Solution 3:

2\. 由于独立 $\bar{X}-\bar{Y} \sim N(0, \frac{2\sigma^2}{n})$ ~~其~~，其分布函数记为 $\Phi_{0,\frac{2\sigma^2}{n}}(x)$

$P(|\bar{X}-\bar{Y}|>\sigma) \approx 0.01 \iff P(|\bar{X}-\bar{Y}| \le \sigma) \approx 0.99$

$P(|\bar{X}-\bar{Y}| \le \sigma) = 2\Phi_{0,\frac{2\sigma^2}{n}}(\sigma) - 1 = 2\Phi(\frac{\sigma}{\sqrt{\frac{2\sigma^2}{n}}}) - 1 \approx 0.99$

$\Phi(\sqrt{\frac{n}{2}}) \approx 0.995 \Rightarrow \sqrt{\frac{n}{2}} \approx 2.58 \Rightarrow n \approx 13.32$

$\therefore n$ 取 14. $\square$

11\. 由于独立，$X_{n+1} - \bar{X} \sim N(0, \frac{n+1}{n}\sigma^2)$

则有 $\frac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma} \sim N(0,1)$

注意到 $S^2 = \frac{nS_n^2}{n-1}$ $\quad (S^2 = \frac{1}{n-1}\sum(x_i-\bar{x})^2)$

由 $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ 得 $\frac{nS_n^2}{\sigma^2} \sim \chi^2_{n-1}$

由 t 分布定义，及 $\bar{X}$ 与 S^2 独立，或而 $\bar{X}$ 与 S_n^2 独立，可得

$\frac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma}$ 与 $\frac{nS_n^2}{\sigma^2}$ 独立

$$\Rightarrow \frac{\frac{X_{n+1}-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma}}{\sqrt{\frac{nS_n^2}{\sigma^2}/n-1}} = \frac{X_{n+1}-\bar{X}}{S_n}\sqrt{\frac{n-1}{n+1}} \sim t_{n-1}$$

$\square$

12\. $\bar{X}-\mu_1 \sim N(0, \frac{\sigma^2}{m})$ $\quad \bar{Y}-\mu_2 \sim N(0, \frac{\sigma^2}{n})$ 由 $\bar{X}-\mu_1$ 与 $\bar{Y}-\mu_2$ 独立，性

可得 $Z = \alpha(\bar{X}-\mu_1) + \beta(\bar{Y}-\mu_2) \sim N(0, (\frac{\alpha^2}{m}+\frac{\beta^2}{n})\sigma^2)$

$$\Rightarrow \frac{Z}{\sqrt{(\frac{\alpha^2}{m}+\frac{\beta^2}{n})}\sigma} \sim N(0,1)$$

又由 $S_{X+Y}^2 = \frac{mS_{1m}^2 + nS_{2n}^2}{m+n-2} \sim \chi^2_{m+n-2}$ 且由上题 $\bar{X}$ 与 S_{1m}^2, S_{2n}^2 独立，$\bar{Y}$ 与 S_{1m}^2, S_{2n}^2 独立

$\Rightarrow \frac{Z}{\sqrt{(\frac{\alpha^2}{m}+\frac{\beta^2}{n})}\sigma}$ 与 $\frac{S_{X+Y}^2(m+n-2)}{\sigma^2}$ 独立.

$\therefore$ 由 t 分布定义，$T \sim t_{m+n-2}$.

