7. $W^{2,p}$ Estimate.

Assume F is uniformly elliptic with ellipticity constant λ, Λ. F, f continuous in x. and $F(0,x) \equiv 0$ $(\forall x \in \Omega)$. Assume all solutions here are bounded in its domain.

$$\beta(x,x_0) := \beta_F(x,x_0) = \sup_{M \in S\setminus\{0\}} \frac{|F(M,x)-F(M,x_0)|}{\|M\|}$$

$$\beta(x) := \beta(x,0)$$

We say $F(D^2 w, x_0)=0$ has $C^{1,1}$ interior estimates (with constant C_e) $(x_0 \in B_1)$ if for any $w_0 \in C(\partial B_1)$, $\exists(!)$ a solution $w \in C^2(B_1)\cap C(\overline{B_1})$ of

$$\begin{cases} F(D^2 w, x_0) = 0 & \forall x \in B_1 \\ w(x) = w_0(x) & x \in \partial B_1 \end{cases}$$

Such that $\|w\|_{C^{1,1}(B_{1/2})} \le C_e \|w_0\|_{L^\infty(\partial B_1)}$

By $\begin{cases} F(0,x)=0 \\ F(D^2 w, x) = f(x) \end{cases} \Longrightarrow w \in S(\frac{\lambda}{n}, \Lambda, f)$

By $\begin{cases} F(D^2 w, x_0)=0 \\ F(0,x) = 0 \end{cases} \Longrightarrow w \in S(\frac{\lambda}{n}, \Lambda, 0) \xrightarrow[MP]{ABP} \|w\|_{L^\infty(B_1)} \le C \|w_0\|_{L^\infty(\partial B_1)}$ for some universal C.

Theorem 1. $F(D^2 u, x) = f(x)$ (B_1) (viscosity solution). $F(0,x) \equiv 0$ (B_1)

$F(D^2 w, x_0)$ has $C^{1,1}$ interior estimates (with constant C_e) for any $x_0 \in B_1$. $n < p < +\infty$. $f \in L^p(B_1)$

Then $\exists$ constants β_0, C, depending on n, λ, Λ, C_e and p. if:

$$\left(\fint_{B_r(x_0)} \beta(x,x_0)^n \right)^{\frac{1}{n}} \le \beta_0 \quad \forall B_r(x_0) \subseteq B_1$$

Then $u \in W^{2,p}(B_{\frac{1}{2}})$ & $\|u\|_{W^{2,p}(B_{\frac{1}{2}})} \le C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right)$.

Some Facts maybe used below.

1. $G(M,x) := \frac{1}{K} F(KM,x) \Longrightarrow \beta_G(x,x_0) = \sup_{M \in S\setminus\{0\}} \frac{|F(KM,x)-F(KM,x_0)|}{K\|M\|} = \beta_F(x,x_0)$

2. If $F(D^2w, x_0)=0$ has $C^{1,1}$ interior estimate for some $x_0 \in B_1$ (with constant C_e)
$\Rightarrow$ $G(D^2w, x_0)=0$ is also. (with same constant C_e)

For solving $\begin{cases} G(D^2w, x_0)=0 & B_1 \\ w = w_0 & \partial B_1 \end{cases}$ (I) we solve $\begin{cases} F(D^2w, x_0)=0 & \text{in } B_1 \\ w = K w_0 & \partial B_1 \end{cases}$ (II)

(and $\|w\|_{C^{1,1}(B_{\frac12})} \le C_e K \|w_0\|_{L^\infty(\partial B_1)}$)

Then $u = \frac{1}{K} w$ solve (I) and $\|u\|_{C^{1,1}(B_{\frac12})} \le C_e \|w_0\|_{L^\infty(\partial B_1)}$ □

3. If $F(D^2w, x_0)$ has $C^{1,1}$ interior estimate for some $x_0 \in B_1$ (with constant C_e)

$\Rightarrow$ $\begin{cases} F(D^2w, x_0)=0 & B_r \\ w = w_0 & \partial B_r \end{cases}$ ($x_0 \in B_1$) have a solution w.

