Inseparable fibrations of genus-one curves for the Albanese morphism

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Contents

The Albanese morphism

Let k be a perfect filed. Let X be a proper geometrically integral variety over k. We mark on X a k-point.

Theorem

There exists an abelian variety $\mathrm{Alb}(X)$ *together with a morphism* $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ *satisfies the following universal property*

- *for every pointed morphism* $f: X \to A$ *to an abelian variety, there exists a homomorphism* $h\colon \mathrm{Alb}(X) \to A$ *such that* $f = h \circ \mathrm{alb}_X$*.*

Albanese morphism for X with $-K_X$ nef over $\mathbb{C}$

Let X be a projective manifold with $-K_X$ nef. Let $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$ be the Albanese morphism.

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(Kawamata 1981) when $K_X \equiv 0$, alb_X is a fiber space (i.e., surjective with connected fibers).

(Qi Zhang 1996) alb_X is a fiber space.

(Qi Zhang 2005) alb_X is a fiber space for a log canonical pair.

(Lu; Tu; Zhang; Zheng; 2010) alb_X is flat with reduced fibers.

(Junyan Cao 2019) alb_X is locally trivial (in analytic topology).

$\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ for X with $-K_X$ nef and $\mathrm{char}\, k > 0$

In the following, k will be an algebraically closed field of characteristic $p > 0$.

Theorem (Yuan Wang 2022)

Let X/k be a normal projective 3-fold and $\Delta \geq 0$ with coefficients ≤ 1 such that $-(K_X + \Delta)$ semiample. Assume $p > \max\{\frac{2}{\delta}, i_{bpf}(-(K_X + \Delta))\}$, where δ the minimal non-zero coefficient of Δ; and that X_0 normal and $(X_0, \Delta|_{X_0})$ F-pure. Then alb_X is surjective.

Theorem (Ejiri 2019)

Let X be a normal projective variety over k such that $-(K_X + \Delta)$ is nef whose Cartier index not divisible by p. Denote $X_{\bar{\eta}}$ the geometric generic fiber of alb_X over its image. If $(X_{\bar{\eta}}, \Delta|_{X_{\bar{\eta}}})$ is F-pure, then alb_X is an algebraic fiber space.

Theorem (Patakfalvi; Zdanowicz; arxiv/2019.12)

Let $f : X \to T$ be a fibration from a normal projective variety onto a smooth variety such that $-K_{X/T}$ is nef and $\mathbb{Q}$-Cartier. Assume that X_t is strongly F-regular for $t \in T$ general.

1. *Then f is equidimensional.*
2. *If moreover X is Cohen-Macaulay so that f is flat, then every geometric fiber of f is reduced.*

The general fiber could be quite singular

Example

There exists a smooth surface S with $K_S \equiv 0$ such that each closed fiber of $\mathrm{alb}_S \colon S \to \mathrm{Alb}(S)$ is a cusp curve.
These are just Quasi-elliptic surfaces.

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Example

There exist smooth threefolds X with $-K_X$ nef such that the each closed fiber of $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$ is a non-reduced curve.

Let X be a variety over an algebraically closed field of characteristic $p > 0$.

Lemma (Moret-Bailly 1979)

There exists a non-isotrivial family $\mathscr{A} \to \mathbb{P}^1_k$ of abelian surfaces.

Let E be a supersingular elliptic curve over k. Let $A = E \times E$. Then A contains a subgroup scheme $\alpha_p^{\oplus 2} \subset A$.

$$\begin{CD} A \times \mathbb{P}^1 @>{/G}>> X \\ @VVV @VV{f}V \\ A @>{\mathrm{Frob}}>> A^p \end{CD}$$

Here we take $G \subset \alpha_p \times \alpha_p \times \mathbb{P}^1 \to \mathbb{P}^1$ defined by the equation $\sigma y = \tau x$.

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$\deg = p$

$\Rightarrow f$ is inseparable

$\deg = p^2$

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Using foliation we can rewrite the above construction. Take a basis $\alpha, \beta \in \mathrm{Lie}(A) \cong H^0(A, \mathcal{T}_A)$ such that $\alpha^p = \beta^p = 0$. Let $\mathcal{F} \subset \mathcal{T}_{A\times\mathbb{P}^1}$ be the foliation generated by

$$\alpha + t\beta$$

By the canonical bundle formula ([Ekedahl 1987]) we have

$$K_{A\times\mathbb{P}^1} = \pi^* K_X + (p-1)\deg\mathcal{F} = \pi^* K_X + (p-1)F$$

where $F = A \times \{\mathrm{pt}\}$. If $p = 3$, then $K_X = 0$.

