EXISTENCE OF SOLUTIONS

LITTLE PRINCE

2020年12月1日

1. Perron Method

1.1. **Introduction.** We are in a position now to approach the question of existence of solutions of the classical Dirichlet problem in arbitrary bounded domains. The treatment here will be accomplished by $Perron's\ method\ of\ subharmonic\ functions$ which relies heavily on the maximum principle and the solvability of the Dirichlet problem in balls. The method has a number of attractive features in that it is elementary, it separates the interior existence problem from that of the boundary behaviour of solutions, and it is easily extended to more general classes of second order elliptic equations.

1.2. **Part 1:Perlimiary.** Let Ω be a bounded domain in $\mathbb{R}^n$ and φ be a continuous function on $\partial\Omega$.
Consider

$$\begin{aligned} \triangle u &= 0 \quad \text{in} \quad \Omega \\ u &= \varphi \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1.1}$$

If Ω is a ball, then the solution of (2.1) is given by the Poisson formula. We now solve (2.1) by Perron's method. The maximum principle plays an essential role. In discussions below, we avoid mean value properties of harmonic functions.
We first define continuous subharmonic and superharmonic functions based on the maximum principle.

Definition 1.1. *A $C(\Omega)$ function u will be called subharmonic (superharmonic) in Ω if for every ball $B \subset\subset \Omega$ and every function h harmonic in B satisfying $u \leq (\geq) h$ on ∂B, we also have $u \leq (\geq) h$ in B.*

We now prove a maximum principle for such subharmonic and superharmonic functions.

Lemma 1.1. *If u is subharmonic in Ω and v is superharmonic in Ω with $v \geq u$ on $\partial\Omega$. Then either $v > u$ in Ω or $v \equiv u$.*

Proof. Suppose $u - v$ attains its nonnegative maximum of $\overline{\Omega}$ in some point of Ω. Set $M = \max_{\overline{\Omega}}(u - v) \geq 0$ and $\Sigma = \{x \in \Omega : u(x) - v(x) = M\} \subset \Omega$. It is nonempty and

relatively closed in Ω, so we only need to show Σ is open.
Now we show that Σ is open. For $x_0 \in \Sigma$, we take $B = B_r(x_0) \subset\subset \Omega$. Letting $\overline{u}, \overline{v}$ denote the harmonic functions respectively equal to u, v on ∂B, one sees that $M \geq sup_{\partial B}(\overline{u} - \overline{v}) \geq (\overline{u} - \overline{v})(x_0) = M$, and hence the equality holds throughout. By the strong maximum principle for harmonic functions, it follows that $u - v \equiv M$ on ∂B. This holds for any $0 < \rho \leq r$, then $u - v \equiv M$ in B and hence $B \subset \Sigma$. In conclusion, $u - v \equiv M \geq 0$ in Ω. By $v \geq u$ on $\partial\Omega$, we have $u = v$ in Ω. □

Before we start our discussion of Perron's method, we demonstrate how to generate bigger subharmonic functions from existing subharmonic functions.

Lemma 1.2. *Let u be subharmonic in Ω and $B \subset\subset \Omega$. Denote by $\overline{u}$ the harmonic function in B (given by the Poisson integral of u on ∂B) satisfying $\overline{u} = u$ on ∂B. We define in Ω the harmonic lifting of u (in B) by*

$$U(x) = \begin{cases} \overline{u}(x), & x \in B \\ u(x), & x \in \Omega \setminus B \end{cases} \tag{1.2}$$

Proof. For consider an arbitrary ball $\tilde{B} \subset\subset \Omega$ and let h be a harmonic function in $\tilde{B}$ satisfying $h \geq U$ on $\partial\tilde{B}$. Since $u \leq U$ in $\tilde{B}$ (in fact, we have $u \leq U$ in Ω) we have $u \leq h$ in $\tilde{B}$ and hence $U \leq h$ in $\tilde{B} \setminus B$. Also since U is harmonic in $B \cap \tilde{B}$, we have by the maximum principle $U \leq h$ in $\tilde{B} \cap B$. Consequently $U \leq h$ in $\tilde{B}$ and U is subharmonic in Ω. □

Next, we say that the class of subharmonic functions S is closed by taking the maximum among finite by many functions in S.

