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内容

1. 解决上次没讲清楚的问题

2. 证 $\mathring{H}^p_k(M) = H^p_k(M)$

3. Sobolev Embeddings (SE)

① SE $\Rightarrow$ 测地球的体积有一致下界

② compact manifold

- SE are valid.
- Rellich-Kondrakov 定理
- poincaré 不等式

  Sobolev-Poincaré 不等式.

3.23 seminar

recall last time:

1. $C^{k,\alpha}$ harmonic radius.

· 单射半径有一致下界. Ricci 曲率及其共变导数有界 ⇒ 调和半径有一致下界.

2. packing lemma:

(M^n, g) complete. $Ricci \geq kg$ for some constant $k \in \mathbb{R}$.

Let $\rho > 0$ given. Then $\exists \{x_i\} \subset M$ s.t. for any $r \geq \rho$:

(i) $\{B_{x_i}(r)\}$ is a uniformly locally finite covering of M

(ii) for any $i \neq j$. $B_{x_i}(\rho/2) \cap B_{x_j}(\rho/2) = \phi$.

3. Sobolev space.

$\vec{L}^p(M)$. (M^n, g) Rie. manifold. $1 \leq p < \infty$

Def 1. X rough vector field. i.e. $X: M \to TM$ $X(p) \in T_pM$.

不谈 X 的连续性, 光滑性.

if for each local chart $(U, \varphi; x^i)$ $X|_U = X^i \frac{\partial}{\partial x^i}$. $X^i \circ \varphi^{-1}$ is a measurable function on $\mathbb{R}^n$ $\forall i = 1, 2, \cdots, n$. then we say that X is measurable on M.

Remark: if X is a measurable vector field, then $|X|^p$ is a measurable function on M. Hence we define

$\vec{L}^p(M) = \{ X$ is a measurable vector field: $\int_M |X|^p dv(g) < \infty \}$.

$L^p(M)$ is Banach space.

相似地，对张量场我们也可以定义 L^p 空间，并证明它们是 Banach 空间

Sobolev spaces

Let (M,g) be a Rie. manifold. For k an integer and $u \in C^\infty(M)$. $\nabla^k u$ denotes the kth covariant derivative of u,

$$|\nabla^k u|^2 = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots j_k}$$

For k an integer and $p \geq 1$ real, we denote by $C_k^p(M)$ the space of smooth functions $u \in C^\infty(M)$ such that $|\nabla^j u| \in L^p(M)$ for any $j = 0, 1, \cdots, k$. Hence

$$C_k^p(M) = \{ u \in C^\infty(M) : \int_M |\nabla^j u|^p dv(g) < \infty, \ \forall j = 0, 1, \cdots, k \}$$

Def. The Sobolev space $H_k^p(M)$ is the completion of $C_k^p(M)$ with respect to the norm

$$\|u\|_{H_k^p} = \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^p dv(g) \right)^{1/p}$$

完备化后的空间

$$\{ C_k^p(M) \text{ 中的 Cauchy 列} \} / \sim$$

$\sim$: $\{u_m\}$, $\{u_m'\}$ 都为 $C_k^p(M)$ 中的 Cauchy 列

$$\{u_m\} \sim \{u_m'\} \iff \|u_m - u_m'\|_{H_k^p} \to 0 \text{ as } m \to \infty$$

i

定义集合 $\{ u : u \text{ 为 } C_k^p(M) \text{ 中 Cauchy 列的 } L^p \text{ limit} \}$

obviously, i surjective.

claim: i is injective.

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设 $\{u_m\}$ $\{v_m\}$ 为 $C_k^p(M)$ 中 Cauchy 列. $u_m \xrightarrow{L^p} u$. $v_m \xrightarrow{L^p} u$

证 $\|u_m - v_m\|_{H_k^p} \to 0$ as $m \to \infty$. 令 $w_m = u_m - v_m$

只需证 $\|w_m\|_{H_k^p} = \sum_{j=0}^{k} (\int_M |\nabla^j w_m|^p \, dv(g))^{1/p} \to 0$.

由于 $\{w_m\}$ 为 $C_k^p(M)$ 中的 Cauchy 列. 则由 L^p 的完备性.

存在 $w^1, \cdots, w^k$ s.t. $\nabla^1 w_m \to w^1, \cdots, \nabla^k w_m \to w^k$.

$\int_M \langle \nabla w_m, X \rangle = -\int_M w_m \cdot \mathrm{div} X$ 令 $m \to \infty$

$\Rightarrow \int_M \langle w^1, X \rangle = 0 \quad \Rightarrow w^1 = 0$ a.e

$$\begin{aligned}
\int_M (\nabla^2 w_m)(X, Y) &= \int_M \nabla_X \nabla_Y w_m - (\nabla_X Y)(w_m) \\
&= \int_M \nabla_X (\langle \nabla w_m, Y \rangle) - \langle \nabla w_m, \nabla_X Y \rangle \\
&= \int_M \nabla_X (w_m \cdot \mathrm{div} Y) - w_m \, \mathrm{div}(\nabla_X Y) \\
&= \int_M \nabla_X w_m \cdot \mathrm{div} Y + w_m \, X(\mathrm{div} Y) - w_m \, \mathrm{div}(\nabla_X Y) \\
&= \int_M \langle \nabla w_m, X \cdot \mathrm{div} Y \rangle + w_m \, X(\mathrm{div} Y) - w_m \, \mathrm{div}(\nabla_X Y) \\
&= \int_M -w_m \cdot \mathrm{div}(\mathrm{div} Y \cdot X) + w_m \cdot \mathrm{div} Y \cdot X - w_m \, \mathrm{div}(\nabla_X Y)
\end{aligned}$$

令 $m \to \infty$.

