq 0$.

$Z = \frac{X_{(n)}}{X_{(1)}} = \frac{Y_{(n)}}{Y_{(1)}}$ 分布与 θ 无关.

找常数 $a<b$ s.t. $P(Z<a) = P(Z<b) > 0$

定义 $\varphi(t) = \begin{cases} 1 & z<a \\ -1 & z>b \\ 0 & 其他 \end{cases}$

有 $E_\theta\varphi(T)=0$ 但 $P(\varphi(T)\neq 0)>0$.

$\therefore (X_{(1)}, X_{(n)})$ 不是完全统计量.

□.