

## Sobolev embeddings for complete manifolds

\* the existence of Sobolev embeddings for complete manifolds

We restrict ourselves to the study of the embeddings  $H_0^1 \subset L^p$ ,  $1 \leq q < n$ ,  $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ .

First we will see that, contrary to the case of compact manifolds, there exist complete manifolds for which the Sobolev embeddings are false. The existence of such manifolds is based on lemma 3.2.

**Lemma 3.2\*** Let  $(M, g)$  be a complete Riemannian  $n$ -manifold. Suppose that the embedding  $H_0^1(M) \subset L^p(M)$  is valid for some  $1 \leq q < n$  and  $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ . Then for any  $r > 0$  there exists a positive constant  $v = v(M, g, r)$  such that for any  $x \in M$ ,  $\text{Vol}(B_x(r)) \geq v$ .

On the other hand, we will see that the question of the existence of Sobolev embeddings for complete manifolds with Ricci curvature bounded from below is completely settled. For such manifolds the Sobolev embeddings are valid if and only if there is a uniform lower bound with respect to the center point  $x$  for the volume of balls  $B_x(r)$ ,  $r > 0$  arbitrary.

**prop\*** For any integer  $n \geq 3$ , there exist complete  $n$ -manifolds  $(M, g)$  for which the whole scale of the Sobolev embeddings  $H_0^1 \subset L^p$  is false, namely for which for any  $1 \leq q < n$  and  $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ ,  $H_0^1(M) \not\subset L^p(M)$ .

**Theorem** Let  $(M, g)$  be a complete Riemannian  $n$ -manifold. Suppose that the Ricci curvature of  $(M, g)$  is bounded from below and that  $\inf_{x \in M} \text{Vol}(B_x(r)) > 0$ . Then the Sobolev embeddings are valid for  $M$ .

**Remark:** Before we start with the proof of theorem, note that by Gromov's result the assumption  $\inf_{x \in M} \text{Vol}(B_x(r)) > 0$  implies that for any  $r > 0$  there exists a positive constant  $v_r$  such that for any  $x \in M$ ,  $\text{Vol}(B_x(r)) \geq v_r$ .

We just have to prove that  $H_0^1(M) \subset L^{\frac{n}{n-1}}(M)$ .

The first step of the proof of theorem we present is the following lemma of Coulhon and Saloff-Coste.

**Lemma** Let  $(M, g)$  be a complete Riemannian  $n$ -manifold such that  $\text{Ric} \geq kg$  for some  $k \in \mathbb{R}$ , and let  $R > 0$ . There exists a positive constant  $C = C(n, k, R)$ , such that for any  $r \in (0, R)$ , and any  $u \in D(M)$ ,

$$\int_M |\underline{u} - \bar{u}_r| dv(g) \leq C \cdot r \int_M |\nabla u|^2 dv(g) \quad \text{global result.}$$

two functions

where  $\bar{u}_r(x) = \frac{1}{\text{Vol}(B_x(r))} \int_{B_x(r)} u dv(g)$ ,  $x \in M$ .

**proof.** Let  $(M, g)$  be a complete Riemannian  $n$ -manifold such that  $\text{Ric} \geq kg$  for some  $k \in \mathbb{R}$ , and let  $R > 0$ . By the work of Buser, there exists a positive constant  $C = C(n, k, R)$ , such that for any  $x \in M$ , any  $r \in (0, 2R)$ , and any  $u \in C^0(B_x(r))$ ,

$$\int_{B_x(r)} |u - \bar{u}_r(x)| dv(g) \leq C \cdot r \int_{B_x(r)} |\nabla u|^2 dv(g) \quad * \quad \text{暂时承认.}$$

function      constant.      local result.

packing lemma  
local result  $\rightarrow$  global result.