$\Rightarrow \lim_{m \to \infty} \int_M (\nabla^2 w_m)(X, Y) = 0 \quad \forall X, Y$ compact support.

question: $\int_M (\nabla^2 w_m)(X, Y) \to \int_M w^2 (X, Y)$? $(*)$

in fact, since $|(\nabla^2 u)(X, Y)| \leq |X| \, |Y| \, |\nabla^2 u|$.

$\Rightarrow (*) \checkmark$.

于是 $\int_M w^2 (X, Y) = 0 \quad \forall X, Y \quad \Rightarrow w^2 = 0$

相似地 可证 $\nabla^k w_m \to 0$. as $m \to \infty$.

$\Rightarrow$ the claim is proved.

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弱梯度.

$H_1^p(M)$: completion of $C_1^p(M)$

$u \in H_1^p(M)$. i.e. there exist a Cauchy sequence $\{u_m\}$ in $H_1^p(M)$ s.t. $u = \lim_{m\to\infty} u_m$ in L^2.

$\nabla u_m \to X$ in $\vec{L}^p(M)$

$\Rightarrow \int_M \langle X, Y\rangle = -\int_M u \,\mathrm{div} Y \quad \forall Y$

记 $\underline{X = Du}$. i.e. $\int_M \langle Du, Y\rangle = -\int_M u \,\mathrm{div} Y$.

称 Du 为 u 的弱梯度.

4. If M is compact, $H_k^p(M)$ does not depend on the Rie. metric.
证明只需对 $C_k^p(M)$ 中的函数来讨论即可.

5. Let (M,g) be a Rie. manifold and $u: M \to R$ a Lipschitzian function on M which equals zero outside a compact subset of M. Then $u \in \mathring{H}_1^p(M)$ for any $p \geq 1$.

6. If (M,g) is complete, then for all $p \geq 1$, $\mathring{H}_1^p(M) = H_1^p(M)$.

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Theorem. Let (M,g) be a complete Rie. manifold with positive injectivity radius and let $k\geq 2$ be an integer. Suppose that there exists a positive constant C such that for any $j=0,\dots,k-2$, $|\nabla^j Ric|\leq C$. Then for any $p\geq 1$, $\mathring{H}^p_k(M)=H^p_k(M)$.

proof. $inj_{(M,g)}>0$. $\exists\, C>0$ s.t. $|\nabla^j Ric_{(M,g)}|\leq C$, $\forall j=0,\dots,k-2$

$\Rightarrow$ for any real number $Q>1$ and $\alpha\in(0,1)$, the harmonic radius $r_H=r_H(Q,k-1,\alpha)>0$. fixed $Q=4$. $\alpha=\frac{1}{2}$.

$\Rightarrow$ For any $x\in M$ one then has that there exists a harmonic chart $\phi: B_x(r_H)\to\mathbb{R}^n$ satisfying

1) $4^{-1}\delta_{ij}\leq g_{ij}\leq 4\delta_{ij}$ as bilinear forms

2) $\displaystyle\sum_{1\leq|\beta|\leq k-1} r_H^{|\beta|}\sup_y|\partial_\beta g_{ij}(y)| + \sum_{|\beta|=k-1} r_H^{k-1+\alpha}\sup_{y\neq z}\frac{|\partial_\beta g_{ij}(y)-\partial_\beta g_{ij}(z)|}{d_g(y,z)^\alpha}\leq Q-1$

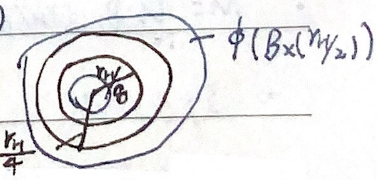
(without loss of generality we can also assume that $\phi(x)=0$).

In particular, we get that for any $r\leq r_H$

$$B_0^e(r/2)\subset\phi(B_x(r))\subset B_0^e(2r).$$

Let $\beta\in D(\mathbb{R}^n)$ be such that $0\leq\beta\leq 1$. $\beta=1$ on $B_0^e(r_H/8)$

$spt\,\beta\subset B_0^e(r_H/4)$



we get that $\beta\circ\phi\in D(M)$ satisfies

$0\leq\beta\circ\phi\leq 1$. $\beta\circ\phi=1$ on $B_x(r_H/16)$, and $spt\,\beta\circ\phi\subset B_x(r_H/2)$

From now on let (x_i) be a sequence of points of M such that $M=\bigcup_{i=1}^{\infty}B_{x_i}(r_H/16)$, and such that the covering $(B_{x_i}(r_H/2))$ is uniformly locally finite. Let $\phi_i : \underline{B_{x_i}(r_H)} \to R^n$ be as above and set $\beta_i = \beta \circ \phi_i$.

