Solution 2.

习题一

5. (1) $\mathcal{X} = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i = 0$ 或 $1, i = 1,2,3,4,5\}$

概率分布，$P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4, X_5 = x_5)$

$$= \begin{cases} p^k(1-p)^{5-k} & \text{当} \sum_{i=1}^{5} x_i = k.\ k = 0,1,2,3,4,5.\ x_i = 0 \text{或} 1, i = 1,2,3,4,5 \\ 0 & \end{cases}$$

(2). $X_1 + X_2$, $\min_{1 \le i \le 5} X_i$ 是统计量 仅依赖样本 $X_1, X_2, \cdots X_5$

$E(X_1) = p$, $D(X_1) = p - p^2$

$\Rightarrow$ $X_5 + 2p$. $X_5 - E(X_1)$, $(X_5 - X_1)^2 / D(X_1)$ 不是统计量 含未知参数 p.

(3). $$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{(-\infty, x]}(X_i)$$

$$= \begin{cases} 0 & x < 0 \\ \frac{n-m}{n} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

□

6. Pf: $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{1}{n} \sum_{i=1}^{n} (a x_i + b) = a\bar{x} + b$

$S_y^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 = \frac{1}{n-1} \sum_{i=1}^{n} (a x_i + b - a\bar{x} - b)^2 = a^2 S_x^2$

作变换 $Y_i = \frac{X_i}{10} - 50$

$\Rightarrow$ $\bar{Y} = 4$, $S_y^2 = 21$

$\Rightarrow$ $\bar{x} = 540$ $S_x^2 = 2100$

□

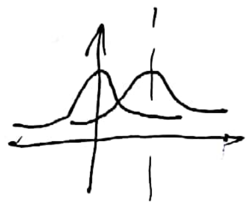
7. 顺序统计量为

$X_{(1)}=-0.5,\ X_{(2)}=-0.2,\ X_{(3)}=0.1,\ X_{(4)}=0.2,\ X_{(5)}=0.4$

$X_{(6)}=0.5\quad X_{(7)}=0.7\quad X_{(8)}=0.7\quad X_{(9)}=1.5\quad X_{(10)}=2.5$

10. 由正态函数特征 $\mu=0$ 时 $P(|\bar{x}|<c)$ 最大.

此时 $\bar{x}\sim N(0,\frac{1}{100})$ 分布函数记为 $\Phi_{\frac{1}{100}}(x)$.



$P(|\bar{x}|\le c)\le 0.05$

即 $2\Phi_{\frac{1}{100}}(c)-1\le 0.05$

$\Phi_{\frac{1}{100}}(c)\le 0.525$

$\Phi(10c)\le 0.525$

$\Rightarrow\ 10c\le 0.06 \Rightarrow c\le 0.006$.

□.

12. ① $P(|\bar{x}-0.5|\le 0.1)\ge 0.9$

$\Leftrightarrow\ P(|\bar{x}-0.5|\ge 0.1)\le 0.1$

由 Chebyshev Inequality

$P(|\bar{x}-0.5|\ge 0.1)=P(|n\bar{x}-0.5n|\ge 0.1n)$

$\le \dfrac{D(\sum_{i=1}^{n}X_i)}{(0.1n)^2}=\dfrac{\frac{n}{4}}{(0.1n)^2}\le 0.1 \Rightarrow n\ge 250$

② 由 CLT $\quad P(|\bar{x}-0.5|<0.1)$

$\dfrac{\sum X_i-0.5n}{\sqrt{n}\,\sigma}\sim N(0,1)\quad = P\left(-\dfrac{0.1\sqrt{n}}{\sigma}<\dfrac{\sum X_i-0.5n}{\sqrt{n}\cdot\sigma}<\dfrac{0.1\sqrt{n}}{\sigma}\right)$

$=2\Phi\left(\dfrac{0.1\sqrt{n}}{\sigma}\right)-1=2\Phi(0.2\sqrt{n})-1\ge 0.9$

$\Rightarrow 0.2\sqrt{n}\ge 1.64\quad\Rightarrow n\ge 67.24\quad n_{min}=68$.

□

习题2.