Recall the packing lemma: Let  $(M, g)$  be a complete Riemannian  $n$ -manifold with Ricci curvature bounded from below by some  $k \in \mathbb{R}$ , and let  $p > 0$  be given. There exists a sequence  $(x_i)$  of points of  $M$  such that for any  $r \geq p$ :

(i) the family  $(B_{x_i}(r))$  is a uniformly locally finite covering of  $M$ , and there is an upper bound for  $N$  in terms of  $n, p, r$  and  $k$ . Moreover,  $\text{Card} \{i \text{ s.t. } x \in B_{x_i}(r)\} \leq \left(\frac{8r}{p}\right)^n e^{4\sqrt{(n-1)|k|} \cdot r}$ .

(ii) for any  $i \neq j$ ,  $B_{x_i}(p/2) \cap B_{x_j}(p/2) = \emptyset$ .

Let  $r \in (0, R)$  be given and let  $(x_i)_{i \in I}$  be a sequence of points of  $M$  such that simultaneously  $M = \bigcup_{i \in I} B_{x_i}(r)$  and  $\text{Card} \{i \in I \text{ s.t. } x \in B_{x_i}(2r)\} \leq 16^n e^{8\sqrt{(n-1)|k|} \cdot r} \leq 16^n e^{8\sqrt{(n-1)|k|} \cdot R} := N = N(n, k, R)$ .

Let  $u \in \mathcal{D}(M)$ . We have

$$\begin{aligned} \int_M |u - \bar{u}_r| \, dV(g) &\leq \sum_i \int_{B_{x_i}(r)} |u - \bar{u}_r| \, dV(g) \\ &\stackrel{\text{global}}{\leq} \sum_i \int_{B_{x_i}(r)} |u - \bar{u}_r(x_i)| \, dV(g) + \sum_i \int_{B_{x_i}(r)} |\bar{u}_r(x_i) - \bar{u}_{2r}(x_i)| \, dV(g) \\ &\stackrel{\text{local}}{\leq} \sum_i \int_{B_{x_i}(r)} |u - \bar{u}_r(x_i)| \, dV(g) + \sum_i \int_{B_{x_i}(r)} |\bar{u}_r(x_i) - \bar{u}_{2r}(x_i)| \, dV(g) \\ &\quad + \sum_i \int_{B_{x_i}(r)} |\bar{u}_r - \bar{u}_{2r}(x_i)| \, dV(g) \end{aligned}$$

*Handwritten notes:*  
 -  $\bar{u}_r(x_i) - \bar{u}_{2r}(x_i)$  is constant.  
 -  $x \in B_{x_i}(r) \rightarrow B_{x_i}(r) \subset B_{x_i}(2r)$   
 -  $B_{x_i}(2r)$  is the integral on  $B_{x_i}(r)$ .  
 - 找个更大的球过渡一下, 方便做估计.  
 - 拆分成: 凑出这一项用 (\*)

By \*, we get that

$$\sum_i \int_{B_{x_i}(r)} |u - \bar{u}_r(x_i)| \, dV(g) \leq C \cdot \sum_i r \int_{B_{x_i}(2r)} |Du| \, dV(g) \leq N \cdot C \cdot r \int_M |Du| \, dV(g)$$

$$\begin{aligned} \text{while } \sum_i \int_{B_{x_i}(r)} |\bar{u}_r(x_i) - \bar{u}_{2r}(x_i)| \, dV(g) &= \sum_i \int_{B_{x_i}(r)} |\bar{u}_r(x_i) - \bar{u}_{2r}(x_i)| \, dV(g) \\ &= \sum_i \int_{B_{x_i}(r)} \frac{1}{\text{Vol}(B_{x_i}(r))} \int_{B_{x_i}(r)} (u(x) - \bar{u}_{2r}(x_i)) \, dV(g) \\ &\leq \sum_i \int_{B_{x_i}(r)} |u(x) - \bar{u}_{2r}(x_i)| \, dV(g) \\ &\leq \sum_i C \cdot 2r \cdot \int_{B_{x_i}(2r)} |Du| \, dV(g) \leq 2CrN \int_M |Du| \, dV(g). \end{aligned}$$