& $\|w\|^*_{C^{1,1}(\bar B_{\frac r2})} \le C_e \|w_0\|_{L^\infty(\partial B_r)}$

$$\|w\|^*_{C^{1,1}(\bar B_{\frac r2})} := \|w\|_{L^\infty(B_{\frac r2})} + \frac{r}{2}\|Dw\|_{L^\infty(B_{\frac r2})} + \left(\frac r2\right)^2 \|D^2w\|_{L^\infty(B_{\frac r2})}.$$

By: If w solves $\begin{cases} F(D^2w, x_0)=0 & B_1 \\ w(x) = w_0(rx) r^{-2} & \partial B_1 \end{cases}$ ($\Rightarrow \|w\|_{C^{1,1}(B_{\frac12})} \le C_e \|w_0\|_{L^\infty(\partial B_r)} \cdot r^{-2}$.) (*)

here $w_0 \in C(\partial B_r)$

$\tilde w(x) := r^2 w\left(\frac{x}{r}\right)$ $x \in B_r$ ($\|\tilde w\|_{L^\infty(B_{\frac r2})} = r^2 \|w\|_{L^\infty(B_{\frac12})}$, $\|D\tilde w\|_{L^\infty(B_{\frac r2})} = r\|Dw\|_{L^\infty(B_{\frac12})}$

solves $\begin{cases} F(D^2\tilde w, x_0)=0 & B_r \\ \tilde w = w_0 & \partial B_r \end{cases}$ $\|D^2\tilde w\|_{L^\infty(B_{\frac r2})} = \|D^2 w\|_{L^\infty(B_{\frac12})}$)

and. (*) $\Leftrightarrow$ $r^{-2}\|\tilde w\|_{L^\infty(B_{\frac r2})} + r^{-1}\|D\tilde w\|_{L^\infty(B_{\frac r2})} + \|D^2\tilde w\|_{L^\infty(B_{\frac r2})} \le r^{-2} \cdot C_e \|w_0\|_{L^\infty(\partial B_r)}$

$\Leftrightarrow$ $\|\tilde w\|^*_{C^{1,1}(B_{\frac r2})} \le C_e \|w_0\|_{L^\infty(\partial B_r)}$. □

4. Q is a dyadic cube (in Q_1) $Q = Q_{\frac{1}{2^i}}(x_0)$ for some $i \ge 0$ $x_0 \in Q_1$
$\Rightarrow$: $Q_{\frac{k\sqrt n}{2^i}}(x_0) \subseteq Q_{k\sqrt n}$ (for $k \ge 2$)

& $\tilde Q \subseteq Q_{\frac{3}{2^i}}(x_0)$ (i.e. $\tilde Q$ incudes in 3-extension of $Q_{\frac{1}{2^i}}(x_0)$).

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By Facts above, we only need to prove:

Prop 2. $F(D^2u, x) = f(x)$ (viscosity) $F(D^2\omega, x_0)$ has $C^{1,1}$ interior estimates (with constant C_e) for $\underbrace{}_{(B_{8\sqrt{n}})}$

any $x_0 \in B_{8\sqrt{n}}$. Let $n < p < +\infty$, and assume that, $\|u\|_{L^\infty(B_{8\sqrt{n}})} \le 1$, $\|f\|_{L^p(B_{8\sqrt{n}})} \le \varepsilon$ &

$$\left(⨍_{B_r(x_0)} \beta(x, x_0)^n \right)^{\frac{1}{n}} \le \varepsilon$$

for any $B_r(x_0) \subseteq B_{8\sqrt{n}}$. Then $u \in W^{2,p}(B_{\frac{1}{2}})$ & $\|u\|_{W^{2,p}(B_{\frac{1}{2}})} \le C$. (1).

ε, C depend only on λ, Λ, C_e and p.

If prop 2 holds. $v(x) := \dfrac{u(x)}{\|u\|_{L^\infty(B_{8\sqrt{n}})} + \varepsilon^{-1}\|f\|_{L^p(B_{8\sqrt{n}})}}$ $\|v\|_{L^\infty(B_{8\sqrt{n}})} \le 1$, $\|\frac{1}{K}f\|_{L^p(B_{8\sqrt{n}})} \le \varepsilon$

Solve $\frac{1}{K}F(KD^2v, x) = \frac{1}{K}f(x)$ & $G(D^2\omega, x_0)$ has $C^{1,1}$ Interior estimate (Fact 1)

$G(D^2v, x) :=$ $(K = \|u\|_\infty + \varepsilon^{-1}\|f\|_{L^p})$ $\Longrightarrow$: $\|v\|_{W^{2,p}(B_{\frac{1}{2}})} \le C$

i.e. $\|u\|_{W^{2,p}(B_{\frac{1}{2}})} \le C(\|u\|_{L^\infty(B_{8\sqrt{n}})} + \|f\|_{L^p(B_{8\sqrt{n}})})$

C depends only on n, λ, Λ, C_e, p. $\square$

Notation. Ω is a bounded domain. H is open subset in Ω $M > 0$ $u \in C(\Omega)$.