Lemma 1.3. *Let $u_1, u_2, \cdots, u_N$ be subharmonic in Ω. Then the function $u(x) = \max\{u_1(x), u_2(x), \cdots, u_N(x)\}$ is also subharmonic in Ω.*

Proof. This is an easy consequence of the definition of subharmonicity. □

Remark 1.4. *Corresponding results for superharmonic functions are obtained by replacing u by $-u$ in lemma 2.1, 2.1, 2.3.*

1.3. **Part 2:Solving the Dirichlet problem.** Now let Ω be bounded and φ be a bounded function on $\partial\Omega$.

Definition 1.2. *A $C(\bar{\Omega})$ subharmonic function u is called a subfunction relative to φ if it satisfies $u \leq \varphi$ on $\partial\Omega$. Similarly, a $C(\bar{\Omega})$ superharmonic function u is called a superfunction relative to φ if it satisfies $u \geq \varphi$ on $\partial\Omega$.*

By the maximum principle(lemma 2.1), every subfunction is less that or equal to every superfunction. In particular, constant functions $\leq \inf_{\partial\Omega} \varphi$ ($\geq \sup_{\partial\Omega} \varphi$) are subfunctions(superfunctions). Let S_φ denote the set of subfunctions relative to φ. The basic result of the Perron method is contained in the following theorem.

Theorem 1.5. *The function $u(x) = \sup_{v \in S_\varphi} v(x)$ is harmonic in Ω.*
Tools: the harmonic lifting, the compactness of bounded harmonic function, the maximum principle

Proof. First we explain the ideas. We need to prove that u is harmonic. Since harmonicity is a local property, we only need to prove that u is locally consistent with a harmonic function.
By the maximum principle, any function $v \in S_\varphi$ satisfies $v \leq \sup_{\partial\Omega} \varphi$, so that u is well-defined. Let y be an arbitrary fixed point of Ω. By the definition of u, there exists a sequence $\{v_n\} \subset S_\varphi$ such that $v_n(y) \to u(y)$. By replacing v_n with $\max(v_n, \inf \varphi)$, we may assume that the sequence $\{v_n\}$ is bounded. Now choose R so that the ball $B = B_R(y) \subset\subset \Omega$ and defined V_n to be the harmonic lifting of v_n in B. Then $V_n \in S_\varphi$, $\{V_n(y)\} \to u(y)$ and by the compactness of bounded harmonic functions the sequence $\{V_n\}$ contains a subsequence $\{V_{n_k}\}$ converging uniformly in any ball $B_\rho(y)$ with $\rho < R$ to a function v that is harmonic in B. v 仅仅生存在B上。Clearly $v \leq u$ in B and $v(y) = u(y)$. We claim now that in fact $v = u$ in B. For suppose $v(z) < u(z)$ at some $z \in B$. Then there exists a function $\overline{u} \in S_\varphi$ such that $v(z) < \overline{u}(z)$. Defining $w_k = \max(\overline{u}, V_{n_k})$ and also the harmonic liftings W_k, we obtain as before a subsequence of the sequence $\{W_{n_k}\}$ converging to a harmonic function w satisfying $v \leq w \leq u$ in B and $v(y) = w(y) = u(y)$. But then by the maximum principle we must have $v = w$ in B. This contradicts the definition of $\overline{u}$ and hence u is harmonic in Ω. □

The preceding result exhibits a harmonic function which is a prospective solution(called the Perron solution) of the classical Dirichlet problem: $\triangle u = 0, u = \varphi$ on $\partial\Omega$. Indeed, if the Dirichlet problem is solvable, its solution is identical with the Perron solution. For let w be the presumed solution. Then clearly $w \in S_\varphi$ and by the maximum principle(lemma 2.1) $w \geq u$ for all $u \in S_\varphi$.

