The Exclusivity Principle Determines the Correlation Monogamy

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Introduction.—One of the characteristic features of classical correlation is its shareability among many parties, but the situation for quantum correlation is very different, it cannot be shared freely. The limitation on the shareability of quantum correlation is now known as monogamy relation, see, for example, Refs.[1–3]. Since it was first qualitatively formulated by Coffman, Kundu and Wootters for quantum entanglement correlations using concurrence[4], monogamy as a quantum phenomenon has been studied in many types of correlations, such as Einstein-Podolsky-Rosen (EPR) steering[5], Bell nonlocality[6–8], contextuality[8, 9], and contextuality-nonlocality[8, 10]. Some experimental verifications for these monogamy relations are also reported[11]. From a practical perspective, monogamy has widespread applications over many areas of physics, including the derivation of security of quantum key distribution[12], determining quantum critical point[13], diagnosing topological edge states[14], even in the arguments of firewall problem of black holes[16, 17].

We provide an operational criterion for monogamy: if the fractional packing number of the graph corresponding to the union of event sets of several physical experiments does not exceed the sum of independence numbers of each individual experiment graph, then these experiments are monogamous. As applications of this observation, several examples are provided, including the monogamy relations of Swetlichny’s genuine nonlocality. We also give the necessary and sufficient condition for several experiments to be monogamous: several experiments are monogamous if and only if the Lovász number the union exclusive graph is less than or equal to the sum of independence numbers of each exclusive graph.

In the typical correlation test scenario, some physicists run an experiment, in which a set of physical events \( \mathcal{E} = \{ e_i \} \) occur with the respective probabilities \( \mathcal{C}(\mathcal{E}) = \{ p(e_i) \} \). Hereinafter, a physical event which we denote as \( a_1, \ldots, a_n, x_1, \ldots, x_n \) is obtaining \( a_1, \ldots, a_n \) upon measuring \( x_1, \ldots, x_n \) on a physical system, assume that the preparation of the system is reproducible, then by repeating the experiment many times, we can collect the probabilities of \( p(a_1, \ldots, a_n|x_1, \ldots, x_n) \) for each event \( a_1, \ldots, a_n, x_1, \ldots, x_n \). To check if the observed statistics \( \mathcal{C}(\mathcal{E}) \), which we refer to as the experimental correlation or physical behavior as in some literature[7], satisfies some physical principle \( P \) (or some theoretical model \( T = \{ P_1, \ldots, P_n \} \) which is nothing more than an assemblage of physical principles), we can calculate a test parameter \( \mathcal{I}_P \), which is a (not necessarily) real-valued bounded function with variables being all involved probabilities, and find out if the resulting value lies in the range allowed by the principle \( P \).

In many cases, the \( P \)-character range \( \mathcal{R}_P \subseteq \mathbb{R} \) is just an interval. For instance, in Bell’s local hidden variable (LHV)[18] test scenario, if test parameter is chosen as Clauser-Horne-Shimony-Holt (CHSH) parameter[19], the LHV-range is \( \mathcal{R}_{LHV} = [-2, 2] \); in Kochen-Specker’s non-contextual hidden variable (NCHV) test scenario[20], if the test parameter is chosen as Klyachko-Can-Binicioğlu-Shumovsky (KCBS) parameter[21], its NCHV-range is \( \mathcal{R}_{NCHV} = [-3, 5] \). The ranges allowed by quantum mechanics of CHSH and KCBS parameters are \([−2\sqrt{2}, 2\sqrt{2}] \supseteq \mathcal{R}_{LHV} \) and \([5 − 4\sqrt{5}, 5] \supseteq \mathcal{R}_{NCHV} \), which indicates that quantum mechanics is beyond LHV model and NCHV model, but the violated values stop at a bound which is now generally referred to as Tsirelson’s bound or quantum bound[22]. Many efforts have been made to answer the question what is the physical principle that prevents quantum mechanics from having a larger violation than the one of quantum
mechanics. Information causality[23], local orthogonal principle[24], measurement sharpness[25], and exclusivity principle (EP)[26, 27] provide us with some rationales for why limits on quantum mechanics may exist. Among all of them, more and more evidences suggest that EP is suitable as a fundamental assumption of quantum mechanics. In this letter, we concern a very relevant but somewhat weaker problem: why different types of monogamy may exist in a given theoretical framework, how can we determine if several experiments are monogamous or not.

Suppose we are running two experiments $E = \{e_i|p(e_i)\}$ and $U = \{u_i|p(u_i)\}$ simultaneously in a theoretical framework $\mathcal{T}$, which means that statistics $p(e_i)$ and $p(u_i)$ are obtained under the constraint of theory $\mathcal{T}$, e.g., in quantum mechanics it is $p(e_i) = \text{tr}(\rho_{e_i} \rho)$ with $\rho_{e_i}$ the measurement corresponding to event $e_i$ and $\rho$ the prepared state and analogously for $p(u_i)$. In order to check if correlations $C(E)$ and $C(U)$ satisfy principle $P$ or not, we need to check that if $I_1(p(e_i)) \in [r^2_p, R^2_p]$ and $I_2(p(u_i)) \in [r^2_p, R^2_p]$ or not, where $r^2_p (R^2_p)$ is the low (upper) bound allowed by $P$-principle. Without losing of generality, hereinafter we assume that each test parameter $I$ is positive, since $I$ is a bounded function with lower bound $l$ (this bound is calculated only under the constraints imposed by probability theory, viz., Kolmogorov axioms), we can replace $I$ with $I' = I - l$, the replacement have no influence on the checking result. Two experiments are no-P monogamous in $T$ theory if