$\underline{G}_M(u, H) := \underline{G}_M(u) := \{x_0 \in H ;\ \exists P$ concave paraboloid of opening M such that $P(x_0) = u(x_0)$, $P(x) \le u(x)$ in $H\}$.

$\underline{A}_M(u) := H \setminus \underline{G}_M(u)$

$\overline{G}_M(u) := \underline{G}_M(-u)$ $\overline{A}_M(u) := H \setminus \overline{G}_M(u)$.

$G_M(u) := \underline{G}_M(u) \cap \overline{G}_M(u)$. $A_M(u) := H \setminus G_M(u)$.

$\Theta(x) := \Theta(u, B_{\frac{1}{2}}) = \inf\{M : x \in G_M(B_{\frac{1}{2}})\} \in [0, \infty]$

$\forall a > 0$, $d_f(a) := |\{x \in \Omega \mid |f| > a\}|$ for $f \in Mes(\Omega) := \{$all measurable functions in $\Omega\}$ call distribution function of f in Ω.

For the proof of prop 2, we need some lemmas below.

Lemma 3. Let g be a nonnegative and measurable function in Ω.

$\forall p \in (0, \infty)$ $\eta > 0$. $M > 1$.

$$g \in L^p(\Omega) \Longleftrightarrow S := \sum_{k=1}^{\infty} M^{pk} d_g(\eta M^k) < +\infty$$

and $C^{-1}S \le \|g\|^p_{L^p(\Omega)} \le C(|\Omega| + S)$ $C = \max\{$

prof: $\int_\Omega g^p\,dx = \sum_{k=0}^{\infty} \int_{\{x\in\Omega \mid g\in(\eta M^k, \eta M^{k+1}]\} := E_k(\eta)} g^p\,dx$

$\le \eta^p M^p \cdot |\Omega| + \sum_{k\geq 1}^{\infty} \int_{E_k(\eta)} g^p\,dx$

$\le \eta^p M^p |\Omega| + \sum_{k\geq 1}^{\infty} (\eta M^{k+1})^p\, d_g(\eta M^k)$

$\le \eta^p M^p \cdot (|\Omega| + S)$

converse: $\int_\Omega g^p \geq \sum_{k=0}^{\infty} (\eta M^k)^p \left(d_g(\eta M^k) - d_g(\eta M^{k+1}) \right)$

$= \sum_{k=0}^{\infty} (\eta M^k)^p d_g(\eta M^k) - \sum_{k=1}^{\infty} (\eta M^{k-1})^p d_g(M^k \eta)$

$= \eta^p d_g(\eta) + (1-\frac{1}{M^p}) \sum_{k\geq 1} (\eta M^k)^p d_g(M^k\eta)$

$\geq (1-M^{-p})\eta^p S$

chose $C = \max\{\eta^p M^p, [(1-M^{-p})\eta^p]^{-1}\}$

$\Rightarrow C^{-1} S \le \int g^p \le C(|\Omega|+S)$ □

Lemma 4. $1<p\le\infty$, $u\in C(\Omega)$ $\varepsilon>0$ define: $\Theta(u,\varepsilon)(x) = \Theta(u, \Omega\cap B_\varepsilon(x))(x)$ $x\in\Omega$.
Assume $\Theta(u,\varepsilon)\in L^p(\Omega)$, Then $D^2u\in L^p(\Omega)$ and

$$\|D^2 u\|_{L^p(\Omega)} \le 2\|\Theta(u,\varepsilon)\|_{L^p(\Omega)}.$$

prof: Denote $\Delta_h^2 u(x_0) = [u(x_0+h)+u(x_0-h)-2u(x_0)]/|h|^2$ $h\in\mathbb{R}^n$.