In the Perron method the study of boundary behaviour of the solution is essentially separate from the existence problem. The continuous assumption of boundary values is connected to the geometric properties of the boundary through the concept of barrier function.

Definition 1.3. *Let ξ be a point of $\partial\Omega$. Then a $C(\bar{\Omega})$ function $w = w_\xi$ is called a barrier at ξ relative to Ω if:*
(i) w is superharmonic in Ω;
(ii) $w > 0$ in $\bar{\Omega} \setminus \{\xi\}$; $w(\xi) = 0$.

An important feature of the barrier concept is that it is a local property of the boundary $\partial\Omega$. Namely, let us define w to be a local barrier at $\xi \in \partial\Omega$ if there is a neighborhood N of ξ such that w satisfies the above definition in $\Omega \cap N$. Then a barrier at ξ relative to Ω can be defined as follows. Let B be a ball satisfying $\xi \in B \subset\subset N$ and $m = \inf_{N \setminus B} w > 0$. The function

$$\overline{w}(x) = \begin{cases} \min(m, w(x)), & x \in \bar{\Omega} \cap B \\ m, & x \in \bar{\Omega} \setminus B \end{cases} \tag{1.3}$$

is then a barrier at ξ relative to Ω .

Lemma 1.6. *Prove that $\overline{w}$ defined above is a barrier at ξ relative to Ω .*

Proof. First, it is easy to see that $\overline{w}$ is continuous in $\bar{\Omega}$ and property (ii) is immediate. So we only need to prove that $\overline{w}$ is superharmonic in Ω. In fact, for $\tilde{B} \subset\subset \Omega$, $\overline{w} \geq h$ on $\partial\tilde{B}$ and h is harmonic in $\tilde{B}$. Since $\overline{w} \geq h$ on $\partial\tilde{B}$ and $m \geq \overline{w}$ in $\bar{\Omega}$, we have $m \geq h$ in $\tilde{B}$ and hence $\overline{w} \geq h$ in $\tilde{B} \setminus B$. Also since $\overline{w} \geq h$ on $\partial(\tilde{B} \cap B)$, we have $m \geq h$ on $\partial(\tilde{B} \cap B)$ and hence $m \geq h$ on $\tilde{B} \cap B$. On the other hand, since $\overline{w} \geq h$ on $\partial(\tilde{B} \cap B)$, we have $w \geq h$ on $\partial(\tilde{B} \cap B)$ and hence $w \geq h$ on $\tilde{B} \cap B$ by the maximum principle. Consequently $\overline{w} \geq h$ in $\tilde{B} \cap B$ and hence $\overline{w} \geq h$ in $\tilde{B}$. □

Definition 1.4. *A boundary point will be called regular (with respect to the Laplacian) if there exists a barrier at that point.*

The connection between the barriar and boundary behavior of solutions is contained in the following.

Lemma 1.7. *Let u be the harmonic function defined in Ω by the Perron method. If ξ is a regular boundary point of Ω and φ is continuous at ξ, then $u(x) \to \varphi(\xi)$ as $x \to \xi$.*

Proof. Given $\varepsilon > 0$, and let $M = \sup_{\partial\Omega} |\varphi|$. Since ξ is a regular boundary point, there is a barrier w at ξ and, by virtue of the continuity of φ, there are constants $\delta(\varepsilon)$ and $k(\delta)$ such that $|\varphi(x) - \varphi(\xi)| < \varepsilon$ if $|x - \xi| < \delta$, and $kw(x) \geq 2M$ if $|x - \xi| \geq \delta$. The functions $\varphi(\xi) + \varepsilon + kw, \varphi(\xi) - \varepsilon - kw$ are respectively superfunction and subfunction relative to φ. Hence from the definition of u and the fact that every superfunction domimates every subfunction, we have in Ω,

$$\varphi(\xi) - \varepsilon - kw(x) \leq u(x) \leq \varphi(\xi) + \varepsilon + kw(x) \tag{1.4}$$

i.e.

$$|u(x) - \varphi(\xi)| \leq \varepsilon + kw(x). \tag{1.5}$$

Since $w(x) \to 0$ as $x \to \xi$, we obtain $u(x) \to \varphi(\xi)$ as $x \to \xi$. □

We now give an equivalent characterization of existence of solutions of the classical Dirichlet problem in arbitrary bounded domains.