*Handwritten notes:*  
 - 展开  
 - 放大成  $B_{x_i}(2r)$  用 (\*)

Independently, we have

$$\begin{aligned} \sum_i \int_{B_{x_i}(r)} |\bar{u}_r - \bar{u}_{2r}(x_i)| \, dV(g) &\leq \sum_i \int_{x \in B_{x_i}(r)} \left\{ \frac{1}{\text{Vol}(B_{x_i}(r))} \int_{y \in B_{x_i}(r)} |u(y) - \bar{u}_{2r}(x_i)| \, dV(g(y)) \right\} dV(g(x)) \\ &= \sum_i \int_{x \in B_{x_i}(r)} \frac{1}{\text{Vol}(B_{x_i}(r))} \int_{B_{x_i}(2r)} |u(y) - \bar{u}_{2r}(x_i)| \, dV(g(y)) \, dV(g(x)) \\ &\leq \sum_i \int_{B_{x_i}(2r)} |u(y) - \bar{u}_{2r}(x_i)| \, dV(g(y)) \int_{B_{x_i}(r)} \frac{1}{\text{Vol}(B_{x_i}(r))} \, dV(g(x)) \\ &\leq \sum_i 2Cr \int_{B_{x_i}(2r)} |Du| \, dV(g) \int_{B_{x_i}(r)} \frac{1}{\text{Vol}(B_{x_i}(r))} \, dV(g(x)) \end{aligned}$$

while, by Gromov's result,

$$\frac{1}{\text{Vol}(B_{x_i}(r))} \leq \frac{K}{\text{Vol}(B_{x_i}(2r))}$$

where  $K = K(n, k, R) = 2^n e^{2\sqrt{(n-1)|k|} \cdot R}$ . Since  $x \in B_{x_i}(r)$  implies that  $B_{x_i}(r)$  is a subset of  $B_{x_i}(2r)$ ,

we get that

$$\int_{B_{x_i}(r)} \frac{1}{\text{Vol}(B_{x_i}(r))} \, dV(g(x)) \leq \int_{B_{x_i}(r)} \frac{K}{\text{Vol}(B_{x_i}(2r))} \, dV(g(x)) \leq \int_{B_{x_i}(r)} \frac{K}{\text{Vol}(B_{x_i}(r))} \, dV(g(x)) = K.$$

$$\int_{\Omega} \int_{B_{x_i}(r)} |\bar{u}_r - \bar{u}_{2r}(x_i)| \, d\text{Vol}_g \leq k \geq c r \cdot N \int_{\Omega} |\text{Vol}_g| \, d\text{Vol}_g$$

and for any  $u \in D(u)$ ,

$$\int_{\Omega} |u - \bar{u}_r| \, d\text{Vol}_g \leq 3(1+k)Ncr \int_{\Omega} |\text{Vol}_g| \, d\text{Vol}_g$$

This ends the proof of the Lemma.

Let us now prove the following.

**Lemma** Let  $(M, g)$  be a complete Riemannian  $n$ -manifold. Suppose that its Ricci curvature satisfies  $\text{Ric} \geq kg$  for some  $k \in \mathbb{R}$  and suppose that there exists  $v > 0$  such that  $\text{Vol}_g(B_{x_i}(1)) \geq v$  for any  $x \in M$ . There exist two positive constants  $C = C(n, k, v)$  and  $\eta = \eta(n, k, v)$ , such that for any open subset  $\Omega$  of  $M$  with smooth boundary and compact closure, if  $\text{Vol}_g(\Omega) \leq \eta$ , then  $\text{Vol}_g(\Omega)^{\frac{n-1}{n}} \leq C \cdot \text{Area}_g(\partial\Omega)$ .

关于小体积区域的结果 证明很“分析” 强!!!