Theorem (Ejiri, Patakfalvi; arxiv/2023.05)

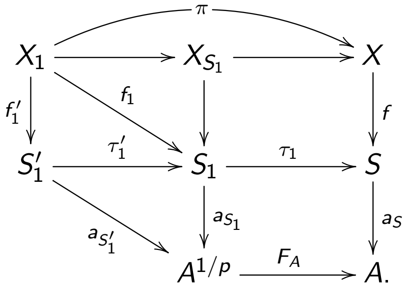
Let (X, Δ) be a projective strongly F-regular pair such that $-(K_X + \Delta)$ is nef whose Cartier index not divisible by p. Then

1. alb_X *is surjective.*
2. *There are no* alb_X*-exceptional divisors, that is, for every prime divisor* E *on* X*, the codimension of* $f(\mathrm{Supp}\, E)$ *is at most one.*
3. *Let :* $X \xrightarrow{f} Y \xrightarrow{g} A$ *be the Stein factorization of* alb_X*. Then* g *is purely inseparable.*

Theorem (Chen; Wang; Zhang; arxiv/2023.08)

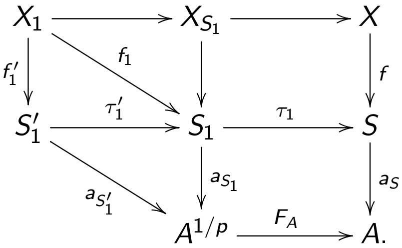
Assume that a singular variety admits a resolution of singularities. Let (X, Δ) be a projective normal $\mathbb{Q}$-factorial klt pair. Assume that $-(K_X + \Delta)$ is nef. If the Albanese morphism $\mathrm{alb}_X \colon X \to A$ is of relative dimension one over the image $\mathrm{alb}_X(X)$. Then the Albanese morphism alb_X is a fibration.

The proof



(0) Assume that f is inseparable to begin with.
(1) reduce to the case that $\kappa(S) > 0$.
(2) reduce to the case that a_S is inseparable.
(3) reduce to the case that S_1 is an abelian variety.
(4) Using foliation to discuss the left situation

(1) reduce to the case that $\kappa(S) > 0$



Proposition (HPZ19 smooth case)

Assume that a singular variety admits a resolution of singularities. Let X be a normal projective varieties of maximal Albanese dimension. If $K_X \equiv 0$ then X is isomorphic to an abelian variety.

(2) reduce to the case that a_S is inseparable

To deal with the separable case, we develop a canonical bundle formula in the following form:

Theorem

Let (X, Δ) be a pair and let $f : X \to S$ be an inseparable fibration of genus-one curve. Assume that

- (C1) (X, Δ) *is lc (on the generic fiber);*
- (C2) *there exists a $\mathbb{Q}$-divisor D on S such that $K_X + \Delta \sim_{\mathbb{Q}} f^*D$;*
- (C3) *S is of m.A.d., and the Albanese morphism $a_S : S \to A$ is separable.*

Then $D \succeq_{\mathbb{Q}} \frac{1}{2p} K_S$. In particular, $\kappa(S, D) \geq \kappa(S)$.

(2) reduce to the case that a_S is inseparable

This is done by taking a Frobenius pullback and obtain a fibration $f_1\colon X_1 \to S_1$ of genus-0-curves. Note that

$$\pi_1^* K_X \sim_{\mathbb{Q}} K_{X_1} + (p-1)\det(\Omega^1_{X_1/X})$$

and by the work [JiWal21], $|\det(\Omega^1_{X_1/X})|$ gives a nontrivial horizontal linear system. And we reduced to prove the following:

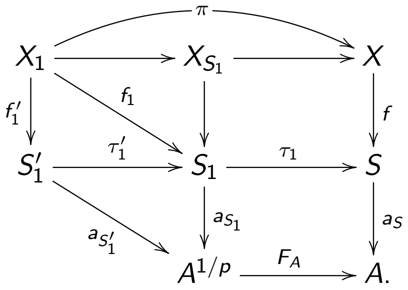
Theorem (A canonical bundle formula of genus-0 curves)

Let $f\colon X \to S$ be a fibration of genus-0 curves. Assume that $K_X + \mathfrak{M} + \Delta \sim_{\mathbb{Q}} f^*D$ where $\mathfrak{M}$ is a movable horizontal linear system.