Theorem 1.8. *The classical Dirichlet problem in a bounded domain is solvable for arbitrary continuous boundary values if and only if the boundary points are all regular.*

Proof. If the boundary values φ are continuous and the boundary $\partial\Omega$ consists of regular points, the preceding lemma states that the harmonic function provided by the Perron method solves the Dirichlet problem. Conversely, suppose that the Dirichlet problem is solvable for all continuous boundary values. Let $\xi \in \partial\Omega$, then the function $\varphi(x) = |x - \xi|$ is continuous on $\partial\Omega$ and the harmonic function solving the Dirichlet problem in Ω with boundary values φ is obviously a barrier at ξ. Hence ξ is regular, as are all points of $\partial\Omega$.

□

1.4. **Part 3:Some examples of domains which the boundary points are regular.** IMPORTANT QUESTION REMAINS: For what domains are the boundary points regular? It turns out that general sufficient conditions can be stated in terms of local geometric properties of the boundary. We mention some of these conditions below.
¶ $n = 2$, consider a boundary point z_0 of a bounded domain Ω and take the origin at z_0 with polar coordinates r, θ. Suppose there is a neighborhood N of z_0 such that a single valued branch of θ is defined in $\Omega \cap N$, or in component of $\Omega \cap N$ having z_0 on its boundary. One sees that

$$w = -Re\frac{1}{\log z} = -\frac{\log r}{\log^2 r + \theta^2} \tag{1.6}$$

is a local barrier at z_0 and hence z_0 is a regular point. More generally, the same barrier shows that the boundary value problem is solvable if every component of the complement of the domain consists of more than a single point.
Question: for disc with center removed, is the classical Dirichlet problem solvable?
¶ For higher dimensions a simple sufficient condition for solvability in a bounded domain $\Omega \subset \mathbb{R}^n$ is that Ω satisfy the *exterior sphere condition*; that is, for every point $\xi \in \partial\Omega$, there exists a ball $B = B_R(y)$ satisfying $\bar{B} \cap \bar{\Omega} = \{\xi\}$. If such a condition is fulfilled, then the function w given by

$$w(x) = \begin{cases} R^{2-n} - |x-y|^{2-n}, & n \geq 3 \\ \log \frac{|x-y|}{R}, & n = 2 \end{cases} \tag{1.7}$$

will be a barrier at ξ. Consequently the boundary points of a domain with C^2 boundary are all regular points.
¶ The Dirichlet problem is solvable for any domain Ω satisfying an *exterior cone condition*; that is, for every point $\xi \in \partial\Omega$ there exists a finite right circular cone K, with vertex ξ, satisfying $\bar{K} \cap \bar{\Omega} = \{\xi\}$.

2. Variational Method

2020年12月1日

2.1. **Introduction.** This Lecture, we discuss the Dirichlet problem for elliptic equations of divergence form and prove the existence of weak solutions.

2.2. **Part 1:Perlimiary.** Let Ω be a bounded domain in $\mathbb{R}^n$ and a_{ij}, b_i, and c be bounded functions in Ω.
Consider the differential operator

$$Lu = -D_j(a_{ij}D_iu) + b_iD_iu + cu. \tag{2.1}$$

We always assume that

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{2.2}$$

for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Definition 2.1. *Let $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$. Then u is a weak solution of $Lu = f$ in Ω if*

$$\int_\Omega (a_{ij} D_i u D_j \varphi + b_i D_i u \varphi + cu\varphi) dx = \int_\Omega f\varphi dx \tag{2.3}$$

for any $\varphi \in H_0^1(\Omega)$.

