30. $H_0: \sigma^2 \le \sigma_0^2 \longleftrightarrow H_1: \sigma^2 > \sigma_0^2$

记 $\theta = (\mu, \sigma^2)$　θ 的似然函数为

$$f(x,\theta) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}.$$

$$\Theta = \{\theta = (\mu, \sigma^2): -\infty < \mu < +\infty, \sigma^2 > 0\}$$

$$\Theta_0 = \{\theta = (\mu, \sigma^2): -\infty < \mu < +\infty, \sigma^2 \le \sigma_0^2\}$$

在 Θ 上，μ, σ^2 的 MLE：$\hat{\mu} = \bar{x}$，$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$

故有 $\sup\limits_{\theta\in\Theta} f(x,\theta) = \left(\frac{2\pi e}{n}\right)^{-\frac{n}{2}}\left(\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{-\frac{n}{2}}$

$$L_{\Theta_0}(x) = \sup_{\sigma^2 \le \sigma_0^2} (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}$$

令 $g(\sigma^2) = \ln\left[(2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}\right]$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\{x_i-\bar{x}\}^2$$

求导 $\Rightarrow$ ~~$L_{\Theta_0}(x) =$~~　$\hat{\mu}' = \bar{x}$，$\hat{\sigma^2}' = \begin{cases} \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2, & \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 \le \sigma_0^2 \\ \sigma_0^2, & \text{else} \end{cases}$

$$\therefore \lambda(x) = \begin{cases} 1 & \sigma_0^2 \ge \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n} \\ \left(\frac{e}{n}\right)^{-\frac{n}{2}}\left[\frac{1}{\sigma_0^2}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{-\frac{n}{2}} \exp\left\{\frac{1}{\sigma_0^2}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}, & \sigma_0^2 < \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n} \end{cases}$$

令 $\xi = \frac{1}{\sigma_0^2}\sum_{i=1}^{n}(x_i-\bar{x})^2$，$g(\xi) = \xi^{-\frac{n}{2}} e^{\frac{\xi}{2}}\left(\frac{e}{n}\right)^{-\frac{n}{2}}$

$\xi > 0$ 时　$g(\xi)$ 在 $(0,n)\downarrow$，$(n,+\infty)\nearrow$

则 $\lambda(x)$ 在 $(0,n]$ 为 1，在 $(n,+\infty)\nearrow$

故 $P(g(\xi) \le c) = P(\xi \le k)$ 且 ξ 在 H_0 成立时服从 χ^2_{n-1}

故 $k = \chi^2_{n-1}(\alpha)$　$\therefore$　$\varphi(x) = \begin{cases} 1, & \frac{\sum(x_i-\bar{x})^2}{\sigma_0^2} > \chi^2_{n-1}(\alpha) \\ 0, & \text{else.} \end{cases}$

□

31. $f(x,y,\theta)=(2\pi\sigma^2)^{-\frac{m+n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{m}(x_i-\mu_1)^2+\sum_{i=1}^{n}(y_i-\mu_2)^2\right]\right\}$

$\Theta=\{\theta=(\mu_1,\mu_2,\sigma^2):-\infty<\mu_1,\mu_2<+\infty,\ \sigma^2>0\}$

由指数族可求得

Θ上，$\hat{\mu}_1=\bar{x}$，$\hat{\mu}_2=\bar{y}$．$\hat{\sigma}^2=\frac{\sum_{i=1}^{m}(x_i-\bar{x})^2+\sum_{i=1}^{n}(y_i-\bar{y})^2}{m+n}$

Θ_0上，$\hat{\mu}'=\frac{m\bar{x}+n\bar{y}}{m+n}$，$\hat{\sigma}^{2\prime}=\frac{\sum_{i=1}^{m}(x_i-\hat{\mu}')^2+\sum_{i=1}^{n}(y_i-\hat{\mu}')^2}{m+n}$

