4TH HOMEWORK REFERENCE SOLUTION

2.42

(1)Use definition. The last formula does not contain λ.

$$\begin{aligned}
&P(X_1 = k_1, ..., X_n = k_n | T = t) \\
&= \frac{P(X_1 = k_1, ..., X_n = k_n, T = t)}{P(T = t)} \\
&= \frac{P(X_1 = k_1, ..., X_n = t - \Sigma_{i=1}^{n-1} k_i)}{P(T = t)} \\
&= \frac{e^{-\lambda}\frac{\lambda^{k_1}}{k_1!} \cdot ...... \cdot e^{-\lambda}\frac{\lambda^{t-\Sigma_{i=1}^{n-1}k_i}}{(t-\Sigma_{i=1}^{n-1}k_i)!}}{P(T = t)} \\
&= \frac{(e^{-\lambda})^n \cdot \frac{\lambda^t}{k_1! \cdot ... \cdot (t-\Sigma_{i=1}^{n-1}k_i)!}}{e^{n\lambda} \cdot \frac{(n\lambda)^t}{t!}} \\
&= \frac{t!}{k_1! \cdot ... \cdot (t - \Sigma_{i=1}^{n-1} k_i))! \cdot n^t}
\end{aligned}$$

(2)Use factorization theorem.

$$\begin{aligned}
&P(X_1 = k_1, ..., X_n = k_n) \\
&= \frac{e^{-\lambda} \cdot \lambda^{k_1}}{k_1!} \cdot ... \cdot \frac{e^{-\lambda} \cdot \lambda^{k_n}}{k_n!} \\
&= e^{-n\lambda} \cdot \lambda^{\Sigma_{i=1}^{n} k_i} \cdot \frac{1}{k_1! \cdot ... \cdot k_n!} \\
&= g(t(k_1, ..., k_n), \lambda) \cdot h(k_1, ..., k_n)
\end{aligned}$$

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2.43

Set $X_1, X_2, X_3, ..., X_n$ i.i.d Geo(p)

(1)Definition

$$\begin{aligned}
&P(X_1 = k_1, ..., X_n = k_n | T = t) \\
&= \frac{P(X_1 = k_1, ..., X_n = k_n, T = t)}{P(T = t)} \\
&= \frac{(1-p)^{k_1-1}p \cdot ... \cdot (1-p)^{t-\Sigma_{i=1}^{i=n-1}k_i-1}p}{C_{t-1}^{n-1} \cdot (1-p)^{t-n} \cdot p^n} \\
&= \frac{1}{C_{t-1}^{n-1}}
\end{aligned}$$

(2)Factorization Theorem

$$
\begin{aligned}
& P(X_1 = k_1, ..., X_n = k_n) \\
= & (1-p)^{(k_1+...+k_n)-n} \cdot p^n \cdot 1 \\
= & g(t(k_1, ..., k_n), p) \cdot h(k_1, ..., k_n)
\end{aligned}
$$

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2.46

$$
\begin{aligned}
& f(x_1, x_2, x_3, ..., x_n, \theta) \\
= & (\frac{1}{\sqrt{2\pi}\theta})^n \cdot e^{\Sigma_{i=1}^n - \frac{(x_i-\theta)^2}{2\theta^2}} \\
= & (\frac{1}{\sqrt{2\pi}\theta})^n \cdot e^{n(\frac{\bar{x}}{\theta}-\frac{1}{2})} \cdot e^{-\frac{1}{2\theta^2}\Sigma_{i=1}^n x_i^2}
\end{aligned}
$$

However, $\Sigma_{i=1}^n x_i^2$ could not be expressed as function of $\bar{x}$. According to factorization theorem, $\bar{x}$ is not sufficient statistic.

2.47

$$
\begin{aligned}
& f(x_1, x_2, ..., x_m, y_1, y_2, ..., y_n) \\
= & (\frac{1}{\sqrt{2\pi}\sigma})^{m+n} \cdot e^{-\frac{1}{2\sigma^2}\cdot[\Sigma_{i=1}^m (x_i-a)^2 + \Sigma_{j=1}^n (y_j-b)^2]} \\
= & (\frac{1}{\sqrt{2\pi}\sigma})^{m+n} \cdot e^{-\frac{1}{2\sigma^2}\cdot[\Sigma_{i=1}^m (x_i-\bar{x}+\bar{x}-a)^2 + \Sigma_{j=1}^n (y_j-\bar{y}+\bar{y}-b)^2]} \\
= & (\frac{1}{\sqrt{2\pi}\sigma})^{m+n} \cdot e^{-\frac{1}{2\sigma^2}\cdot[\Sigma_{i=1}^m (x_i-\bar{x})^2 + \Sigma_{j=1}^n (y_j-\bar{y})^2 + m(\bar{x}-a)^2 + n(\bar{y}-b)^2]} \\
= & (\frac{1}{\sqrt{2\pi}\sigma})^{m+n} \cdot e^{-\frac{1}{2\sigma^2}\cdot[(n+m-2)s^2 + m(\bar{x}-a)^2 + n(\bar{y}-b)^2]} \\
= & h(x, y) \cdot g_{a,b,\sigma^2}(\bar{x}, \bar{y}, s^2)
\end{aligned}
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Therefore, according to factorization theorem, $(\bar{X},\ \bar{Y},\ s^2)$ is sufficient statistic.