Next, we define

$$B[u, v] = \int_\Omega (a_{ij} D_i u D_j v + b_i D_i uv + cuv) dx \tag{2.4}$$

for any $u, v \in H_0^1(\Omega)$. We call B the bilinear form associated with the operator L. If $a_{ij} = a_{ji}$ and $b_i = 0$, then B is symmetric, i.e.

$$B[u, v] = B[v, u] \;\; for \;\; any \;\; u, v \in H_0^1(\Omega). \tag{2.5}$$

We now solve the Dirichlet problem in the weak sense for a special class of elliptic operators. We recall that the standard $H_0^1(\Omega)$ inner product is defined by

$$(u, v)_{H_0^1(\Omega)} = \int_\Omega (uv + \nabla u \cdot \nabla v) dx. \tag{2.6}$$

2.3. **Part2:Existence of weak solutions.** We assume H is a real Hilbert space, with norm $\| \;\|$ and inner product $(,)$. We let $\langle , \rangle$ denote the pairing of H with its dual space.

Theorem 2.1. *(Lax-Milgram Theorem) Assume that*

$$B : H \times B \to \mathbb{R}$$

is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that

$$|B[u, v]| \le \alpha \|u\| \;\|v\| \quad (u, v \in H)$$

and

$$\beta \|u\|^2 \le B[u, u] \quad (u \in H)$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H.
Then there exists a unique element $u \in H$ such that

$$B[u, v] = \langle f, v \rangle \tag{2.7}$$

for all $v \in H$.

Proof. Step 1. For each fixed $u \in H$, the mapping $v \mapsto B[u, v]$ is a bounded linear functional on H, whence the Riesz Representation Theorem asserts the existence of a unique element $w \in H$ satisfying

$$B[u, v] = (w, v) \quad (v \in H) \tag{2.8}$$

Let us write $Au = w$ whenever (2.8) holds, so that

$$B[u, v] = (Au, v) \quad (u, v \in H).$$

Step 2. We first claim $A : H \to H$ is a bounded linear operator.

Step 3. Next we assert

$$\begin{cases} A \ is \ one - to - one \ and \\ R(A), the \ range \ of \ A, is \ closed \ in \ H. \end{cases}$$

In fact, we have

$$\beta\|u\| \leq \|Au\|.$$

Step 4. We demonstrate now

$$R(A) = H. \tag{2.9}$$

Step 5. Next, we observe once more from the the Riesz Representation Theorem

$$\langle f, v\rangle = (w, v) \quad (\forall u \in H)$$

for some element $w \in H$. Then we can find $u \in H$ satisfying $Au = w$ by (2.9). Then

$$B[u, v] = (Au, v) = (w, v) = \langle f, v\rangle \quad (v \in H)$$

and this is (2.7).

Step 6. Finally, we show there is at most one element $u \in H$ verifying (2.7). □

Remark 2.2. *If the bilinear form $B[,]$ is symmetric, that is, if*

$$B[u, v] = B[v, u] \quad (u, v \in H),$$

we can fashion a much simpler proof by noting $((u, v)) := B[u, v]$ is a new inner product on H, to which the Riesz Representation Theorem directly applies. Consequently, the $Lax - Milgram$ Theorem is primarily significant in that it does not require symmetric of $B[,]$.

We return now to the specific bilinear form $B[,]$ defined by the formula (2.4), and try to verify the hypothesis of the $Lax - Milgram\ Theorem$.

Theorem 2.3. *Energy estimates There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

$$|B[u, v]| \leq \alpha\|u\|_{H_0^1(\Omega)} \ \|v\|_{H_0^1(\Omega)} \tag{2.10}$$

and

$$\beta\|u\|^2_{H_0^1(\Omega)} \leq B[u, u] + \gamma\|u\|^2_{L^2(\Omega)} \tag{2.11}$$

for all $u, v \in H_0^1(\Omega)$.

Proof. Step 1. It is easy to verify (2.10) by the Holder's inequality.