代回，$\Rightarrow$

$\lambda(\vec{x})=\frac{L_{\Theta}(x)}{L_{\Theta_0}(x)}=\left[\frac{\sum_{i=1}^{m}(x_i-\bar{x})^2+\sum_{i=1}^{n}(y_i-\bar{y})^2}{\sum_{i=1}^{m}(x_i-\hat{\mu})^2+\sum_{i=1}^{n}(y_i-\hat{\mu})^2}\right]^{-\frac{m+n}{2}}$

$\sum_{i=1}^{m}(x_i-\hat{\mu})^2=\sum_{i=1}^{m}(x_i-\bar{x})^2+m(\bar{x}-\hat{\mu})^2=\sum_{i=1}^{m}(x_i-\bar{x})^2+\frac{mn^2(\bar{x}-\bar{y})^2}{(m+n)^2}$

同理 $\sum_{i=1}^{n}(y_i-\hat{\mu})^2=\sum_{i=1}^{n}(y_i-\bar{y})^2+\frac{m^2n(\bar{x}-\bar{y})^2}{(m+n)^2}$

$\Rightarrow\lambda(x,y)=\left(1+\frac{mn}{m+n}\cdot\frac{(\bar{x}-\bar{y})^2}{\sum(x_i-\bar{x})^2+\sum(y_i-\bar{y})^2}\right)^{\frac{m+n}{2}}$

令$T=\frac{\bar{x}-\bar{y}}{\sqrt{\frac{1}{m}+\frac{1}{n}}\sqrt{\sum(x_i-\bar{x})^2+\sum(y_i-\bar{y})^2/m+n-2}}$ $\Rightarrow$ 时$T\sim t_{m+n-2}$
（上下独立）．

$P(|T|>d)=\alpha\Rightarrow d=t_{m+n-2}(\frac{\alpha}{2})$

$\therefore\varphi(x)=\begin{cases}0, & |T|<t_{m+n-2}(\frac{\alpha}{2})\\ 1 & |T|>t_{m+n-2}(\frac{\alpha}{2})\end{cases}$

口．

34. ~~$f(x,y,\theta)$~~

令 $\sum_{i=1}^{m}(x_i-\bar{x})^2=A$，$\sum_{i=1}^{n}(y_i-\bar{y})^2=B$．

$$f(x,y,\theta)=(2\pi)^{-\frac{m+n}{2}}(\sigma_1^2)^{-\frac{m}{2}}(\sigma_2^2)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{m}(x_i-\mu_1)^2-\frac{1}{2\sigma_2^2}\sum_{i=1}^{n}(y_i-\mu_2)^2\right\}$$

由前：

Θ 上，$\hat{\mu}_1=\bar{x}$，$\hat{\mu}_2=\bar{y}$，$\hat{\sigma}_1^2=\frac{A}{m}$，$\hat{\sigma}_2=\frac{B}{n}$

Θ_0 上，$\hat{\mu}_1=\bar{x}$，$\hat{\mu}_2=\bar{y}$　　设 $\sigma_1^2=c\sigma_2^2=c\sigma^2$

$$\hat{\sigma}^2=\frac{\sum(x_i-\bar{x})^2+c\sum(y_i-\bar{y})^2}{c(m+n)}$$

$$L_{\Theta_0}(x)=(2\pi e)^{-\frac{m+n}{2}}(\hat{\sigma}^2)^{-\frac{m+n}{2}}c^{-\frac{m}{2}}$$

~~$\lambda(x)=\frac{(\hat{\sigma}_1^2)^{\frac{m}{2}}(\hat{\sigma}_2^2)^{-\frac{n}{2}}}{(\hat{\sigma}^2)^{-\frac{m+n}{2}}c^{-\frac{m}{2}}}$~~

$$\lambda(x)=\left(\frac{m+n}{n}\right)^{-\frac{n}{2}}\left(\frac{m+n}{m}\right)^{-\frac{n}{2}}c^{-\frac{n}{2}}\left(\frac{A}{A+cB}\right)^{-\frac{m}{2}}\left(\frac{B}{A+cB}\right)^{-\frac{n}{2}}$$

令 $F=\frac{A/\sqrt{c}m}{B/n}$　则 $\theta\in\Theta_0$　$F\sim F_{m-1,n-1}$

$\Rightarrow \lambda(x,y)=c(m,n)\left(1+\frac{n\sqrt{c}}{mF}\right)\left(c+\frac{m\sqrt{c}}{n}F\right)$，随 F 先减后增