m=0 $|\beta_i| \le 1$.

m=1. $|\nabla \beta_i|^2 = \frac{\partial \beta_i}{\partial x^k} g^{kl} \frac{\partial \beta_i}{\partial x^l} = \frac{\partial \beta}{\partial x^k} \circ \phi_i \, g^{kl} \frac{\partial \beta}{\partial x^k} \circ \phi_i \le 4 \cdot \sum_{k=1}^n \left(\frac{\partial \beta}{\partial x^k} \circ \phi_i\right)^2$

m=2 $|\nabla^2 \beta_i|^2 = \langle \frac{\partial^2 \beta_i}{\partial x^k \partial x^l} - \Gamma_{kl}^m \frac{\partial \beta_i}{\partial x^m}, \frac{\partial^2 \beta_i}{\partial x^p \partial x^q} - \Gamma_{pq}^r \frac{\partial \beta_i}{\partial x^r} \rangle$

$= \left(\frac{\partial^2 \beta}{\partial x^k \partial x^l} \circ \phi_i - \underline{\Gamma_{kl}^m} \frac{\partial \beta}{\partial x^m} \circ \phi_i\right) \left(\frac{\partial^2 \beta}{\partial x^p \partial x^q} \circ \phi_i - \Gamma_{pq}^r \frac{\partial \beta}{\partial x^r} \circ \phi_i\right) g^{kp} g^{lq}$

度量系数一阶导的控制

我们有 k-1 阶导的控制 => $|\nabla^m \beta_i|^2$ 有控制.

综上所述. there exists $C > 0$ such that for any i and any $m = 0, \dots, k$ $|\nabla^m \beta_i| \le C$.

Let us now set $\eta_i = \beta_i / \sum_j \beta_j$, then (η_i) is a smooth partition of unity subordinate to the covering $(B_{x_i}(r_H/2))$, since this covering is uniformly locally finite, one easily obtains that there exists some constant $\tilde{C} \ge 1$ such that for any $m = 0, \dots, k$ $\sum_{i=1}^\infty |\nabla^m \eta_i| \le \tilde{C}$.

- $M = \bigcup_{i=1}^\infty B_{x_i}(r_H/16)$ $\beta_i \equiv 1$ on $B_{x_i}(r_H/16)$

=> $\sum_j \beta_j \ge 1$; $(B_{x_i}(r_H/2))$ locally finite => $\sum_j \beta_j$ 为有限和.

=> η_i 有意义.

m=1 $|\nabla \eta_i|^2 = \left(\frac{\beta_i}{\sum_j \beta_j}\right)_k g^{kl} \left(\frac{\beta_i}{\sum_j \beta_j}\right)_l$

$\le 4 \sum_k \left(\frac{\beta_i}{\sum_j \beta_j}\right)_k^2$

$= 4 \sum_k \frac{\left(\frac{\partial \beta}{\partial x^k} \circ \phi_i \sum_j \beta_j - \sum_j \frac{\partial \beta}{\partial x^k} \circ \phi_j \cdot \beta_i\right)^2}{\left(\sum_j \beta_j\right)^4}$

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$\leq 4\cdot 2\left\{\Sigma_k \left(\frac{\partial\beta}{\partial x^k}\circ\phi_i\ \Sigma_j \beta_j\right)^2 + \Sigma_k\left(\Sigma_j \frac{\partial\beta}{\partial x^k}\circ\phi_j\cdot\beta_i\right)^2\right\}$ 有一致界与i无关

+ uniformly locally finite $\Rightarrow \Sigma_i |\nabla^m \eta_i| \leq \tilde{C}$

Now, fixed $u\in C_k^p(M)$ where $p\geq 1$ is some given real number.

idea: given $\varepsilon>0$. find $u_0\in D(M)$ s.t. $\|u_0-u\|_{H_k^p}<\varepsilon$.

$u\in C_k^p(M)$. Let $\Omega\subset M$ be some bounded subset of M such that

$$\sum_{m=0}^{k} C_{k+1}^{m+1}\left(\int_{M\setminus\Omega}|\nabla^m u|^p dv(g)\right)^{1/p} < \varepsilon/\tilde{C}$$

where $$C_{k+1}^{m+1} = \frac{(k+1)!}{(m+1)!\,(k-m)!}$$

Since the covering $(B_{x_i}(r_H/2))$ is uniformly locally finite, one easily obtains that there exist some integer N such that for any $i\geq N+1$, $B_{x_i}(r_H/2)\cap\Omega=\phi$.