Step 2. It is easy to verify (2.11) by the uniformly elliptic condition and Cauchy's inequality. □

Remark 2.4. *Observe now that if $\gamma > 0$ in these energy estimates, then $B[,]$ does not precisely satisfy then hypotheses of the $Lax - Milgram\ Theorem$. The following existence assertion for weak solutions must confront this possibility:*

Theorem 2.5. *There is a number $\gamma \geq 0$ such that for each*

$$\mu \geq \gamma$$

and each function

$$f \in L^2(\Omega),$$

there exists a unique weak solution $u \in H_0^1(\Omega)$ of the boundary-value problem

$$\begin{cases} Lu + \mu u = f \ \ in \ \ \Omega \\ u = 0 \ \ on \ \ \partial\Omega. \end{cases} \tag{2.12}$$

Proof. Step 1. Take γ from Theorem 3.3, let $\mu \geq \gamma$, and define then the bilinear form

$$B_\mu[u,v] := B[u,v] + \mu(u,v) \quad (u, v \in H_0^1(\Omega))$$

which corresponds (3.1) to the operator $L_\mu u := Lu + \mu u$. Then $B_\mu[,]$ satisfies the hypotheses of the $Lax - Milgram\ Theorem$.

Step 2. Now fix $f \in L^2(\Omega)$ and set $\langle f, v\rangle := (f,v)_{L^2(\Omega)}$. This is a bounded linear functional on $L^2(\Omega)$ and thus on $H_0^1(\Omega)$.

We apply the $Lax-Milgram\ Theorem$ to find a unique function $u \in H_0^1(\Omega)$ satisfying

$$B_\mu[u,v] = \langle f, v\rangle \quad (v \in H_0^1(\Omega));$$

u is consequently the unique weak solution of (2.12). □

In the rest of this lecture, we use a minimizing process to solve the Dirichlet problem on the boundary domain with the homogeneous boundary value. Suppose $f \in L^2(\Omega)$ Define

$$J(u) := \frac{1}{2}\int_\Omega (a_{ij}D_iuD_ju + cu^2)dx + \int_\Omega uf dx. \tag{2.13}$$

Theorem 2.6. *Let $a_{ij} = a_{ji}$ and $c \geq 0$. Then J admits a minimizer $u \in H_0^1(\Omega)$.*

It is easy to check that the minimizer u is a weak solution of

$$Lu = -D_j(a_{ij}D_iu)cu = -f \ \ in \ \ \Omega.$$

Proof. We first prove that J has a lower bound in $H_0^1(\Omega)$. We calculate

$$\begin{aligned} \int_\Omega |u|f dx &\leq \Big(\int_\Omega u^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega f^2 dx\Big)^{\frac{1}{2}} \\ &\leq \sqrt{C}\Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega f^2 dx\Big)^{\frac{1}{2}} \\ &\leq \frac{\lambda}{4}\int_\Omega |\nabla u|^2 dx + C\frac{1}{\lambda}\int_\Omega f^2 dx. \end{aligned}$$

Hence for any $u \in H_0^1(\Omega)$,

$$J(u) \geq \frac{\lambda}{4}\int_\Omega |\nabla u|^2 dx - C\frac{1}{\lambda}\int_\Omega f^2 dx. \tag{2.14}$$

and in particular

$$J(u) \geq -C\frac{1}{\lambda}\int_{\Omega} f^2 dx.$$

Therefore, J has a lower bound in $H_0^1(\Omega)$. We set

$$J_0 := \inf\{J(u) : u \in H_0^1(\Omega)\}.$$

Next, we prove that J_0 is attained by some $u \in H_0^1(\Omega)$. We consider a minimizing sequence $\{u_k\} \subset H_0^1(\Omega)$ with $J(u_k) \to J_0$ as $k \to \infty$. By (2.14) we have

$$\int_{\Omega} |\nabla u_k|^2 dx \leq \frac{4}{\lambda} J(u_k) + 4C\frac{1}{\lambda^2}\int_{\Omega} f^2 dx.$$