$$\mathcal{M}(I_1, I_2) \subseteq [m_p, M_p] \subseteq [M(R^1_p, R^2_p), M(R^1_p, R^2_p)]$$  \hspace{1cm} (1)

where $\mathcal{M}(I_1, I_2)$ is a monotonically increasing function which we refer to as the monogamy function and is often chosen as $I_1 + I_2$, "$\subseteq"$ indicates the tight bound allowed by $T$ theory, and the respective lower (upper) bound $m_p(M_p)$ is referred to as monogamy score. The monogamy relation (1) means that if the experimental correlation $C(E)$ is a no-P correlation, viz., $I_1(p(e_i)) \not\in [r^2_p, R^2_p]$, then the correlation $C(U)$ must be $P$-correlation, viz., the value of $I_2$ must lie in the $P$-character interval $[r^2_p, R^2_p]$. We take the monogamy of nonlocality as an example, if Alice implement two CHSH-type Bell experiments $E_{AB}$ and $E_{AC}$ with Bob and Charlie simultaneously, then monogamy relation reads $T^{CHSH}_{AB} + T^{CHSH}_{AC} \leq \{ -4, 4 \}$, viz, if Alice and Bob observe the violation of $T^{CHSH}_{AB} \in \{ -2, 2 \}$ then Alice and Bob must not observe the violation and vice versa.

Since quantum mechanics is also a theoretical model consisting of a set of physical principles (actually, we haven’t yet found out what these principles are), to give an explanation of the origin of no-P monogamy in quantum mechanics, we must explain what constitute principle of quantum mechanics can be used to derive the monogamy relation. There are some trials to explain the origin of different kinds of monogamy, for instance, the principle of no-disturbance[8, 9](or more restricted no-signaling principle in Bell’s scenario[6, 8]) can be used to derive contextuality monogamy, correlation complementarity can be used to derive nonlocality monogamy[28] and the Lorentz invariance in Bloch representation can be used to derive entanglement monogamy[29]. But all these attempts provide only partial answer, and in many cases their derivation can not give the tight monogamy bound in quantum mechanics.

In this letter, we analyze the quantum correlation relations in detail and we find many important monogamies (monogamy of quantum contextuality, nonlocality, genuine nonlocality and contextuality-nonlocality, etc.) appearing in quantum theory can be derived from EP. We prove that the monogamy derived from EP is tight, viz., the monogamy bound restricted by EP meets the bound restricted by quantum mechanics. We give the necessary and sufficient conditions for some experiments to be monogamous and as an application, we provide an operational criterion to determine if several experiments are monogamy. This sheds new light on the relationship between EP and quantum mechanics.

We begin with a brief review of the EP.

Exclusivity as a physical principle.—Suppose that we are running an experiment $E = \{e_i\}$, the test parameter for a physical property may have many types of formulations: it can be a sum $\sum_i w_i p(e_i)$ where each weight $w_i$ is a real number, like in Bell inequality, noncontextuality inequality; or some other function such as Shannon entropy, Tsallis entropy, Rényi entropy and so on. Here, we focus on the sum-type test parameter, which is also the most studied form since Bell’s seminal work[18]. A standard correlation test inequality of the experiment $E$ is of the form:

$$I_E = \sum_i w(e_i)p(e_i) \leq R_C \leq R_Q \leq R_{SQ},$$  \hspace{1cm} (2)

in which $R_C$, $R_Q$ and $R_{SQ}$ represent the classical bound, Tsirelson’s bound and supraquantum bound respectively. We assume each $w(e_i)$ to be a positive real number, this can be done by substituting probabilities of events which have a negative $w_i$ by unity minus the probability of the opposite events. Hereinafter, for simplicity, we will assume that all $w(e_i)$ equal to 1, and all results can be applied to the general case by simply substituting all graph terms with the weighted graph terms. To explain why quantum mechanical violation of the inequality stops at the Tsirelson’s bound, Cabello[26] and Yan[27] suggest the EP which can be summarized as:

Exclusivity principle(EP): the sum of probabilities of pairwise exclusive events cannot exceed 1.

Note that EP can be applied to a given target experiment without considering other experiments that may be carried out in other part of the universe, we refer to this kind of application of EP as weak EP, relatively, it can also be applied to an extendeded experiments that
shed some light on the correlations of target experiment, we refer to this case as strong EP. Tsirelson’s bounds of CHSH inequality[19] and KCBS inequality[21] can be derived from EP[26, 27], more exactly, it should be strong EP.

In Ref.[30], Cabello, Severini, and Winter provide a graph theoretical approach to the correlation test experiment $\mathcal{E}$, it can be sketched as:

$$\begin{align*}
\text{inequality } \mathcal{I}_E &: R_C \leq R_Q \leq R_{SQ} \\
\text{experiment } \mathcal{E} &: \\
\text{graph } G_E &: \alpha(G_E) \leq \vartheta(G_E) \leq \alpha^*(G_E)
\end{align*}$$

where we associate an events graph $G_E$ to each correlation experiment $\mathcal{E}$, in which a vertex represents an event $e_i$ and an edge represents an exclusive pair $(e_i, e_j)$. Correlation test parameter can be considered as a linear function $\mathcal{I}_E$ of probabilities of related events. The classical bound $R_C$ is given by the independence number $\alpha(G)$ of the exclusive graph $G_E$, which is the cardinality of the largest independent vertex set. The Tsirelson’s bound is given by the Lovász number $\vartheta(G_E)$ of the exclusive graph, which is defined as $\vartheta(G_E) = \max \sum_{i \in V(G_E)} |\phi(v_i)|^2$ where the maximum is taken over all orthonormal representations of the complete graph of $G_E$. The bound given by EP is then the fractional packing number $\alpha^*(G_E)$ of the exclusivity graph. See supplementary material [31] for the involved graph theoretical terminologies in more detail.