3. (1). $N(a,\sigma^2)$ $\quad f(t)=Ee^{itX}=e^{iat-\frac{1}{2}\sigma^2t^2}$

$$f_{\bar X}(t)=Ee^{i\frac{t}{n}\sum X_j}\overset{\text{独立}}{=}\prod_{j=1}^{n}Ee^{i\frac{t}{n}X_j}=e^{iat-\frac{1}{2}\sigma^2\frac{t^2}{n}}$$

$\therefore\ \bar X\sim N(a,\frac{\sigma^2}{n})$.

(2). $$f_{n\bar X}(t)=Ee^{it(X_1+\cdots+X_n)}=\prod_{k=1}^{n}f_{X_k}(t)$$
$$=e^{n\lambda(e^{it}-1)}$$

$\Rightarrow n\bar X\sim \text{Possion}(n\lambda)$ $\quad\Rightarrow P(n\bar X=t)=\dfrac{(n\lambda)^t}{t!}e^{-n\lambda}$ $\quad t=0,1,2,\cdots$

$$P(\bar X=\frac{t}{n})=\frac{(n\lambda)^t}{t!}e^{-n\lambda},\ t=0,1,2,\cdots$$

(3). 指数分布特征函数 $f_X(t)=(1-\frac{it}{\lambda})^{-1}$

$$f_{\bar X}(t)\overset{\text{独立}}{=}f^n(\frac{t}{n})=(1-\frac{it}{n\lambda})^{-n}$$

$\Rightarrow \bar X\sim$ ~~$\Gamma(n\lambda,n)$~~ $\Gamma(n,n\lambda)$. □

4. $$S_n^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar X)^2=\frac{1}{n}\sum_{i=1}^{n}X_i^2-\bar X^2$$

又 $X_i\sim b(1,p)$ $\quad X_i$ 取 $0,1\Rightarrow X_i^2=X_i$

$\therefore\ S_n^2=\bar X-\bar X^2$

$X_1,\dots,X_n\sim b(1,p)$ i.i.d $\quad \sum X_i\sim b(n,p)$

$P(\sum X_i=k)=\binom{n}{k}p^k(1-p)^{n-k},\ k=0,1,\cdots,n$

$P(\bar X=\frac{k}{n})=\binom{n}{k}p^k(1-p)^{n-k}.\ k=0,1,\cdots,n$

① ~~n为奇数时~~ $n=2l+1$ 时

$$P(S_n^2=\frac{k}{n}(1-\frac{k}{n}))=P(\bar X=\frac{k}{n})+P(\bar X=1-\frac{k}{n})$$
$$=\binom{n}{k}[p^k(1-p)^{n-k}+p^{n-k}(1-p)^k]\quad k=0,1,\cdots,l$$

② ~~n为偶数时~~ $n=2l$ 时

$$P(S_n^2=\frac{k}{n}(1-\frac{k}{n}))=\begin{cases}\binom{n}{k}(p^k(1-p)^{n-k}+p^{n-k}(1-p)^k), & k=0,\cdots,l-1\\ \binom{n}{k}p^k(1-p)^{n-k}, & k=l\end{cases}$$

9. (1). 略.

(2) $X_{(6)}$ pdf

$$f_6(x)=\frac{10!}{(6-1)!(10-6)!}(F(x))^5[1-F(x)]^4 f(x)$$

$$=6\cdot\binom{10}{6}(F(x))^5[1-F(x)]^4 f(x)$$

$\cancel{E(F}$ $E(F(X_{(6)}))=\int F(x)\cdot f_6(x)dx$

$$=\cancel{6}\int 6\binom{10}{6}(F(x))^6[1-F(x)]^4 dF(x)$$

$$=6\frac{10!}{6!\cdot 4!}\cdot B(7,5)=6\cdot\frac{10!}{6!\cdot 4!}\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}$$

$$=6\frac{10!}{6!4!}\times\frac{6!\,4!}{11!}=\frac{6}{11}$$

同理 $E\{F^2(X_{(6)})\}=\frac{7}{22}$

$$\Rightarrow Var(F(X_{(6)}))=\frac{5}{242}=\frac{7}{22}-\left(\frac{6}{11}\right)^2$$

(3) $n=10$.

$$F_{X_{(6)}}(x)=6\binom{10}{6}\int_0^{F(0.2)}t^5(1-t)^4dt$$

代入 $\cancel{x}=0.2$

$\Rightarrow F(0.2)=0.5793 \Rightarrow F_{X_{(6)}}(x)=0.5804$.