Then we have that $\|u\|_{H_0^1(\Omega)}$ is bounded. Hence we may assume $u_k \to u_0 \in H_0^1(\Omega)$ strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$. Hence we have

$$J(u_0) \leq \liminf_{k\to\infty} J(u_k)$$

This implies $J(u_0) = J_0$. We conclude that J_0 is attained in $H_0^1(\Omega)$. □

Remark 2.7. *In fact, by J is a convex functional and $u_k \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, we have*

$$J(u_0) \leq \liminf_{k\to\infty} J(u_k)$$

3. Continuity Method

2020年12月8日

3.1. **Introduction.** This Lecture, we discuss how to solve the Dirichlet problems by the method of continuity. We illustrate this method by solving the Dirichlet problem for uniformly elliptic equationa on $C^{2,\alpha}-$*domains* by assuming that a similar problem for the Laplace equation can be solved(但事实上，我们目前只承认了在球上有这样的结果,一般地,Kellogg'theorem). The method of continuity can be applies to nonlinear elliptic equations. The crucial ingredient is a priori estimates.

3.2. **Par1.** Let Ω be a bounded domain in $\mathbb{R}^n$ and let a_{ij}, b_i, and c be defined in Ω, with $a_{ij} = a_{ji}$. We consider the differential operator

$$Lu = a_{ij}D_{ij}u) + b_iD_iu + cu. \tag{3.1}$$

for any $u \in C^2(\Omega)$. We always assume that

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Now we state a general existence result for solutions of the Dirichlet problem with $C^{2,\alpha}$ boundary values for general uniformly elliptic equations with C^α coefficients.

Theorem 3.1. *Let $L \equiv a^{ij}D_{ij} + b^iD_i + c$ be uniformly elliptic with coefficients in $C^\alpha(\bar{\Omega})$ in a bounded $C^{2,\alpha}$ domain Ω. Assume that $c \leq 0$ in Ω. Then $Lu = f$ in $\Omega, u = \varphi$, in $\partial\Omega$, has a unique $C^{2,\alpha}(\bar{\Omega})$ solution for all $f \in C^\alpha(\bar{\Omega}), \varphi \in C^{2,\alpha}(\bar{\Omega})$.*

4. Compactness Methods

2020年12月8日

4.1. **Introduction.** This Lecture, we discuss several methods to solve nonlinear elliptic differential equations. All these methods involve the compactness of the Holder functions: A bounded sequence of Holder functions has a subsequence convergent to a Holder function.

4.2. **Part1.**

Theorem 4.1. *Let Ω be a bounded $C^{2,\alpha}$-domain in $\mathbb{R}^n$ and f be a C^1-function in $\bar{\Omega} \times \mathbb{R}$. Suppose $\underline{u}, \overline{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfy $\underline{u} \leq \overline{u}$,*

$$\Delta \underline{u} \geq f(x, \underline{u}) \text{ in } \Omega, \ \ \underline{u} \leq 0 \text{ on } \partial\Omega, \text{ and } \Delta \overline{u} \geq f(x, \overline{u}) \text{ in } \Omega, \ \ \overline{u} \leq 0 \text{ on } \partial\Omega. \tag{4.1}$$

Then there exists a solution $u \in C^{2,\alpha}(\bar{\Omega})$ of

$$\begin{aligned} \Delta u &= f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, &\ \ \underline{u} \leq u \leq \overline{u} \text{ in } \Omega. \end{aligned} \tag{4.2}$$

Corollary 4.2. *Let Ω be a bounded $C^{2,\alpha}$-domain in $\mathbb{R}^n$ and f be a C^1-function in $\bar{\Omega} \times \mathbb{R}$. Then there exists a solution $u \in C^{2,\alpha}(\bar{\Omega})$ of*

$$\begin{aligned} \Delta u &= f(x, u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.3}$$

Remark 4.3. *Corollary still holds if we assume f is C^1 in $\bar{\Omega} \times \mathbb{R}$ and satisfies*

$$|f(x, z)| \leq C(1 + |z|^\tau) \text{ for any } (x, z) \in \bar{\Omega} \times \mathbb{R} \tag{4.4}$$

for some $C > 0$ and $\tau \in [0, 1)$.