Here we take the CHSH inequality

$$\sum_{a \oplus b = xy \atop x, y = 0, 1} p(a, b|x, y) \leq 2 + \sqrt{2} \leq 4, \quad (3)$$

and KCBS inequality

$$\sum_{a \oplus b = i \atop i = 1}^5 p(a, b|i, i + 1) \leq 4 \leq 2 \sqrt{5} \leq 5, \quad (4)$$

as two important examples. As depicted in Fig 1, the exclusive graph of CHSH and KCBS experiments are a 4-Möbius ladder $M_4$ and 5-prism graph $Y_5$, it is obvious that graph theoretical terms coincide with the inequality bounds [31]. Note as a consequence of symmetry of prism graph $Y_5$, this KCBS inequality can be simplified into a five-vertex pentagon graph inequality

$$\sum_{i = 1}^5 p(0, 1|i, i + 1) \leq 2 \sqrt{5} \leq 5/2. \quad (5)$$

These two inequalities can be extended to a unified $n$-cycle inequality[32, 33] as

$$\mathcal{I}_n \leq n - 1 \leq \begin{cases} 
2n \cos \left(\frac{\pi}{n}\right) & n \in 2\mathbb{N} + 1, \\
\frac{n \cos \left(\frac{\pi}{n}\right) + 1}{n \cos \left(\frac{\pi}{n}\right)} & n \in 2\mathbb{N},
\end{cases} \quad (6)$$

where $\mathcal{I}_n$ is of the form $\sum_{i \neq j \in E} \sum_{a, b \oplus 0 \neq 01 \atop i, j} p(ab|ii + 1) + \sum_{i \neq j \in E} \sum_{a \oplus b \neq 0 \atop i, j} p(ab|jj + 1)$ or of its complementary form $\sum_{i \neq j \in E} \sum_{a \oplus b \neq 0 \atop i, j} p(ab|ii + 1) + \sum_{i \neq j \in E} \sum_{a \oplus b \neq 0 \atop i, j} p(ab|jj + 1)$ for some $j \in \{1, \cdots, n\}$, and we make the convention that $n + 1 = 1$. The graph of odd $n$-cycle inequality is a prism graph $Y_n$ which is isomorphic to the 2$\nu$-vertex $(2n, n)$ circulant graph $C_{2n}(2, n)$, the graph of even $n$-cycle inequality is a Möbius ladder graph $M_n$ which is isomorphic to the 2$\nu$-vertex $(1, n)$ circulant graph $C_{2n}(1, n)$, their independence numbers, Lovász numbers and fractional numbers coincide with each physical correspondence.

**Origin of monogamy.—** An interesting aspect of these correlation inequalities is that the violations are monogamous, viz., violation of one inequality may lead to an unviolated value of the other inequality. It is a fundamental problem to pinpoint what part of quantum mechanics is responsible for these constraints. Here we proffer an explanation: monogamy is a consequence of EP.

Consider two laboratories are running two testable correlation experiments with respective exclusivity set $\mathcal{E}_1(i = 1, 2)$, where the word *testable* means that the graph of the exclusivity set $G_{\mathcal{E}_1}$ has a distinct independence number, Lovász number and fractional packing number, and each graph theoretical term coincides with the physical bound of corresponding testing parameter $\mathcal{I}_{\mathcal{E}_1}$, this is equivalent to say that each exclusivity graph $G_{\mathcal{E}_1}$ contains, as induced subgraphs, odd cycles on five or more vertices and/or their complements[34]. If two experiments are implemented simultaneously, they can be regarded as an integral experiment $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. Then there will be some exclusivity pairs $(e_i, u_j)$ with $e_i \in \mathcal{E}_1$ and $u_j \in \mathcal{E}_2$, which do not appear if we regard them as two independent experiments (vìz., if they are not implemented simultaneously).

As depicted in Fig. 2, if two non-contextuality experiments $\mathcal{E}_{\text{KCBS}} = \{0|1, 0|1, 0|1, 0|1, 0|1\}$ and $\mathcal{E}_{\text{KCBS'}} = \{0|1, 0|1, 0|1, 0|1, 0|1\}$ (where we assume that the triple $1, 1', 2'$ are exclusive, viz., they can not get 0 outcomes simultaneously, as is also the triple $4, 5, 5'$) are not implemented simultaneously, their corresponding graph is just two pentagons, the red and black one as in Fig. 2(a), but if they are imple-
mented simultaneously, there will be some new exclusive pairs (four green edges as in Fig. 2) like (01|12, 01|1’2’) and so on which does not exist before. Thus we will get the graph corresponding to $\mathcal{E}_{\text{CBS}} \sqcup \mathcal{E}_{\text{CBS}'}$ just as the whole graph in Fig. 2(b). It is these new exclusive pairs which are responsible for monogamy relations.