Set $u_0=(1-\eta)u$ where $1-\eta=\Sigma_{i=1}^N \eta_i$. Then $u_0\in D(M)$ and

$$\|u-u_0\|_{H_k^p} \leq \sum_{m=0}^{k}\|\nabla^m(\eta u)\|_{L^p}.$$

$$|\nabla^m \eta u| \leq \sum_{j=0}^{m} C_m^j |\nabla^j\eta||\nabla^{m-j}u|$$

and since $\operatorname{supp}\eta\subset\Omega^c$ and $\Sigma_i|\nabla^j\eta_i|\leq\tilde{C}$ for any $j=0,\dots,k$,

we get that

$$\|\nabla^m(\eta u)\|_{L^p} \leq \left(\int_M |\nabla^m(\eta u)|^p dv(g)\right)^{1/p}$$

$$\leq\left(\int_M\left(\Sigma_{j=0}^m C_m^j|\nabla^j\eta||\nabla^{m-j}u|\right)^p dv(g)\right)^{1/p}$$

$$\leq \Sigma_{j=0}^m C_m^j\,\tilde{C}\left(\int_{M\setminus\Omega}|\nabla^j u|^p dv(g)\right)^{1/p}$$

in fact. $j=0$ $|\eta|\leq 1$. $j\geq 1$. $|\nabla^j\eta| = |\Sigma_{i=1}^N \nabla^j\eta_i| \leq \Sigma_{i=1}^N|\nabla^j\eta_i|\leq\tilde{C}$

Note that $\sum_{j=m}^{K} C_j^m = C_{K+1}^{m+1}$ for any $0 \le m \le K$, we get that

$$\|u-u_0\|_{H_k^p(M)} \le \tilde{C} \sum_{m=0}^{K} \sum_{j=0}^{m} C_m^j \left(\int_{M\setminus\Omega} |\nabla^j u|^p dv(g) \right)^{1/p}$$

$$= \tilde{C} \sum_{m=0}^{K} \left(\sum_{j=m}^{K} C_j^m \right) \left(\int_{M\setminus\Omega} |\nabla^m u|^p dv(g) \right)^{1/p}$$

$$= \tilde{C} \sum_{m=0}^{K} C_{K+1}^{m+1} \left(\int_{M\setminus\Omega} |\nabla^m u|^p dv(g) \right)^{1/p} < \varepsilon$$ #

prop. Let (M,g) be a complete Rie. manifold whose Ricci curvature is bounded from below and whose injectivity radius is positive. Then $\mathring{H}_2^2(M) = H_2^2(M)$.

proof. Let $K_2^2(M)$ be the completion of

$\tilde{C}_2^2(M) = \{u \in C^\infty(M) : u, |\nabla u|, \Delta_g u \in L^2(M)\}$

w.r.t. $\|u\|_{K_2^2(M)} = (\int_M u^2 dv(g))^{1/2} + (\int_M |\nabla u|^2 dv(g))^{1/2} + (\int_M |\Delta_g u|^2 dv(g))^{1/2}$

and Let $\mathring{K}_2^2(M)$ be the closure of $D(M)$ in $K_2^2(M)$.

$Ric \ge \lambda g$, $inj_{(M,g)} > 0$

$\Rightarrow$ for any real numbers $Q > 1$ and $\alpha \in (0,1)$, the harmonic radius $r_H = r_H(Q, 0, \alpha) > 0$. Noting that in a harmonic coordinate chart, $\Delta_g u = -g^{ij}\partial_{ij}u$ for any $u \in C^\infty(M)$.

similar argument to those used in the proof of the above theorem prove that $\mathring{K}_2^2(M) = K_2^2(M)$.

Independently, one clearly has that for any $u \in C^\infty(M)$, $|\Delta_g u|^2 \le n |\nabla^2 u|^2$. Hence $H_2^2(M) \subset K_2^2(M)$ and the embedding is continuous.

Finally, by the Bocher Lichnerowicz-Weitzenböck formula,

for any $u \in D(M)$ $\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 + \langle D(\Delta u), Du \rangle + Ric(\nabla u, \nabla u)$

$\int_M |\nabla^2 u|^2 = \int_M |\Delta_g u|^2 dv(g) - \int_M Ric(\nabla u, \nabla u) dv(g)$

$\leq \int_M |\Delta_g u|^2 dv(g) + \int_M |\lambda| \, |\nabla u|^2 dv(g)$

$\Rightarrow \|u\|_{H^2_2} \leq (1 + \sqrt{|\lambda|}) \|u\|_{K^2_2}$ for any $u \in D(M)$.

$\Rightarrow \mathring{H}^2_2(M) = \mathring{K}^2_2(M)$

$\mathring{H}^2_2(M) \subset H^2_2(M) \subset K^2_2(M) = \mathring{K}^2_2(M) = \mathring{H}^2_2(M)$ #

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Sobolev embeddings:

(*) For p, q two real numbers with $1 \le q < p$, and k, m two integers with $0 \le m < k$, if $1/p = 1/q - (k-m)/n$ then $H_k^q(M) \subset H_m^p(M)$.

Lemma 3.1 Let (M^n, g) be complete Rie. manifold. Suppose that the embedding $H_1^1(M) \subset L^{\frac{n}{n-1}}(M)$ is valid, then (*) are valid.

Remark. take $k = q = 1$, $m = 0$, in the (*), we have

$$H_1^1(M) \subset L^{\frac{n}{n-1}}(M).$$

proof. 我们证 $1 < q < n$ $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ $H_1^q(M) \subset L^p(M)$.