故 $\lambda\le c \Leftrightarrow k_1\le F\le k_2$　取 $\begin{cases}P(F>k_2)=\frac{\alpha}{2}\\ P(F<k_1)=\frac{\alpha}{2}\end{cases}$

$$\varphi(x)=\begin{cases}0, & F_{m-1,n-1}(1-\frac{\alpha}{2})\le F\le F_{m-1,n-1}(\frac{\alpha}{2})\\ 1, & \text{else}.\end{cases}$$

□.

38. 利用定理5.3.1

$$L(x,p)=\binom{n}{\sum x_i}p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}\quad x_i\in\{0,1\},\ i=1,\cdots,n.$$

$$\Theta=\{p: p\in[0,1]\}\quad \Theta_0=\{p_0\}$$

$$\Rightarrow L_{\Theta}(x,p)=\binom{n}{\sum x_i}(\bar{x})^{\sum x_i}(1-\bar{x})^{n-\sum x_i}$$

$$L_{\Theta_0}(x,p)=\binom{n}{\sum x_i}p_0^{\sum x_i}(1-p_0)^{n-\sum x_i}$$

$$\therefore\ \lambda(x)=\left(\frac{\bar{x}}{p_0}\right)^{\sum x_i}\left(\frac{1-\bar{x}}{1-p_0}\right)^{n-\sum x_i}$$

Θ 维数 1　Θ_0 维数 0.

$$\therefore\ 2\ln\lambda(x)\xrightarrow{\mathcal{L}}\chi_1^2$$

$$\Rightarrow\ \varphi(x)=\begin{cases}0, & \lambda(x)\le e^{\frac{\chi_1^2(\alpha)}{2}}\\ 1 & \lambda(x)>\cancel{\chi_1^2(\alpha)}\,e^{\frac{\chi_1^2(\alpha)}{2}}\end{cases}$$

$\square$

41. 由因子分解定理 $f(x,\theta)=g(t(x),\theta)h(x)$

$\Rightarrow f(x,\theta_1)=g(t(x),\theta_1)h(x)$

$f(x,\theta_0)=g(t(x),\theta_0)h(x)$

$\Rightarrow \dfrac{f(x,\theta_1)}{f(x,\theta_0)}=\dfrac{g(t(x),\theta_1)}{g(t(x),\theta_0)}\triangleq h(t(x))$

故由NP定理确定的

$\varphi(x)=\begin{cases}1 & h(t(x))>c\\ r & h(t(x))=c\\ 0 & h(t(x))<c\end{cases}$ 为T的函数。

□.

42. (1) $x_1,\cdots,x_n$ 联合p.d.f $f(\vec{x},\theta)=\frac{1}{\theta^n}I[x_{(n)}<\theta]$

$\Rightarrow \lambda(x)=\dfrac{f(x,\theta_1)}{f(x,\theta_0)}=\begin{cases}\frac{1}{2^n} & 0<x_{(n)}<1\\ +\infty & x_{(n)}\geq 1\end{cases}$ （定义 $\frac{a}{0}=\infty$）

$\Rightarrow \varphi(x)=\begin{cases}1, & \lambda(x)>c'\\ 0, & \lambda(x)\leq c'\end{cases}=\begin{cases}1, & x_{(n)}>c\\ 0, & x_{(n)}\leq c.\end{cases}$

$x_{(n)}$ p.d.f $g_\theta(t)=\dfrac{nt^{n-1}}{\theta^n}I_{\{0<t<\theta\}}$

$\Rightarrow E_{\theta_0}[\varphi(x)]=\int_c^{\theta_0}\dfrac{nt^{n-1}}{\theta_0^n}dt=1-\dfrac{c^n}{\theta_0^n}=1-c^n=\alpha$

$\Rightarrow c=\sqrt[n]{1-\alpha}$

$\therefore \varphi(x)=\begin{cases}1 & x_{(n)}>\sqrt[n]{1-\alpha}\\ 0 & x_{(n)}\leq\sqrt[n]{1-\alpha}\end{cases}$ 为水平α UMPT