$\square$

13. $X_1,\cdots,X_n$ iid $\sim N(a,\sigma^2)$

令 $\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \sqrt{\frac{n-1}{n}} & -\frac{1}{\sqrt{n(n-1)}} & \cdots & -\frac{1}{\sqrt{n(n-1)}} \\ \vdots & & c_{ij} & \vdots \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$ 记 $Y=AX$

即 $Y_1=\sqrt{n}\bar{X}$ $Y_2=\frac{\sqrt{n}}{\sqrt{n-1}}(X_1-\bar{X})$ $Y_i=\sum_{j=1}^{n} c_{ij}X_j$ $(i=3,4,\cdots,n)$

取 A 为正交阵 正交变换保距 $\sum_{i=1}^{n} x_i^2=\sum_{i=1}^{n} Y_i^2$

$(n-1)S^2=\sum_{i=1}^{n}(X_i-\bar{X})^2=\sum_{i=1}^{n} X_i^2-n\bar{X}^2=\sum_{i=2}^{n} Y_i^2$

由定理 2.2.3, $\frac{(n-1)S^2}{\sigma^2}\sim\chi^2_{n-1}$

$$X_1-\bar{X}=\frac{n-1}{n}X_1-\sum_{i=2}^{n}\frac{1}{n}X_i\sim N\left(0,\frac{n-1}{n}\sigma^2\right)\left(\frac{(n-1)^2}{n^2}+\frac{n-1}{n^2}=\frac{n-1}{n}\right)$$

$\Rightarrow Y_2=\frac{\sqrt{n}}{\sqrt{n-1}}(X_1-\bar{X})\sim N(0,\sigma^2)$

$$\xi=\frac{X_1-\bar{X}}{S}=\frac{\sqrt{\frac{n-1}{n}}Y_2}{\frac{1}{\sqrt{n-1}}\sqrt{\sum_{i=2}^{n}Y_i^2}}=\frac{(n-1)Y_2}{\sqrt{n}\sqrt{\sum_{i=2}^{n}Y_i^2}}$$

有 $\frac{1}{\xi^2}=\frac{n}{(n-1)^2}\left(1+\frac{\sum_{i=3}^{n}Y_i^2}{Y_2^2}\right)$ 有 $\frac{\sum_{i=3}^{n}Y_i^2}{\sigma^2}\sim\chi^2_{n-2}$

取 $\eta=\frac{\frac{Y_2}{\sigma}}{\sqrt{\frac{\sum_{i=3}^{n}Y_i^2}{\sigma^2(n-2)}}}=\frac{\sqrt{n-2}\,Y_2}{\sum_{i=3}^{n}Y_i^2}\sim t_{n-2}$.

$\Rightarrow \frac{1}{\xi^2}=\frac{n}{(n-1)^2}\left(1+\frac{n-2}{\eta^2}\right)$, ξ 与 η 同号

即 $\eta=\sqrt{\frac{n-2}{\frac{(n-1)^2}{n\xi^2}-1}}\cdot\mathrm{sgn}(\xi)$

~~对~~ $\cancel{x<0,\ F(x)=P(\xi<x)=P\left(\eta<-\sqrt{\frac{n-2}{\frac{(n-1)^2}{nx^2}-1}}\right)=\int}$

由变量代换公式可得

$$f(\xi)=\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)\sqrt{\pi}}\left[\frac{(n-1)^2}{n}-\xi^2\right]^{\frac{n-4}{2}}\left[\frac{n}{(n-1)^2}\right]^{\frac{n-3}{2}}$$

D.

14. $\xi = \sum_{i=1}^{n}\frac{X_i^2}{\sigma_i^2} + Z^2\sum_{i=1}^{n}\frac{1}{\sigma_i^2} - 2Z\sum_{i=1}^{n}\frac{X_i}{\sigma_i^2}$

$= \sum_{i=1}^{n}\frac{X_i^2}{\sigma_i^2} - \sum_{i=1}^{n}\frac{1}{\sigma_i^2}Z^2$

则取 $A = \begin{pmatrix} \frac{\frac{1}{\sigma_1}}{\sqrt{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}} & \frac{\frac{1}{\sigma_2}}{\sqrt{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}} & \cdots & \frac{\frac{1}{\sigma_n}}{\sqrt{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}} \\ \vdots & & & \vdots \end{pmatrix}$ 为正交阵(Schmidt正交化保证).