**proof.** By Gromov's result, for any  $x \in M$  and any  $0 < r < R$ ,

$$\text{Vol}_g(B_{x_i}(r)) \geq \left\{ \frac{1}{R^n} e^{-\sqrt{(n-1)|k|} \cdot R} \text{Vol}_g(B_{x_i}(R)) \right\} r^n$$

Fix  $R=1$ . Then we get that for any  $x \in M$  and any  $r \in (0, 1)$ ,  $\text{Vol}_g(B_{x_i}(r)) \geq e^{-\sqrt{(n-1)|k|}} v r^n$

Set  $\eta = \frac{1}{16} e^{-\sqrt{(n-1)|k|}} v$  and  $c_1 = e^{-\sqrt{(n-1)|k|}} v$ . Let  $\Omega$  be some open subset of  $M$  with smooth boundary,

compact closure, and such that  $\text{Vol}_g(\Omega) \leq \eta$ . For sufficiently small  $\varepsilon > 0$  consider the function

$$u_\varepsilon(x) = 1 \text{ if } x \in \Omega$$

$$u_\varepsilon(x) = 1 - \frac{1}{2} d_g(x, \partial\Omega) \text{ if } x \in \Omega^\varepsilon \text{ and } d_g(x, \partial\Omega) \leq \varepsilon$$

$$u_\varepsilon(x) = 0 \text{ if } x \in \Omega^\varepsilon \text{ and } d_g(x, \partial\Omega) \geq \varepsilon.$$

Then  $u_\varepsilon$  is Lipschitz for every  $\varepsilon$  and one easily sees that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon \, d\text{Vol}_g = \text{Vol}_g(\Omega)$$

$$\text{and } |Du_\varepsilon| \leq \frac{1}{2} \text{ if } x \in \Omega^\varepsilon \text{ and } d_g(x, \partial\Omega) < \varepsilon$$

$$|Du_\varepsilon| = 0 \text{ otherwise. 比较}$$

which implies that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |Du_\varepsilon| \, d\text{Vol}_g = \text{Area}_g(\partial\Omega).$$

Furthermore, for every  $\varepsilon > 0$ ,  $\text{Vol}_g(\Omega) = \text{Vol}_g(\{x \in M : u_\varepsilon(x) \geq \frac{1}{2}\})$ , and for any  $\varepsilon > 0$  and any  $r > 0$ ,

$$\text{Vol}_g(\{x \in M : u_\varepsilon(x) \geq \frac{1}{2}\}) = \text{Vol}_g(\{x \in M : |u_\varepsilon(x) - \bar{u}_{\varepsilon, r}(x)| \geq \frac{1}{2}\}) + \text{Vol}_g(\{x \in M : \bar{u}_{\varepsilon, r}(x) \geq \frac{1}{2}\})$$

估计. 策略: 想办法为含  $r$  一项. (特别选取  $r$ )

where  $\bar{u}_{\varepsilon, r}(x) = \frac{1}{\text{Vol}_g(B_{x_i}(r))} \int_{B_{x_i}(r)} u_\varepsilon \, d\text{Vol}_g$ . Now, note that for  $r > 0$  and  $\varepsilon \ll 1$ ,  $\bar{u}_{\varepsilon, r}(x) \leq \frac{2 \text{Vol}_g(\Omega)}{\text{Vol}_g(B_{x_i}(r))}$ .

Fix  $r = \left(\frac{8 \text{Vol}_g(\Omega)}{c_1}\right)^{\frac{1}{n}}$ . Since  $\text{Vol}_g(\Omega) \leq \eta = \frac{c_1}{16}$ , we get that  $\text{Vol}_g(B_{x_i}(r)) \geq c_1 r^n = c_1 \frac{8 \text{Vol}_g(\Omega)}{c_1} = 8 \text{Vol}_g(\Omega)$

$$\Rightarrow \frac{2 \text{Vol}_g(\Omega)}{\text{Vol}_g(B_{x_i}(r))} \leq \frac{1}{4} \Rightarrow \bar{u}_{\varepsilon, r}(x) \leq \frac{1}{4}. \text{ Hence, } \{x \in M : \bar{u}_{\varepsilon, r}(x) \geq \frac{1}{2}\} = \emptyset.$$