We are now in position to explain what is monogamy, and in what sense can a physical principle be regarded as the origin of it. Suppose that we are concerned about a physical principle $P$, like no-disturbance, no-signaling, and exclusivity. The principle will give some constraints while calculating the test parameter $I_{\mathcal{E}_i}$, the maximal value of the $I_{\mathcal{E}_i}$ restricted by the principle $P$ is called the $P$-bound, mathematically, $I_{\mathcal{E}_i}^P \leq R_P$. If $P$ is chosen as classical mechanics, we have the classical bound $R_C$, similarly, for quantum mechanics we have Tsirelson bound $R_Q$. We call two no-$P$ correlations monogamous in quantum mechanics if

$$I_{\mathcal{E}_i} + I_{\mathcal{E}_j} \leq m_P(\mathcal{E}) \leq R_P^1 + R_P^2,$$

where the maximal value of two simultaneous experiments allowed by quantum mechanics is called $P$-monogamy score in quantum mechanics, denoted as $m_P(\mathcal{E})$. This means that in quantum mechanical framework, viz., in our universe, if the first $P$-test experiment get the violated value $I_{\mathcal{E}_i}(\rho(\epsilon)) \geq R_P$, then the second $P$-test inequality cannot be violated. A physical principle $X$ can be regard as the origin of no-$P$ correlation monogamy if the maximal value of $I_{\mathcal{E}_i} + I_{\mathcal{E}_j}$ allowed by the principle $X$ is less than or equal to the $P$-monogamy score, viz., $I_{\mathcal{E}_i} + I_{\mathcal{E}_j} \leq R_X = m_P(\mathcal{E})$.

The appearance of extra exclusive pair when we run several experiments simultaneously make the fractional packing number $\alpha^*(G_{\mathcal{E}})$ which corresponds to the physical bound restricted by EP decrease in compare to the sum of individual fractional packing numbers $\alpha^*(G_{\mathcal{E}_i}) + \alpha^*(G_{\mathcal{E}_j})$, if there are enough exclusivity pair to make $\alpha^*(G_{\mathcal{E}}) \leq R_C^1 + R_C^2 = \alpha(G_{\mathcal{E}_i}) + \alpha(G_{\mathcal{E}_j})$, then we get the monogamy relation. So monogamy is quantitative relation to evaluate the exclusive degree of two experiments. Above explanation can be summarized as the following result:

**Theorem 1.** Given several disjoint experimental event sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$ with exclusive graphs $G_{\mathcal{E}_1}, \ldots, G_{\mathcal{E}_n}$ respectively, we can make them into an assemblage of events $\mathcal{E} = \sqcup_{i=1}^n \mathcal{E}_i$ with exclusive graph $G_\mathcal{E}$, if we have the relation

$$\alpha^*(G_\mathcal{E}) \leq \sum_{i=1}^n \alpha(G_{\mathcal{E}_i}),$$

then these experiments are monogamous.

See supplemental material for the detailed proof. Since EP only concerns about the exclusive relation between physical events, it can be used in any type of correlation test scenario. To illustrate how EP can be utilized to establish monogamy relations, we give some examples.

**Example 1. Monogamy of nonlocality:** consider three Bell-CHSH correlation test experiments $\mathcal{I}_{AB}$, $\mathcal{I}_{BC}$ and $\mathcal{I}_{CA}$ among Alice, Bob and Charlie, with the corresponding exclusive sets of each run of experiment are:

$$\mathcal{E}_{AB} = \{00|0_A0_B, 00|0_B1_A, 01|1_A1_B, 00|1_B0_A, 11|0_A0_B, 11|0_B1_A, 10|1_A1_B, 11|1_B0_A\}$$

$$\mathcal{E}_{BC} = \{01|0_A0_C, 00|0_C1_A, 00|0_A1_C, 00|1_A0_C, 10|0_A0_C, 11|0_C1_A, 11|1_A1_C, 11|1_C0_A\}$$

$$\mathcal{E}_{CA} = \{00|0_C0_B, 01|0_C1_A, 01|1_C1_A, 00|1_B0_C, 11|0_C0_B, 10|0_C1_A, 11|1_A1_C, 11|1_B0_C\}$$

The graph of each of them is a 4-Möbius ladder $M_4$, we know that the classical bound, Tsirelson’s bound and exclusivity bound of three inequalities are $\alpha(M_4) = 3$, $\psi(M_4) = 2 + \sqrt{2}$ and $\alpha^*(M_4) = 4$ respectively. The power of EP shows up when we apply it to the overall exclusive set $\mathcal{E} = \mathcal{E}_{AB} \sqcup \mathcal{E}_{BC} \sqcup \mathcal{E}_{CA}$ with the corresponding testing parameter $I = I_{\mathcal{I}_{AB}} + I_{\mathcal{I}_{BC}} + I_{\mathcal{I}_{CA}}$. There are some exclusive relations between the events of three different sets of events, for example $00|0_A0_B$ from $\mathcal{E}_{AB}$, $10|1_A0_C$ from $\mathcal{E}_{BC}$ and $11|0_C0_B$ from $\mathcal{E}_{CA}$ form a complete exclusive graph $K_3$. In fact, each column of the first three columns of the right hand side of Eq. (9) can be packed into two 3-complete graphs $K_3$, and the last column can be packed into 3 2-complete graphs $K_2$, thus the graph $G_E$ is packed into six $K_3$ graph and three $K_2$ graphs. The fractional packing number of the graph $G_E$ is therefore $9$, i.e. $I_{\mathcal{I}_{AB}} + I_{\mathcal{I}_{BC}} + I_{\mathcal{I}_{CA}} \leq 9 = R_{C_{AB}}^B + R_{C_{BC}}^B + R_{C_{CA}}^B$, this is exactly a monogamy relation.