1. *f is separable;*
2. *f is inseparable, and S is of maximal Albanese dimension;*

Then there exists a number $t > 0$ such that $D \succeq_{\mathbb{Q}} \frac{1}{2}K_S$.

(3) reduce to the case that S_1 is an abelian variety



By taking plenty times of base change, we reduce to the case that the fibration f_n is separable. Thus the following three is the main ingredients:

- a canonical formula of quasi-elliptic curves;
- a canonical formula of conic bundles;
- positivity of the relative canonical bundle.

(3) reduce to the case that S_1 is an abelian variety

We deduce a canonical formula in the following form as a conjunction to the one of Witaszek:

Theorem (a canonical formula of quasi-elliptic curves)

Let $f: X \to S$ be a fibration of quasi-elliptic curves. Let $\tau_1: S_1 \to S$ be a finite purely inseparable morphism of height one. Assume that $(X_{K(S)}, \Delta_{K(S)})$ is lc and $K_X + \Delta \sim_{\mathbb{Q}} f^*D$ for some $\mathbb{Q}$-divisor D on S.

Then there exist finite morphisms
$$\begin{array}{ccc} \bar{T}' & \xrightarrow{\tau'} & \bar{T} \\ \downarrow{\scriptstyle \tau_1'} & & \downarrow{\scriptstyle \tau} \\ S_1 & \xrightarrow{\tau_1} & S \end{array}$$
and an effective $\mathbb{Q}$-divisor $E_{\bar{T}'}$ on $\bar{T}'$ such that

$$(\tau_1' \circ \tau_1)^* D \sim_{\mathbb{Q}} aK_{\bar{T}'} + b\tau'^* K_{\bar{T}} + c\tau_1'^*(\tau_1^* K_S - K_{S_1}) + E_{\bar{T}'}$$

where $a, b, c \geq 0$ are rational numbers.

(3) reduce to the case that S_1 is an abelian variety

Proposition (A canonical bundle formula of conic bundles)

Let $f\colon X \to S$ be a fibration of curves onto S of maximal Albanese dimension. Let $\mathfrak{M}$ a horizontal movable linear system without fixed components such that $-(K_X + \mathfrak{M} + \Delta)$ is nef. Assume that

- *either $(X_{K(S)}, \Delta_{K(S)})$ is klt, or if T is a (the unique) horizontal irreducible component of Δ with coefficient one then $\deg_{K(S)} T = 1$ and the restriction $T|_{T^\nu}$ on the normalization of T is pseudo-effective.*

Then

(i) *S is an abelian variety;*

(ii) *M_0 is semi-ample with numerical dimension $\nu(M_0) = 1$, that is, $|M_0|$ defines a fibration $g\colon X \to \mathbb{P}^1$;*

(iii) *for a general $t \in \mathbb{P}^1$, the fiber of g over t (denoted by G_t) is isomorphic to an abelian variety, and $\Delta|_{G_t} \equiv 0$.*

(4) Using foliation to discuss the left situation

Lemma

Let X be a normal projective variety equipped with two fibrations

- $f: X \to A$, *of relative dimension one onto an abelian variety; and*
- $g: X \to \mathbb{P}^1$, *such that each fiber is dominant over A.*

$$\begin{array}{ccccccc} \mathbb{P}^1 & \xleftarrow{g} & X & \xrightarrow{/\mathcal{F}} & X' & \longrightarrow & X^p \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f^p} \\ & & A & \xrightarrow{\sigma} & S' & \longrightarrow & A^p, \end{array}$$

where $X' \to S' \to A^p$ is the Stein factorization of $X' \to A^p$. Assume that there is a dense open subset $V \subseteq \mathbb{P}^1$ such that for each $t \in V$, the fiber G_t of g is isomorphic to an abelian variety, and $\det(\mathcal{F}|_{G_t}) \equiv 0$. Then S' is an abelian variety.

Thank you!

Thank you for your attention!