Want:  $\{x \in M : \bar{u}_{\varepsilon, r}(x) \geq \frac{1}{2}\} = \emptyset$ .  
 $\uparrow$   
 $\bar{u}_{\varepsilon, r}(x) \leq \frac{2 \text{Vol}_g(\Omega)}{\text{Vol}_g(B_{x_i}(r))} \leq \frac{1}{4}$ .  
 $\uparrow$   
 $\text{Vol}_g(B_{x_i}(r)) \geq 8 \text{Vol}_g(\Omega)$   
 take  $r^n = \frac{8 \text{Vol}_g(\Omega)}{c_1}$

and for every  $0 < \varepsilon < 1$ .  $\text{Vol}_g(\Omega) \leq \text{Vol}_g(\{x \in M: |u_\varepsilon(x) - \bar{u}_{\varepsilon, \nu}(x)| \geq \frac{1}{2}\})$

But 
$$\text{Vol}_g(\{x \in M: |u_\varepsilon(x) - \bar{u}_{\varepsilon, \nu}(x)| \geq \frac{1}{2}\}) = 2 \int_{\{x \in M: |u_\varepsilon(x) - \bar{u}_{\varepsilon, \nu}(x)| \geq \frac{1}{2}\}} \frac{1}{2} d\text{Vol}_g$$

$$\leq 2 \int_M |u_\varepsilon(x) - \bar{u}_{\varepsilon, \nu}(x)| d\text{Vol}_g$$

$\Rightarrow$  there exists a positive constant  $C_2 = C_2(n, k, \nu)$  such that

$$\int_M |u_\varepsilon - \bar{u}_{\varepsilon, \nu}| d\text{Vol}_g \leq C_2 \cdot r \int_M |\nabla u_\varepsilon| d\text{Vol}_g$$

Hence, 
$$\text{Vol}_g(\Omega) \leq 2C_2 \left(\frac{8\text{Vol}_g(\Omega)}{c_1}\right)^{\frac{1}{n}} \lim_{\varepsilon \rightarrow 0} \int_M |\nabla u_\varepsilon| d\text{Vol}_g$$

$$\leq C_3 \text{Vol}_g(\Omega)^{\frac{1}{n}} \text{Area}_g(\partial\Omega)$$

Lemma. Let  $(M, g)$  be a complete Riemannian  $n$ -manifold. Suppose that its Ricci curvature satisfies  $\text{Ric} \geq kg$  for some  $k \in \mathbb{R}$  and suppose that there exists  $\nu > 0$  such that  $\text{Vol}_g(B_x(1)) \geq \nu$  for any  $x \in M$ . There exist two positive constants  $S = S(n, k, \nu)$  and  $A = A(n, k, \nu)$ , such that for any  $x \in M$  and any  $u \in \mathcal{D}(B_x(1))$ .

$$\left(\int_M |u|^{\frac{n}{n-1}} d\text{Vol}_g\right)^{\frac{n-1}{n}} \leq A \int_M |\nabla u| d\text{Vol}_g.$$

proof. Let  $\eta = \eta(n, k, \nu)$  be as in above lemma. By Gromov's result there exists  $\delta = \delta(n, k, \nu)$  such that for any



$x \in M$ ,  $\text{Vol}_g(B_x(\delta)) \leq \eta$ . Let  $x \in M$  and  $u \in \mathcal{D}(B_x(\delta))$ . For  $t > 0$ , let

$$\Omega(t) = \{x \in M: \text{s.t. } |u(x)| > t\} \text{ and } V(t) = \text{Vol}_g(\Omega(t))$$