由此容易得到 $H_k^q(M) \subset H_m^p(M)$.

in fact, if $k=2$, $m=1$ $1 \le q < p$. $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$

当然有 $H_1^q(M) \subset L^p(M)$ i.e. $\exists\, C > 0$ s.t.

$$\left(\int_M u^p \, dv(g)\right)^{1/p} \le C\left\{\left(\int_M u^q \, dv(g)\right)^{1/q} + \left(\int_M |Du|^q \, dv(g)\right)^{1/q}\right\}$$

$$\left(\int_M u^p dv(g)\right)^{1/p} + \left(\int_M |Du|^p dv(g)\right)^{1/p}$$

$$\le C\left(\int_M u^q dv(g)\right)^{1/q} + C\left(\int_M |Du|^q dv(g)\right)^{1/q}$$

$$+ C \int_M |Du|^q dv(g) + C\left(\int_M |D|Du||^q dv(g)\right)^{1/q}$$

$$\le \quad \cdots$$

$$+ \quad \cdots \quad + C\left(\int_M |D^2 u|^q dv(g)\right)^{1/q}$$

$\Rightarrow H_2^q(M) \subset H_1^p(M)$

if $k=2$, $m=0$, $1 \le q < p$ $\frac{1}{p} = \frac{1}{q} - \frac{2}{n}$.

$\Rightarrow \frac{1}{p}+\frac{1}{n}=\frac{1}{q}-\frac{1}{n}:=\frac{1}{p'}$ $\quad 1\le q<p'<p$

$\Rightarrow H_2^q(M)\subset H_1^{p'}(M)$ $\quad$ by $\frac{1}{p'}-\frac{1}{n}=\frac{1}{p}$

$\Rightarrow H_2^q(M)\subset H_1^{p'}(M)\subset L^p(M)$.

Let $A\in\mathbb{R}$ be such that for any $u\in H_1^1(M)$

$$\left(\int_M u^{\frac{n}{n-1}}dv(g)\right)^{\frac{n-1}{n}}\le A\int_M(|\nabla u|+|u|)\,dv(g).$$

Let $1\le q<n$, $1/p=1/q-1/n$, and $u\in D(M)$.

$$\left(\int_M |u|^p dv(g)\right)^{\frac{n-1}{n}}=\left(\int_M |u|^{p\frac{n-1}{n}\frac{n}{n-1}}dv(g)\right)^{\frac{n-1}{n}}$$

$$\le A\int_M\left(|\nabla |u|^{\frac{p(n-1)}{n}}|+|u|^{\frac{p(n-1)}{n}}\right)dv(g)$$

$$=A\int_M\frac{p(n-1)}{n}|u|^{\frac{pn-p-n}{n}}|\nabla u|\,dv(g)+A\int_M|u|^{\frac{p(n-1)}{n}}dv(g)$$

$$\le\frac{Ap(n-1)}{n}\left\{\left(\int_M|u|^{\frac{pn-p-n}{n}q'}dv(g)\right)^{1/q'}\cdot\left(\int_M|\nabla u|^q dv(g)\right)^{1/q}\right\}$$

$$+A\left\{\left(\int_M|u|^{\frac{pn-p-n}{n}q'}dv(g)\right)^{1/q'}\cdot\left(\int_M|u|^q dv(g)\right)^{1/q}\right\}$$

$$=\frac{Ap(n-1)}{n}\left(\int_M|u|^p dv(g)\right)^{1/q'}\left(\int_M|\nabla u|^q dv(g)\right)^{1/q}$$

$$+A\left(\int_M|u|^p dv(g)\right)^{1/q'}\left(\int_M|\nabla u|^q dv(g)\right)^{1/q}$$

$\frac{1}{q'}+\frac{1}{q}=1$ $\quad \frac{1}{p}=\frac{1}{q}-\frac{1}{n}=1-\frac{1}{q'}-\frac{1}{n}$

$$\Rightarrow\left(\int_M|u|^p dv(g)\right)^{1/p}\le\frac{Ap(n-1)}{n}\left(\int_M|\nabla u|^q dv(g)\right)^{1/q}+A\left(\int_M|\nabla u|^q dv(g)\right)^{1/q}\quad\#$$

Remark: $H_1^1(M)\subset L^{\frac{n}{n-1}}(M)\Leftrightarrow H_1^q(M)\subset L^p(M)$ where $1<q<n$. $\frac{1}{p}=\frac{1}{q}-\frac{1}{n}$

in fact, for complete manifolds with Ricci curvature bounded from below, 则 "$\Rightarrow$" 之逆成立. for some $q\in(1,n)$.

Lemma Let (M^n, g) be a complete Rie. manifold. Suppose that the embedding $H_1^q(M) \subset L^p(M)$ is valid for some $1 \le q < n$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$. Then for any $r > 0$ there exists a positive constant $v = v(M, q, r)$ such that for any $x \in M$, $Vol_g(B_x(r)) \ge v$.

proof. · By hypothesis. Let $A > 0$ be such that for any $u \in H_1^q(M)$,

$$\left(\int_M |u|^p dv(g)\right)^{1/p} \le A \left\{ \left(\int_M |\nabla u|^q dv(g)\right)^{1/q} + \left(\int_M |u|^q dv(g)\right)^{1/q} \right\}$$

Let $r > 0$. Let x be some point of M, and let $v \in H_1^q(M)$ be such that $v = 0$ on $M \setminus B_x(r)$. $\|v\|_{H_1^q(M)} \neq 0$.