$Y_1 = \sqrt{\sum_{i=1}^{n}\frac{1}{\sigma_i^2}}\,Z \implies \xi = \sum_{i=1}^{n}\frac{X_i^2}{\sigma_i^2} - Y_1^2$

由正交变换保模长 $\implies \quad = \sum_{i=2}^{n} Y_i^2$

且 $Y_1, Y_2, \cdots, Y_n \sim N(0,1)$ i.i.d.

$\therefore \xi \sim \chi^2_{n-1}$.

□.

16. $Y = X_1 + \cdots + X_n \sim \chi^2_{mn}$

$$f(\bar{X}=x) = f(\sum X_i = nx) = f(Y=nx) = \begin{cases} \frac{n}{2^{\frac{mn}{2}}\Gamma(\frac{mn}{2})}(nx)^{\frac{mn}{2}-1}e^{-\frac{nx}{2}}, & x>0 \\ 0, & x\le 0 \end{cases}$$

□

20. 令 $\vec{Y}=\vec{X}-\vec{\mu}$

由 $Cov(\vec{X}) = \Sigma_{n\times n} > 0$

$\therefore \exists A$ s.t. $\Sigma_{n\times n} = A\cdot A^T$

令 $Z = A^{-1}Y$

由 $Y \sim N(0, \Sigma_{n\times n})$

$\Rightarrow Z \sim N(0, A^{-1}(\Sigma_{n\times n})(A^{-1})^T)$

$\sim N(0, I_{n\times n})$

$\Rightarrow Z_i \sim N(0,1)$ i.i.d. $i=1,2,\cdots,n$

$$U = (\vec{X}-\vec{\mu})^T \Sigma^{-1} (\vec{X}-\vec{\mu})$$

$$= \vec{Y}^T (AA^T)^{-1} Y$$

$$= (A^{-1}Y)^T (A^{-1}Y)$$

$$= Z^T Z$$

$$= \sum_{i=1}^{n} Z_i^2 \Rightarrow U \sim \chi_n^2$$

□

28. 设 X_1 pdf $f_1(x_1)$, X_2 pdf $f_2(x_2)$ 联合 pdf $f(x_1,x_2)$, X_1X_2独立

$$f(x_1,x_2) = f_1(x_1)\cdot f_2(x_2) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot x_1^{\alpha_1-1} x_2^{\alpha_2-1} e^{-\lambda(x_1+x_2)}.$$

令 $\begin{cases} y_1 = x_1+x_2 \\ y_2 = \frac{x_1}{x_1+x_2} \end{cases} \Rightarrow \begin{cases} x_1 = y_1y_2 \\ x_2 = y_1 - y_1y_2 \end{cases}$ $\left|\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)}\right| = y_1 \neq 0$

$$\hat{f}(y_1,y_2) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot (y_1y_2)^{\alpha_1-1} (y_1-y_1y_2)^{\alpha_2-1} e^{-\lambda y_1} \cdot y_1$$

$$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} e^{-\lambda y_1} y_1^{\alpha_1+\alpha_2-1} \frac{1}{B(\alpha_1,\alpha_2)} y_2^{\alpha_1-1}(1-y_2)^{\alpha_2-1}$$

$$= \hat{f}_1(y_1)\hat{f}_2(y_2)$$

$\hat{f}(y_1,y_2)$ 可分离 $\Rightarrow Y_1$ 与 Y_2 相互独立

且 $Y_1 \sim \Gamma(\alpha_1+\alpha_2, \lambda)$, $Y_2 \sim B(\alpha_1,\alpha_2)$.

□.

35. $X_1, \cdots, X_n \overset{iid}{\sim} U(0,\theta)$ i.i.d.

$EX_1 = \frac{\theta}{2} \quad DX_1 = \frac{\theta^2}{12}$

由CLT $$\frac{\bar{X} - \frac{\theta}{2}}{\sqrt{\frac{\theta^2}{12n}}} \longrightarrow N(0,1)$$

$\Rightarrow$ $\bar{X}$ 渐近分布为 $N(\frac{\theta}{2}, \frac{\theta^2}{12n})$

$\square$.