Note that the overall exclusive set is the disjoint union of three exclusive sets, this guarantees that the corresponding testing parameter $I$ is exact the summation of three sub-testing parameters. This 3-loop nonlocality monogamy can also be derived from non-signaling principle[8]. Qin et. al. provide a derivation completely from quantum mechanics[35]. Here we prove that this phenomenon is a consequence of EP.

As a generalization of this example, we apply theorem 1 to Swetlichny’s genuine multipartite nonlocal inequalities [36–39], the CHSH inequality is just a two partite special case of Swetlichny’s inequality.

**Example 2. Monogamy of Swetlichny’s genuine nonlocality:** Suppose 16 experimenters are running three 4-body genuine nonlocality test experiments $\mathcal{I}_4 \leq 12$, see supplementary material [31] or Refs. [36–39] for the explicit expression of the test parameter $\mathcal{I}_4$, two of them have the same expression ant the other is of the complementary form, then we have such a monogamy relation

$$\mathcal{I}^{ABCD} + \mathcal{I}_4^{CDEF} + \mathcal{I}_4^{EABF} \leq 36 = R_C^{ABCD} + R_C^{CDEF} + R_C^{EFAB}.$$

Since we can packing the graph of $\mathcal{E} = \mathcal{E}_{ABCD} \sqcup \mathcal{E}_{CDEF} \sqcup \mathcal{E}_{EFAB}$ into 24 12-complete graphs $K_{12}$ and 12
Example 3. Monogamy of contextuality: One of the initial successes of EP as a fundamental principle of quantum mechanics is that it can be used to derive the Tsirelson’s bound of the KCBS inequality. It is shown [8, 9] that two contextual correlations are monogamous. Here we give a very simple proof based on EP. As depicted in Fig. 2 (b), we have two KCBS inequalities $\mathcal{I}_{KCBS}$ and $\mathcal{I}_{KCBS}$ each of which involves five dichotomic measurements $1, \ldots, 5$ (respectively $1, \ldots, 5'$), as in [9], we assume that the triple $1, 1', 2'$ are exclusive, viz., they can not get 0 outcomes simultaneously, as is also the triple $4, 5, 5'$. Then we have two event sets $\mathcal{E}_{KCBS} = \{01|12, 01|23, 01|34, 01|45, 01|51\}$ and $\mathcal{E}_{KCBS}' = \{01|1'2', 01|2'3', 01|3'4', 01|4'5', 01|5'1'\}$, if we run two experiments simultaneously, we get a set $\mathcal{E} = \mathcal{E}_{KCBS} \cup \mathcal{E}_{KCBS}'$. By carefully analysis of the graph $G_\mathcal{E}$, we find that it can be packed into two $K_3$ graphs and two $K_2$ graphs, thus its packing number is $\alpha^*(G_\mathcal{E}) = 4 = R_C + R_C'$, this means that $\mathcal{I}_{KCBS} + \mathcal{I}_{KCBS} \leq 4 = R_C + R_C'$, we get the monogamy relation.

Actually, adopting proper exclusive assumptions, we can get many monogamy relations using theorem 1, like monogamy relations of n-cycle inequalities. See Supplemental Material [31] for more examples. All these show that theorem 1 is a very general and useful result for monogamy relations, a large number of monogamy relations can be subsumed into this scenario.

Exclusivity principle yields tight monogamy. — Note that in all above examples we focus on the exclusivity relations appear in the isolated exclusivity set, viz., we have some restrictions $\sum_{e_i \in K_n} p(e_i) \leq 1$ for all $K_n \subseteq G_\mathcal{E}$. But EP can be applied to a broader scenario, which we refer to as the strong EP, it will make much more restrictions to calculate the exclusivity bound: $\sum_{e_i} p(e_i) p (\bar{e}_i)$, where $\bar{e}_i$ is corresponding events of $e_i$ in the complementary graph of $G_\mathcal{E}$. Actually, in many cases, it will make the exclusivity bound equal to the quantum bound, this is exactly the meaning that EP tight bound the quantum correlations [27]. We now analyze how this principle can be applied to the monogamy phenomenon to give the tight monogamy bound restricted by quantum mechanics. If two experiments are implemented simultaneously, their monogamy score cannot always meet the bound $R_C + R_C'$. With the strong EP, we have an explicit bound of the monogamy score restricted by quantum mechanics:

Theorem 2. Let $\mathcal{E}_1, \cdots, \mathcal{E}_n$ be several disjoint experimental event sets, the integral event set $\mathcal{E} = \cup_{i=1}^n \mathcal{E}_i$ is the disjoint union of these sets. Their monogamy score is given by theLovász number $\vartheta(G_\mathcal{E})$ of the integral exclusivity graph $G_\mathcal{E}$. These experiments are monogamous if and only if

$$\vartheta(G_\mathcal{E}) \leq \sum_i \alpha(G_\mathcal{E}_i),$$

the monogamy is tight if the equality holds.

See supplementary material for the proof. Note that Theorem 1 is a operational weak version of this theorem, since Lovász number of a graph is more difficult to calculate than the fractional packing number of a graph. If some experiments satisfy the conditions of theorem 1, they of course satisfy the condition of theorem 2, viz., they are monogamous. Actually, theorem 1 is the monogamy relations for any generalize probability theory which obeys EP and theorem 2 is the quantum version.

Discussions and conclusions — In this letter we investigated the monogamy of correlation inequality and indicated that the origin of the phenomenon is the EP. We gave the necessary and sufficient condition for several correlation experiments to be monogamous. Beside, we give some new type of monogamy relations, in particular, we give the monogamy of genuine nonlocality, its prediction of the quantum mechanics.