By Hölder, $\left(\int_M v^q dv(g)\right)^{1/q} \le Vol_g(B_x(r))^{1/n} \left(\int_M v^p dv(g)\right)^{1/p}$

$$\le Vol_g(B_x(r))^{1/n} A \left\{ \left(\int_M |\nabla v|^q\right)^{1/q} + \left(\int_M v^q\right)^{1/q} \right\}$$

多项式

$$\Rightarrow \frac{1}{Vol_g(B_x(r))^{1/n}} - A \le A \frac{\left(\int_M |\nabla v|^q\right)^{1/q}}{\left(\int_M v^q\right)^{1/q}}$$

Fix $x \in M$ and let $R > 0$ be given. Then, either $Vol_g(B_x(R)) > \left(\frac{1}{2A}\right)^n$

either $Vol_g(B_x(R)) \le \left(\frac{1}{2A}\right)^n$

想法：迭代 不希望是多项式. → 单项式

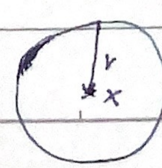
$$Vol_g(B_x(r))^{-\frac{1}{n}} - A \ge \frac{1}{2} Vol_g(B_x(r))^{-\frac{1}{n}}$$

$\iff Vol_g(B_x(r)) \le (2A)^{-n}$ $Vol_g(B_x(r)) > (2A)^{-n}$ 无须细致讨论

$$\frac{1}{2} Vol_g(B_x(r))^{-\frac{1}{n}} \le A \frac{\left(\int_M |\nabla v|^q\right)^{1/q}}{\left(\int_M v^q\right)^{1/q}}$$

$$\Rightarrow \frac{1}{(2A)^q} Vol_g(B_x(r))^{-\frac{q}{n}} \le \frac{\int_M |\nabla v|^q dv(g)}{\int_M v^q dv(g)}$$

From now on. Let



$v(y) = r - d_g(x, y)$ if $d_g(x, y) \le r$

$v(y) = 0$ if $d_g(x, y) \ge r$.

Clearly v is Lipschitzian and $v=0$ on $M \setminus B_x(r)$. Hence, $v \in H_1^q(M)$. As a consequence,

$$\frac{1}{(2A)^q} \mathrm{Vol}_g(B_x(r))^{-\frac{q}{n}} \le \frac{\mathrm{Vol}_g(B_x(r))}{\int_{B_x(r/2)} v^q \, dv(g)} \le \left(\frac{2}{r}\right)^q \frac{\mathrm{Vol}_g(B_x(r))}{\mathrm{Vol}_g(B_x(r/2))}$$

and we get that for any $r \le R$,

$$\mathrm{Vol}_g(B_x(r)) \ge \left(\frac{r}{4A}\right)^{\frac{nq}{n+q}} \mathrm{Vol}_g(B_x(r/2))^{\frac{n}{n+q}} . \quad (*)$$

By induction we then get that for any $m \in \mathbb{N}$,

$$\mathrm{Vol}_g(B_x(R)) \ge \left(\frac{R}{2A}\right)^{q\alpha(m)} \left(\frac{1}{2}\right)^{q\beta(m)} \mathrm{Vol}_g(B_x(R/2^m))^{\gamma(m)} \quad (1)$$

where $\alpha(m) = \sum_{i=1}^{m} \left(\frac{n}{n+q}\right)^i$, $\beta(m) = \sum_{i=1}^{m} i \left(\frac{n}{n+q}\right)^i$, $\gamma(m) = \left(\frac{n}{n+q}\right)^m$

But, we have $\lim_{r \to 0^+} \frac{\mathrm{Vol}_g(B_x(r))}{b_n r^n} = 1$

where b_n is the volume of the euclidean ball of radius one.

Hence, $\lim_{m\to\infty} \mathrm{Vol}_g(B_x(R/2^m))^{\gamma(m)} = 1$

in fact, $\lim_{m\to\infty} \gamma(m) \log \mathrm{Vol}_g(B_x(R/2^m))$

$$= \lim_{m\to\infty} \left(\frac{n}{n+q}\right)^m \log \mathrm{Vol}_g(B_x(R/2^m))$$

$$= \lim_{m\to\infty} \left(\frac{n}{n+q}\right)^m \log b_n \frac{R^n}{2^{mn}}$$

$$= \lim_{m\to\infty} \left(\frac{n}{n+q}\right)^m m(-\log 2^n)$$

$$= 0$$

In addition, we have that $\sum_{i=1}^{\infty} \left(\frac{n}{n+q}\right)^i = \frac{n}{q}$

$$\sum_{i=1}^{\infty} i \left(\frac{n}{n+q}\right)^i = \frac{n(n+q)}{q^2} .$$

As a consequence, letting $m \to \infty$ in (1), $\mathrm{Vol}_g(B_x(R)) \ge \left(\frac{1}{2^{(n+2q)/q} A}\right)^n R^n$ #

The sobolev embedding theorem for $\mathbb{R}^n$.