Clearly,  $V(t) \leq \eta$  for any  $t > 0$ . Then the coarea formula

$$\int_M |\nabla u| d\text{Vol}_g \geq \int_0^\infty \text{Area}(\partial\Omega(t)) dt \geq \frac{1}{c} \int_0^\infty V(t)^{\frac{n-1}{n}} dt$$

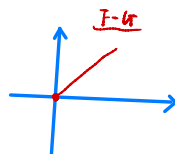
Independently, 
$$\int_M |u|^{\frac{n}{n-1}} d\text{Vol}_g = \frac{n}{n-1} \int_0^\infty t^{\frac{1}{n-1}} V(t) dt$$

Noting that 
$$\int_0^\infty V(t)^{1-\frac{1}{n}} dt \geq \left(\frac{n}{n-1} \int_0^\infty t^{\frac{1}{n-1}} V(t) dt\right)^{1-\frac{1}{n}}$$

set  $F(x) = \left(\int_0^x |u(t)|^{\frac{n}{n-1}} dt\right)^{\frac{n-1}{n}}$ ,  $G(x) = \frac{n}{n-1} \int_0^x t^{\frac{1}{n-1}} V(t) dt$   
 $F'(x) = G'(x)$

Since  $V(t)$  is a decreasing function of  $t$ , we have

$$F(x) = \frac{n}{n-1} \left(\int_0^x |u(t)|^{\frac{n}{n-1}} dt\right)^{\frac{n-1}{n}} \geq \frac{n}{n-1} x^{\frac{1}{n-1}} V(x) = G'(x)$$



分析这个工具很强大。  
值得好好学一学  
加油!

this ends the proof of the lemma.

proof of thm 3.14 Let  $(M, g)$  be a complete Riemannian  $n$ -manifold such that  $\text{Ric} \geq kg$  for some  $k \in \mathbb{R}$  and such that there exists  $\nu > 0$  with

$\text{Vol}_g(B_x(1)) \geq \nu$  for any  $x \in M$ . We want to prove that  $H_1(M) = L^{\frac{n}{n-1}}(M)$ .

Let  $\delta = \delta(n, k, \nu)$  be as in Lemma 3.7 and  $(x_i)$  be a sequence of points of  $M$  such that

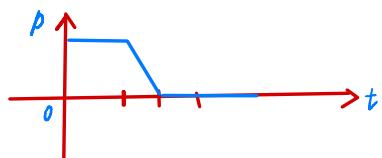
(1)  $M = \bigcup_i B_{x_i}(\delta/4)$  and  $B_{x_i}(\delta/4) \cap B_{x_j}(\delta/4) = \emptyset$ , if  $i \neq j$

(2) there exists  $N = N(n, k, \nu)$  such that each points of  $M$  has a neighborhood which intersects at most  $N$  of the  $B_{x_i}(\delta)$ 's.

Let  $p: [0, \infty) \rightarrow [0, 1]$  be defined by  $p(t) = 1$  if  $0 \leq t \leq \delta/2$ .

$p(t) = 3 - \frac{4}{\delta}t$  if  $\delta/2 \leq t \leq 3\delta/4$

$p(t) = 0$  if  $t \geq 3\delta/4$ .



and let  $\alpha_i(x) = \rho(d_g(x, x_i))$ . Clearly,  $\alpha_i$  is Lipschitz with compact support. Hence,  $\alpha_i$  belongs to  $H^1(M)$ .

Furthermore, since  $\text{supp } \alpha_i = \overline{B_{x_i}(\frac{3r}{4})}$ , we get without any difficulty that  $\alpha_i \in H^1_0(B_{x_i}(r))$ . Let

$\eta_i = \frac{\alpha_i}{\sum_m \alpha_m}$ . Then, since  $|\nabla \alpha_i| \leq \frac{4}{r}$  a.e. we get that for any  $i$ ,  $\eta_i \in H^1_0(B_{x_i}(r))$ ,  $(\eta_i)$  is a partition of unity

subordinate to the covering  $(B_{x_i}(r))$ .  $\nabla \eta_i$  exists almost everywhere, and there exists a positive constant  $H = H(n, k, \nu)$

such that  $|\nabla \eta_i| \leq H$  a.e.