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Supplemental Material: The Exclusivity Principle Determines the Correlation Monogamy

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I. GRAPH THEORETICAL TERMINOLOGIES

A graph $G = (V, E)$ is a pair of sets such that $E \subseteq [V]^2$, where $[V]^2$ denotes the family of 2-element subsets of $V$. We call the set $V = V(G)$ the vertex set of $G$ and $E = E(G)$ the edge set of $G$. Two vertices $v_i$ and $v_j$ are adjacent if there exist an edge $e \in E$ such that $v_i, v_j \in e$. A vertex weight is a map $w : E \to \mathbb{R}$ and a weighted graph is a graph with a vertex weight.

The independent number $\alpha(G)$ of the graph $G$ is the cardinality of the maximum independent vertex set, where independent vertex means that each pair of vertices in the set are not adjacent.

The orthonormal representation $a$ of a graph $G$ is a map $r$ from vertex set $V(G)$ of $G$ to some vector space $W$, for which each vector $r(v_i)$ is a unit vector and $r(v_i)$ and $r(v_j)$ are orthogonal if $v_i$ and $v_j$ are adjacent. For simplicity we will use the same letter $v_i$ to label the vector corresponding to vertex $v_i$. The Lovász number $\vartheta(G)$ is then defined as

$$ \vartheta(G) = \max \sum_i |\langle v_i | \phi \rangle|^2, $$

(1)

where the maximum is taken over all orthonormal representations, i.e., in all dimensions and all orthonormal vector assignments of that dimension, and over all unit vector $\phi$.

The fractional packing number of the graph $G$ is the maximal value $\sum_{v_i \in V(G)} p(v_i)$ with the constraints

$$ \sum_{\varnothing \in K_n \subseteq G} p(v) \leq 1 $$

for all complete subgraphs $K_n$ of $G$, where $p(v)$ is non-negative real number less than one.

Here we list some important properties of independent number $\alpha(G)$, Lovász number $\vartheta(G)$ and fractional packing number $\alpha^*(G)$:

- For any graph $G$, we have $\alpha(G) \leq \vartheta(G) \leq \alpha^*(G)$;
- For the OR product graph $G \ast H$, $\vartheta(G \ast H) = \vartheta(G) \vartheta(H)$;
- For the odd cycle graph $C_n$, $\vartheta(C_n) = \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)}$.

II. PROOF OF THEOREM 1 AND THEOREM 2 IN THE MAIN TEXT

Theorem 1. Given several disjoint experimental event sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$ with exclusive graphs $G_{\mathcal{E}_1}, \ldots, G_{\mathcal{E}_n}$ respectively, we can make them into an assemblage of events $\mathcal{E} = \bigsqcup_{i=1}^n \mathcal{E}_i$ with exclusive graph $G_{\mathcal{E}}$, if we have the relation

$$ \alpha^*(G_{\mathcal{E}}) \leq \sum_{i=1}^n \alpha(G_{\mathcal{E}_i}), $$

(2)

then these experiments are monogamous.

Proof. In fact, we need to translate graph terms into their physical correspondences: $\mathcal{E} = \bigsqcup_{i=1}^n \mathcal{E}_i \leftrightarrow I_\mathcal{E} = \bigsqcup_{i=1}^n I_{\mathcal{E}_i}$, $\alpha^*(G_{\mathcal{E}})$ $\leftrightarrow R_E(I_{\mathcal{E}})$ and $\alpha(G_{\mathcal{E}_i})$ $\leftrightarrow R_C(I_{\mathcal{E}_i})$. Thus we arrive at the monogamy relation: $\sum_{i=1}^n I_{\mathcal{E}_i} \leq R_Q(I_{\mathcal{E}}) \leq R_E(I_{\mathcal{E}}) \leq \sum_{i=1}^n R_C(I_{\mathcal{E}_i})$.

Theorem 2. Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be several disjoint experimental event sets, the integral event set $\mathcal{E} = \bigsqcup_{i=1}^n \mathcal{E}_i$ is the disjoint union of these sets. Their monogamy score

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is given by the Lovász number $\vartheta(G_\mathcal{E})$ of the integral exclusivity graph $G_\mathcal{E}$. These experiments are monogamous if and only if

$$\vartheta(G_\mathcal{E}) \leq \sum_i \alpha(G_{\mathcal{E}_i}), \quad (3)$$

the monogamy is tight if the equality holds.

Proof. Actually we can regard these $n$ experiments as a whole experiment $\mathcal{E}$, the quantum bound $R_Q(\mathcal{E})$ of this experiment is $\vartheta(G_\mathcal{E})$, thus we arrive at $\mathcal{I} = \sum_{i=1}^{n} \mathcal{I}(\mathcal{E}_i) \leq R_Q = \vartheta(G_\mathcal{E}) \leq \sum_{i=1}^{n} \alpha(\mathcal{E}_i) = \sum_{i=1}^{n} R_C(\mathcal{E}_i)$, which is the monogamy relation.