Step 1. $1<p\le\infty$. $\forall \varphi\in C_0^\infty(\Omega)$. $v\in S^{n-1}$

$|\int_\Omega \varphi_{vv}\cdot u| \le \|\Theta(u,\varepsilon)\|_p \|\varphi\|_{p'}$

$K=\operatorname{supp}\varphi \subseteq \Omega$ chose $\delta < \operatorname{dist}(K,\partial\Omega)$

$|\int_\Omega u\varphi_{vv}| = |\int_K u\varphi_{vv}| = \lim_{\delta\to 0} |\int_K u\cdot \Delta^2_{\delta v}\varphi|$

$= \lim_{\delta\to 0} |\int_K \Delta^2_{\delta v} u \cdot \varphi|$

$\le \lim_{\delta\to 0} \int_K |\Theta(u,\varepsilon)|\,|\varphi|$

$\le \|\Theta(u,\varepsilon)\|_p \|\varphi\|_{p'}$.

Step 2. $|\int_\Omega u\varphi_{vj}\,dx| \le 2\cdot \|\Theta(u,\varepsilon)\|_p \|\varphi\|_{p'}$

$\varphi_{vj} = \frac{1}{2}(\varphi_{e_v+e_j,e_v+e_j} - \varphi_{e_v e_v} - \varphi_{e_j e_j}) = \frac{1}{2}(2\varphi_{VV} - \varphi_{vv} - \varphi_{jj})$ $V = \frac{1}{\sqrt{2}}(e_v+e_j)$

By: $|\int_\Omega u\varphi_{vj}| \underset{\text{Step1}}{\le} \frac{1}{2}[2|\int u\varphi_{VV}| + |\int u\varphi_{vv}| + |\int u\varphi_{jj}|] \le 2\|\Theta(u,\varepsilon)\|_p \|\varphi\|_{p'}$

Step 3. $D^2 u$ exists. (WLOG, we prove $D_{vv}u$ exists)

Case1. $1<p<\infty$. $\delta < \mathrm{dist}(K,\partial\Omega)$ By Step 1.

$\|\Delta^2_{\delta e_j} u\|_{L^p(K)} \le \|\theta(u,\Omega)\|_p \Rightarrow \exists V_K \in L^p(K)$ $\Delta^2_{\delta e_j}u \rightharpoonup V_K$ (subsequence)

choose $\bigcup_{n=1}^{\infty} K_n = \Omega$ $K_n \subseteq \mathrm{Int}\, K_{n+1}$ and diagonal argument + Lebesgue's Dominate Theorem

$\exists V\in L^p(\Omega)$. $\Delta^2_{\delta e_j}u \rightharpoonup V$ $(L^p(K))$ $\forall K \overset{cpt}{\subseteq}\Omega$.

Case2. $p=\infty$. $\|\Delta^2_{\delta e_j} u\|_{L^2(K)} \le |\Omega|^{\frac12}\|\theta(u,\Omega)\|_{L^\infty(\Omega)}$ we also have $V\in L^2(\Omega)$ $\Delta^2_{\delta e_j} u\rightharpoonup V$ $(L^2(K))$ $\forall K \Subset_{cpt}\Omega$.

Then: $\int_\Omega u\varphi_{ij} = \lim_{\delta\to0}\int_K \Delta^2_{\delta e_j}u\cdot\varphi = \int_\Omega V\varphi \Rightarrow V = u_{ij}$
$\forall\varphi\in C_0^\infty(\Omega)$,

Step4. $V=D_{ij}u$ $\left|\int_\Omega V\varphi\right| \overset{\text{Step2}}{\le} 2\cdot\|\theta(u,\Omega)\|_p\|\varphi\|_{p'}$

$1<p<\infty$

$$\|V\|_p = \sup_{\substack{\varphi\in C_0^\infty(\Omega)\\ \|\varphi\|_{p'}=1}} \left|\int_\Omega V\varphi\right| \le 2\|\theta(u,\Omega)\|_p .$$

$p=\infty$ $\|V\|_\infty = \lim_{p\to\infty}\|V\|_p \le 2\|\theta(u,\Omega)\|_\infty$. □

Lemma5. Ω is bounded domain st $B_{6\sqrt n}\subseteq\Omega$, $u\in C(\Omega)$, & $\|u\|_{L^\infty(\Omega)}\le 1$, $u\in\bar S(f)$ in $B_{6\sqrt n}$ and $\|f\|_{L^n(B_{6\sqrt n})}\le\delta_0$. Then $|G_M(u,\Omega)\cap Q_1|\ge 1-\sigma$

where $0<\sigma<1$, $\delta_0>0$, $M>1$ are universal constants.

proof: Let φ in Section4. $\omega = u+1+2\varphi$ in $\bar B_{6\sqrt n}$ $\omega\in\bar S(f+2C\xi)$ $(0\le\xi\le1)$