Lemma For any $u \in D(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \le \frac{1}{2} \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial x^i}\right| dx\right)^{1/n} \le \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u| dx$$

Theorem Let $q \in [1, n)$ and let p be such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$. For any $u \in H_1^q(\mathbb{R}^n)$, $\left(\int_{\mathbb{R}^n} |u|^p dx\right)^{1/p} \le \frac{p(n-1)}{2n} \left(\int_{\mathbb{R}^n} |\nabla u|^q\right)^{1/q}$.

Sobolev Embeddings for compact manifolds.

1. the Sobolev embeddings are valid for compact manifolds.
2. Rellich - Kondrakov theorem
3. Poincaré and Sobolev - Poincaré inequalities.

Theorem: Let (M^n, g) be a compact Rie. manifold. (*) are valid.

证 $H_1^1(M) \subset L^{\frac{n}{n-1}}(M)$

proof. M compact ∴ M can be covered by a finite number of charts $(\Omega_m, \phi_m)_{m=1,\dots,N}$ such that for any m the components g_{ij} of g in (Ω_m, ϕ_m) satisfies

$\phi_m : \Omega_m \to \mathbb{R}^n$ local chart

$$2^{-1} \delta_{ij} \le g_{ij} \le 2 \delta_{ij} \quad \text{as bilinear forms}$$

Let (η_m) be a smooth partition of unity subordinate to the covering (Ω_m). For any $u \in C^\infty(M)$ and any m, one then has that

$$\int_M |\eta_m u|^{\frac{n}{n-1}} dv(g) \le 2^{n/2} \int_{\mathbb{R}^n} |(\eta_m u) \circ \phi_m^{-1}|^{\frac{n}{n-1}} dx$$

and $\int_M |D(\eta_m u)|\, dv(g) \geq 2^{-\frac{n+1}{2}} \int_{R^n} |D((\eta_m u)\circ \phi_m^{-1})(x)|\, dx$

By above theorem, we have

$\left(\int_{R^n} |(\eta_m u)\circ\phi_m^{-1}|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq \frac{1}{2} \cdot \int_{R^n} |D((\eta_m u)\circ\phi_m^{-1})|$ for any m.

$$\left(\int_M |u|^{\frac{n}{n-1}} dv(g)\right)^{\frac{n-1}{n}} \leq \sum_{m=1}^{N} \left(\int_M |\eta_m u|^{\frac{n}{n-1}} dv(g)\right)^{\frac{n-1}{n}}$$
$$\leq 2^{n-1} \sum_{m=1}^{N} \int_M |D(\eta_m u)|\, dv(g)$$
$$\leq 2^{n-1} \sum_{m=1}^{N} \int_M (|D\eta_m||u| + \eta_m |Du|)\, dv(g)$$
$$\leq 2^{n-1} \int_M |Du|\, dv(g) + 2^{n-1} \left(\max_M \sum_{m=1}^{N} |D\eta_m|\right) \int_M |u|\, dv(g) \quad \#$$

Let us now discuss the Rellich-Kondrakov theorem.

If (M, g) is compact. $Vol_{(M,g)}$ is finite.

Hence, we get that for any integers $j \geq 0$ and $m \geq 1$, any $q \geq 1$ real, and any p real such that $1 \leq p \leq \frac{nq}{n-mq}$.

$H^q_{j+m}(M) \subset H^p_j(M)$.

The Rellich-Kondrakov theorem then asserts that these embeddings are compact provided $1 \leq p < \frac{nq}{n-mq}$.

proof. one can prove that for any bounded domain Ω of R^n, any $1 \leq q < n$, and any $1 \leq p < nq/(n-q)$, the embedding of $\mathring{H}^q_1(\Omega)$ in $L^p(\Omega)$ is compact.

claim $\mathring{H}^q_m(\Omega) \subset\subset L^p(\Omega)$ $\qquad \frac{1}{p} > \frac{1}{q} - \frac{m}{n} = \underbrace{\frac{1}{q} - \frac{m-1}{n}}_{m\geq 2} - \frac{1}{n}$

by the sobolev embeddings, $\mathring{H}^q_m(\Omega) \subset \mathring{H}^{q'}_m(\Omega)$ $\qquad = \frac{1}{q'} - \frac{1}{n}$

by the above discuss. $\mathring{H}^{q'}_m(\Omega) \subset\subset L^p(\Omega) \Rightarrow \mathring{H}^q_m(\Omega) \subset\subset L^p(\Omega)$.

归纳

$\Rightarrow \quad \mathring{H}^q_{j+m}(\Omega) \subset\subset \mathring{H}^p_j(\Omega)$ 关键 $\mathring{H}^q_1(\Omega) \subset\subset L^p(\Omega)$

Back to compact manifold.

· prove that $H^q_1(M) \subset\subset L^p(M)$

· similarly. $H^q_{j+m}(M) \subset\subset H^p_j(M)$ #

Let us now say some words about the Poincaré and Sobolev-Poincaré inequalities.

Lemma (Poincaré inequalities).