### III. MONOGAMY OF GENUINE NONLOCALITY

Compared with the two-body nonlocality, the $n$-body nonlocal correlations have a more richer structures. Swetlichny(Svetlichny, 1987) argued that there may exist some tripartite correlations which can not be described using the so-called hybrid LHV models:

$$p(a, b, c) = P_{A|BC} \int d\lambda \rho(a|\lambda)p(b, c|\lambda)$$

$$+ P_{C|AB} \int d\lambda \rho(c|\lambda)p(a, b|\lambda)$$

$$+ P_{B|AC} \int d\lambda \rho(b|\lambda)p(a, c|\lambda), \quad (4)$$

where $P_{A|BC} + P_{C|AB} + P_{B|AC} = 1$. He provided an inequality to test this kind of nonlocality and find that some quantum states can violate the inequality. This inequality was latter generalized into the $n$-body case(Collins et al., 2002; Seevinck and Svetlichny, 2002; Bancal et al., 2011), which is of the form

$$S_n^\text{hLV} \leq 2^{-n-1} \leq \sqrt{2} \times 2^{n-1}, \quad (5)$$

where $S_n$ is defined recursively from CHSH parameter $S_2 = 0_{02} + 0_{12} + 1_{02} - 1_{12}$, the subscripts are used to denote different party. The recursive formula is

$$S_n = S_{n-1}1_{n} + S_{n-1}0_{n}, \quad (6)$$

where $S_{n-1}$ is obtained from $S_{n-1}$ by exchanging 0$_i$ and 1$_i$ for all $i = 1, \cdots, n-1$. The Swetlichny inequality can be rewritten as a linear inequality of probabilities of involved events as

$$\mathcal{I}_n \leq 3 \times 2^{n-2} \leq (2 + \sqrt{2}) \times 2^{n-2}, \quad (7)$$

where $\mathcal{I}_n$ takes the form

$$\sum_{(A_1, A_2, A_3) \neq (A_1, A_1, A_1)} p(a_1, \cdots, a_n|A_1, \cdots, A_n)$$

$$+ \sum_{(A_1, A_2, A_3) = (A_1, A_1, A_1)} p(a_1, \cdots, a_n|A_1, \cdots, A_n) \leq (8)$$

The work of Cabello(Cabello, 2015) indicates that quantum bound of $S_n$ can be derived from exclusivity principle.

Here we derive the monogamy of three $S_4$ from exclusivity principle using the theorem 1. The precise formula of $S_4$ is

$$S_4 = |\langle-1111 + 1101 + 0111 + 1011 \\ -0000 + 0010 + 1000 + 0100 \\ -0001 + 0011 + 1001 + 0101 \\ -1110 + 1100 + 0110 + 1010\rangle|. \quad (9)$$

It can be equivalently expressed as a linear inequality of probability of events as

$$\mathcal{I}_4 = \sum_{abcde=000} p(abcd|1111)$$

$$+ \sum_{abcde=001} p(abcd|1101)$$

$$+ \sum_{abcde=010} p(abcd|0111)$$

$$+ \sum_{abcde=011} p(abcd|0011)$$

$$+ \sum_{abcde=100} p(abcd|0010)$$

$$+ \sum_{abcde=101} p(abcd|0010)$$

$$+ \sum_{abcde=110} p(abcd|0010)$$

$$+ \sum_{abcde=111} p(abcd|0010) \leq (10)$$

Suppose twelve physicists $A, \cdots, F$ run three genuine nonlocality test experiment simultaneously, with $\mathcal{I}^{ABCD}_4$ and $\mathcal{I}^{CDEF}_4$ of the same form whilst $\mathcal{I}^{EFAB}_4$ are of the complementary form. Their exclusive event sets are $\mathcal{E}^{ABCD}$, $\mathcal{E}^{CDEF}$ and $\mathcal{E}^{EFAB}$ respectively. The integral event set is $\mathcal{E} = \mathcal{E}^{ABCD} \cup \mathcal{E}^{CDEF} \cup \mathcal{E}^{EFAB}$. We will show that the graph $G_\mathcal{E}$ corresponds to the integral event set can be packed into 24 12-complete graphs $K_{12}$ and 12 8-complete graphs $K_8$. Thus $\vartheta(G_\mathcal{E}) = 36 = 3 \times R_C$ which means that $\mathcal{I}_4^{ABCD} + \mathcal{I}_4^{CDEF} + \mathcal{I}_4^{EFAB} \leq R \leq 3R_C$, they can not simultaneously be violated.

The events in the experiment set $\mathcal{E}$ is listed in Table I, where ‘+’ means that the module 2 sum of all outcomes is equal to 0, and ‘−’ for 1.

Each column with odd number of ‘−’ can be packed into two 12-complete graph $K_{12}$, we take the second row as an example, it can be divided as $\mathcal{E}^{(1)}$ of the form

$$\{ ABCD \quad 0000|1100 \quad 0011|1100 \quad 1010|1000 \quad 1100|1010 \}
$$

$$\{ CDEF \quad 0101|0010 \quad 0110|0010 \quad 1001|0100 \quad 1010|0010 \}
$$

$$\{ EFAB \quad 1101|1110 \quad 1110|1101 \quad 0110|1001 \quad 0110|1011 \}
$$

and $\mathcal{E}^{(2)}$ of the form

$$\{ ABCD \quad 0101|1100 \quad 0110|1100 \quad 1001|1100 \quad 1010|1100 \}
$$

$$\{ CDEF \quad 1100|0010 \quad 1110|0010 \quad 0110|0010 \quad 0110|0010 \}
$$

$$\{ EFAB \quad 0111|1011 \quad 0101|1011 \quad 1011|1011 \quad 1011|1011 \}, \quad (11)$$
TABLE I In this table we list the packing set of the whole experiment set.

<table>
<thead>
<tr>
<th>$S_4$</th>
<th>$S_4'$</th>
<th>$S_4''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-abcd</td>
<td>-abcd</td>
<td>-abcd</td>
</tr>
<tr>
<td>+abcd</td>
<td>+abcd</td>
<td>+abcd</td>
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<td>+abcd</td>
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<td>-abcd</td>
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<tr>
<td>+abcd</td>
<td>+abcd</td>
<td>+abcd</td>
</tr>
</tbody>
</table>

Similarly, we can pack other row into two $K_{12}$ graphs. There are totally 24 $K_{12}$ graphs.