(2). $\lambda(x) = \begin{cases} 2^n, & 0 < X_{(n)} \le \frac{1}{2} \\ 0, & \text{else} \end{cases}$ （定义 $\frac{0}{0} = 0$）

关于 $X_{(n)} \downarrow$

$\therefore \varphi(x) = \begin{cases} 1 & X_{(n)} < c \\ 0 & X_{(n)} \ge c \end{cases}$

$E_\theta[\varphi(x)] = \int_0^c \frac{n t^{n-1}}{\theta_0^n} dt = \frac{c^n}{\theta_0^n} = c^n = \alpha$

$\Rightarrow c = \sqrt[n]{\alpha}$.

$\Rightarrow \varphi(x) = \begin{cases} 1 & X_{(n)} < \sqrt[n]{\alpha} \\ 0 & X_{(n)} \ge \sqrt[n]{\alpha} \end{cases}$ ~~为~~ 水平为 α UMPT.

b

43. $f(x,\mu) = (2\pi)^{-\frac{n}{2}} e^{-\frac{n}{2}\mu^2} \exp\{\mu \cdot \sum x_i\} \cdot \exp\{-\frac{1}{2}\sum x_i^2\}$

$c(\mu) = e^{-\frac{n}{2}\mu^2}$ $Q(\mu) = \mu$, $h(x) = \exp\{-\frac{1}{2}\sum x_i^2\}$, $T(x) = \sum x_i$

$Q(\mu)$ 关于 $\mu \nearrow$ $\Rightarrow$

UMPT 形式 $\varphi(x) = \begin{cases} 1 & T(x) < nc \\ 0, & T(x) \ge nc \end{cases}$

$E_{\mu_0}(\varphi(x)) = P_{\mu_0}(T(x) < nc) = P_{\mu_0}(\bar{x} < c) = \alpha$

$P_{\mu_0}(\sqrt{n}(\bar{x}-\mu_0) < \sqrt{n}(c-\mu_0)) = \alpha \Rightarrow$

$\sqrt{n}(c-\mu_0) = -u_\alpha$

$\Rightarrow c = \mu_0 - \frac{u_\alpha}{\sqrt{n}}$

$\therefore \varphi(x) = \begin{cases} 1, & \bar{x} < \theta_0 - \frac{u_\alpha}{\sqrt{n}} \\ 0, & \text{else} \end{cases}$

日.

46、 $H_0: p \geq \frac{1}{2} \longleftrightarrow H_1: p < \frac{1}{2}$

$$f(x_1, \cdots, x_n, p) = \binom{n}{\sum x_i} p^{\sum x_i} (1-p)^{n-\sum x_i}$$

$$= (1-p)^n \exp\{ \ln\frac{p}{1-p} \cdot \sum x_i \}$$

为指数族 $T(x) = \sum x_i$ $\quad \alpha(p) = \ln\frac{p}{1-p}$ 关于 $p \nearrow$

$$\therefore \varphi(x) = \begin{cases} 1 & , \sum x_i < c \\ r & , \sum x_i = c \\ 0 & , \sum x_i > c \end{cases}$$

由 $\sum x_i \sim b(n, p)$

$$E_{p_0}[\varphi(x)] = \sum_{i=0}^{c-1} \binom{n}{i} p_0^i (1-p_0)^{n-i} + r\binom{n}{c} p_0^c (1-p_0)^{n-c}$$

$$= (\tfrac{1}{2})^n \left[\sum_{i=0}^{c-1} \binom{n}{i} + r\binom{n}{c} \right]$$

c 由 $\frac{1}{2^n}\sum_{i=1}^{c-1}\binom{n}{i} \leq \alpha < \frac{1}{2^n}\sum_{i=1}^{c}\binom{n}{i}$ 确定.

$$r = \frac{\alpha - \frac{1}{2^n}\sum_{i=0}^{c-1}\binom{n}{i}}{\binom{n}{c}\frac{1}{2^n}}$$

□.