Let  $u \in \mathcal{D}(M)$ . We have

$$\left( \int_M |u|^{\frac{n}{n-1}} dv(g) \right)^{\frac{n-1}{n}} \leq \sum_i \left( \int_M |\eta_i u|^{\frac{n}{n-1}} dv(g) \right)^{\frac{n-1}{n}} \leq A \sum_i \int_M |\nabla \eta_i u| dv(g).$$

$$\text{Hence, } \left( \int_M |u|^{\frac{n}{n-1}} dv(g) \right)^{\frac{n-1}{n}} \leq A \sum_i \int_M |\eta_i| |u| + A \sum_i \int_M |\eta_i| |\nabla \eta_i| |u| dv(g) \leq A \int_M |u| + A n H \int_M |u| dv(g).$$

Thm. Let  $(M, g)$  be a complete Riemannian  $n$ -manifold with Ricci curvature bounded from below.

(1) If for some  $1 \leq q < n$  and  $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ ,  $H^1_q(M) \subset L^p(M)$ , then for any  $1 \leq q < n$  and  $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ ,  $H^1_q(M) \subset L^p(M)$ .

In particular,  $H^1(M) \subset L^{\frac{n}{n-1}}(M)$ .

(2) The Sobolev embeddings are valid for  $M$  if and only if  $\inf_{x \in M} \text{Vol}_g(B_{x,r}) > 0$ .

Cor. The Sobolev embeddings are valid for complete manifolds with Ricci curvature bounded from below and positive injectivity radius.

Thm. Let  $(M, g)$  be a complete Riemannian  $n$ -manifold with Ricci curvature bounded from below and such that  $\inf_{x \in M} \text{Vol}_g(B_{x,r}) > 0$ .

For any  $1 \leq q < n$  and any  $p$  such that  $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ ,  $H^1_q(M) \subset L^p(M)$ .

PROOF: Set  $q^* = nq/(n-q)$ , and let  $p \in [q, q^*]$ . As a simple application of Hölder's inequality, one gets that for any  $u \in \mathcal{D}(M)$ ,

$$\left( \int_M |u|^p dv(g) \right)^{1/p} \leq \left( \int_M |u|^q dv(g) \right)^{\alpha/q} \left( \int_M |u|^{q^*} dv(g) \right)^{(1-\alpha)/q^*}$$

where  $\alpha \in [0, 1]$  is given by

$$\alpha = \frac{1/p - 1/q^*}{1/q - 1/q^*}$$

By Theorem 3.2, there exists  $A = A(n, q, k, \nu)$  such that for any  $u \in \mathcal{D}(M)$ ,

$$\left( \int_M |u|^{q^*} dv(g) \right)^{1/q^*} \leq A \left( \int_M |\nabla u|^q dv(g) \right)^{1/q} + A \left( \int_M |u|^q dv(g) \right)^{1/q}$$

Since for any  $x$  and  $y$  nonnegative, and any  $\alpha \in [0, 1]$ ,  $x^\alpha y^{1-\alpha} \leq x + y$ , one gets that for any  $p \in [q, q^*]$ , and any  $u \in \mathcal{D}(M)$ ,

$$\begin{aligned} \left( \int_M |u|^p dv(g) \right)^{1/p} &\leq \left( \int_M |u|^q dv(g) \right)^{1/q} + \left( \int_M |u|^{q^*} dv(g) \right)^{1/q^*} \\ &\leq A \left( \int_M |\nabla u|^q dv(g) \right)^{1/q} + (A+1) \left( \int_M |u|^q dv(g) \right)^{1/q} \end{aligned}$$

Clearly, this proves the proposition.  $\square$