For the row with even number of ‘-’, we can pack it into there $K_8$ graphs, this is obvious, since for each fixed measurement, there are eight outcomes, there events are pairwise exclusive. Thus the row 1, 5, 11 and 14 can be packed into 12 $K_8$ graphs. This completes the proof. In another work Jia et. al. give an more complete investigation of Swetlitchny’s genuine nonlocality, see Ref.(Jia et al., 2017) for detail.

IV. MORE EXAMPLES OF MONOGAMY

In this section, we give the monogamy relations between $n$-cycle non-contextual inequalities, here we derive the monogamy relations from exclusivity principle, which is different from the approach in Ref.(Jia et al., 2016).

(i) Suppose that $n$ Alice do $2m$-cycle correlation test experiments $I_{A_1A_2} , \ldots, I_{A_nA_1}$ as depicted in Fig. 1(b), and the $i$-th Alice choose $m$ observables $1, \ldots, m$. If each pair of $k$-th observables $k_i$ and $k_j$ of two nonadjacent Alice(i.e., $i \neq j \pm 1$) is a xor pair(i.e. events $a | k_i$ and $b | k_j$ are exclusive for $a = b$ ), then we have a loop-type monogamy relation:

$$I_{A_1A_2} + \cdots + I_{A_nA_1} \leq R_E \leq R_C^{2m} + \cdots + R_C^{n1}. \quad (11)$$

(ii) In one-to-many scenario, Alice run $n$ experiments with $n$ Bob, Alice’s observables are chosen from $\{1, \ldots, m_A\}$ and the $j$-th Bob’s observable set is $\{k_j\}_{k_j=1}^m$, again we assume that each pair of $k$-th observables $k_i$ and $k_j$ of two different Bob is a xor pair, then there is a monogamy relation like:

$$I_{AB_1} + I_{AB_2} + \cdots + I_{AB_n} \leq R_E \leq R_C + \cdots + R_C^n. \quad (12)$$

(iii) As depicted in Fig 1(c), we also have a chain type monogamy relation

$$\mathcal{I}_{A_1A_2} + \cdots + \mathcal{I}_{A_{n-1}A_n} \leq R_E \leq R_C^{2} + \cdots + R_C^{n-1}. \quad (13)$$

**Proof.** (i) The event set of each experiment $I_{A_iA_{i+1}}$ is

$$\mathcal{E}_{i,i+1} = \{00|1,1_{i+1}, 00|1,1_{i+1}, 11|1,1_{i+1}, 00|1_{i+1}, 11|1_{i+1}, 11|1_{i+1}, 10|1_{i+1}, 10|1_{i+1}\}.\quad$$

We need to calculate the packing number $\alpha^*(\mathcal{G}_E)$ for the overall event set $\mathcal{E} = \{ \mathcal{E}_{i,i+1} \}$. Note that we can divide $\mathcal{E}$ into $2m$ groups as:

$$\mathcal{E}^1 = \{00|1_{12}, 00|1_{12}, 00|1_{11}, 00|1_{11}, 00|1_{11}, 00|1_{11}, 00|1_{11}, 00|1_{11}\}$$

$$\vdots$$

$$\mathcal{E}^{2m} = \{01|1_{21}, 01|1_{21}, 01|1_{21}, 01|1_{21}, 01|1_{21}, 01|1_{21}, 01|1_{21}, 01|1_{21}\}.$$

Since the packing number of $\mathcal{E}'(j \leq 2m-1) = \frac{2m}{2}$ and for $\mathcal{E}^{2m}$ it is $\alpha^*(\mathcal{G}_E) = \frac{2m}{2} + n$. But note that the sum of all classical bound of these testing inequalities satisfies $\sum R_C^{i+1} = n(2m-1) - (2m-1)\frac{2m}{2} + n$ for $m \geq 2$. Therefore,

$$\mathcal{I}_E = \sum_{i=1}^{n} I_{A_iA_{i+1}} \leq \alpha^*(\mathcal{G}_E) \leq (2m-1)\frac{2m}{2} + n$$

this completes the proof.

In the same spirit as the proof of (i), we collect all events and redivide them into some disjoint subsets, we find that the packing number of the overall event set $\mathcal{E}$ is less than the sum of classical bound of each inequality: $\alpha^*(\mathcal{G}_E) \leq \sum R_C$. We can prove (ii) and (iii).

Here, we use the term Alice and Bob only for convenience of description, they are not assumed to be spatially separated like in Bell’s scenario. To see how the conditions for nonadjacent Alice’s observables can be implemented in quantum mechanics, let us take the $i$-th Alice’s observable set as $\{k_i = 2\langle v_i^k | v_i^k \rangle - 1\}$ and the $j$-th Alice’s observable set as $\{k_j = 2\langle v_j^k | v_j^k \rangle - 1\}$ with
\langle v^k_i | v^k_j \rangle = 0 \text{ for all } k \text{ and } i \neq j \pm 1. \text{ It is obvious that } k_i \text{ and } k_j \text{ are commutative and they cannot simultaneously have the same outcomes. This implementation is a much stronger condition than the one in the derivation of monogamy from no-disturbance principle where } k_i \text{ and } k_j \text{ are merely commutative (Jia et al., 2016).}

REFERENCES