Let (M,g) be a compact Rie. manifold, and let $1 \le q < n$ be a real number. There exists a positive constant $A = A(M,g,q)$ such that for any $u \in H^q_1(M)$,

$$\left(\int_M |u-\bar{u}|^q dv(g)\right)^{1/q} \le A \left(\int_M |\nabla u|^q dv(g)\right)^{1/q}$$

where $\bar{u} = \frac{1}{vol(M,g)} \int_M u \, dv(g)$.

思想：密分法

proof. ①. $q>1$ 只要证 $\inf_{u \in \mathcal{H}} \int_M |\nabla u|^q dv(g) > 0$

where $\mathcal{H} = \{u \in H^q_1(M) : \int_M u^q dv(g) = 1 \text{ and } \int_M u \, dv(g) = 0\}$

取极小化序列 $\{u_k\}$ s.t. $\lim_k \int_M |\nabla u_k|^q dv(g) = \inf_{u \in \mathcal{H}} \int_M |\nabla u|^q dv(g)$

$H^q_1(M)$ is reflexive for $q>1$. ⊛

$\Rightarrow$ $\{u_k\}$ 有子列 在 $H^q_1(M)$ 中弱收敛

在 $L^q(M) \cap L^1(M)$ 中强收敛 仍记为 $\{u_k\}$

Let v be its limit $\Rightarrow v \in \mathcal{H}$

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弱收敛性 + $L^p(M)$ 中强收敛.

$$\Rightarrow \int_M |\nabla v|^q dv(g) \le \varliminf_{k\to\infty} \int_M |\nabla u_k|^q dv(g)$$

$$= \lim_{k\to\infty} \int_M |\nabla u_k|^q dv(g)$$

$$= \inf_{u\in\mathcal{H}} \int_M |\nabla u|^q dv(g).$$

$\Rightarrow$ v 实现了下确界. $v\in\mathcal{H}$ v 不能为常值函数.

$\Rightarrow \inf_{u\in\mathcal{H}} \int_M |\nabla u|^q dv(g) > 0$ 证明中用到 $H_1^q(M)$, $q>1$ 的自反性!

② $q=1$ 怎么呢?

利用事实: 紧流形上存在 Laplacian 的格林函数.

即 $\exists\, G: M\times M \longrightarrow \mathbb{R}$ s.t.

(1) $\forall u\in C^\infty(M)$, $x\in M$

$$u(x) = \frac{1}{\mathrm{vol}(M,g)} \int_M u\, dv(g) + \int_M G(x,y)\, \Delta_g u(y)\, dv_g(y).$$

(2) $G(x,y) = G(y,x)$ and $G(x,y)$ is C^∞ on $M\times M \setminus \Delta$

where Δ is the diagonal $\Delta = \{(x,y)\in M\times M : x=y\}$

(3) $\exists K>0$ s.t. $\forall (x,y)\in M\times M\setminus\Delta$

$$|G(x,y)| \le \frac{K}{r^{n-2}} \quad \text{and} \quad |D_y G(x,y)| \le \frac{K}{r^{n-1}}.$$

where $r = d_g(x,y)$.

From now on let $u\in C_1^q(M)$ be such that $\int_M u\, dv(g) = 0$

$\forall x\in M$, $u(x) = \int_M G(x,y)\, \Delta_g u(y)\, dv_g(y)$

Hence, $|u(x)| \le \int_M |D_y G(x,y)|\ |\nabla u(y)|\, dv_g(y)$

and we have

$\int_M |u(x)| dv_g(x) \le \int_M \int_M |\nabla_y G(x,y)| \, |\nabla u(y)| \, dv_g(x) \, dv_g(y)$

$\le \int_M \underbrace{\int_M \frac{K}{r^{n-1}} dv_g(x)}_{} |\nabla u(y)| \, dv_g(y)$

$\le C \cdot \int_M |\nabla u(y)| \, dv_g(y)$

As a consequence, for any $u \in C_1^q(M)$ & $\int_M u \, dv(g) = 0$

$\int_M |u| \, dv(g) \le C \cdot \int_M |\nabla u| \, dv(g)$ #

prop (Sobolev-Poincaré inequalities).

Let (M,g) be a compact Rie. manifold, and Let p, q two real numbers such that $1 \le q < n$ and $1/p = 1/q - 1/n$. There exists a positive constant $A = A(M, g, q)$ s.t for any $u \in H_1^q(M)$.

$$\left(\int_M |u-\bar{u}|^p dv(g)\right)^{1/p} \le A \left(\int_M |\nabla u|^q dv(g)\right)^{1/q}.$$

where $\bar{u} = \frac{1}{vol(M,g)} \int_M u \, dv(g)$.

proof. $H_1^q(M) \subset L^p(M) \Leftarrow 1 \le q < n,\ \frac{1}{p} = \frac{1}{q} - \frac{1}{n}$

$\left(\int_M |u-\bar{u}|^p dv(g)\right)^{1/p} \le B \left(\int_M |u-\bar{u}|^q dv(g)\right)^{1/q} + B \left(\int_M |\nabla u|^q dv(g)\right)^{1/q}$

$\le B \left(A \int_M |\nabla u|^q dv(g)\right)^{1/q} + B \left(\int_M |\nabla u|^q dv(g)\right)^{1/q}$ #