$\omega\ge0$ on $\partial B_{2\sqrt n}$ $\inf_{Q_3}\omega\le -2$

$$2\overset{\text{ABP}}{\le} C\left[\left(\int_{\{\omega=\Gamma_\omega\}\cap B_{2\sqrt n}} f^n\right)^{\frac1n} + \left(\int_{\{\omega=\Gamma_\omega\}\cap B_{2\sqrt n}}\xi^n\right)^{\frac1n}\right] \le C\left(\delta_0 + |\{\omega=\Gamma_\omega\}\cap Q_1|^{\frac1n}\right)$$

$\delta_0 = C^{-1}$ $\sigma = 1-C^{-n}$ $\Rightarrow 1-\sigma\le|\{\omega=\Gamma_\omega\}\cap Q_1|$ (if $C\le1$ σ can chose any number in $(0,1)$).

Claim: $\{\omega=\Gamma_\omega\}\cap Q_1\subseteq G_M(u,\Omega)\cap Q_1$

$x_0\in LHS \Rightarrow L\le\Gamma_\omega\le\omega$ equal at x_0 in $B_{2\sqrt n}$ ($L<0$ in $\partial B_{2\sqrt n}$ unless $\omega\equiv0$ in $B_{2\sqrt n}$)
(then it's trivial that Lemma5 holds)

$\Rightarrow L-1-2\varphi\le u$ equal at x_0

$L-1-2\varphi = (L-1)(x_0) - 2\left(\varphi(x_0)+\nabla\varphi(x_0)(x-x_0)+\frac12(x-x_0)^T\nabla^2\varphi(\xi_0)(x-x_0)\right)$

$\ge A - 2\nabla\varphi(x_0)(x-x_0) - \frac M2|x-x_0|^2$ $M:=2\sup_{B_{2\sqrt n}}\|\nabla^2\varphi\|$

$:= P_{x_0}(x)$ $P_{x_0}(x_0)=u(x_0)$

Then $P_{x_0} \le L-1-2\varphi \le u$ in $B_{2\sqrt{n}}$.

$P_{x_0}(x) < -1$ $\partial B_{2\sqrt{n}}$ & $P_{x_0}(x_0) \ge -1$ $(=u(x_0))$ & $\{P_{x_0}(x) \ge -1\}$ convex $\Rightarrow P_{x_0}(x) < -1 \le u(x)$ in $\bar{B}_{2\sqrt{n}}^c$

$\Rightarrow x_0 \in G_M(u.\Omega) \cap Q_1 \Rightarrow |\underset{(u,\Omega)}{G_M} \cap Q_1| \ge 1-\sigma$. □

Lemma 6. Ω bounded and $B_{6\sqrt{n}} \subseteq \Omega$. $u \in \bar{S}(f)$ in $B_{6\sqrt{n}}$ if $\|f\|_{L^n(B_{6\sqrt{n}})} \le \delta_0$, $G_1(u.\Omega) \cap Q_3 \ne \phi$ Then
$$|G_M(u.\Omega) \cap Q_1| \ge 1-\sigma.$$

$0<\sigma<1$, $\delta_0>0$. $M>1$ are universal constants.

proof: $x_0 \in G_1(u.\Omega) \cap Q_3$. $\Rightarrow$ $\exists L$ affine. $-\frac{1}{2}|x-x_0|^2 + L(x) \le u(x)$ in Ω $L(x_0) = u(x_0)$

$\Rightarrow P(x) \le V := u(x)/\delta n + \tilde{L}(x)$ $P(x) := 1 - \frac{1}{16n}|x-x_0|^2$

$V(x_0) = 1 = P(x_0)$

$\omega := V+\varphi \in \bar{S}(\frac{f}{\delta n} + C\xi)$ $B_{4\sqrt{n}}$

as in Lemma 5 $|\{\omega = \Gamma_\omega\} \cap Q_1| \ge 1-\sigma$ if $\|f\|_{L^n(B_{6\sqrt{n}})} \le \delta_0$

(σ. δ_0 are universal constant)

Claim: $LHS := Q_1 \cap \{\omega \cap \Gamma_\omega\} \subseteq G_M(u.\Omega) \cap Q_1$ for some universal $M > 1$.

$\forall x_1 \in LHS$. $\exists L_1$ affine. $L_1 \le V+\varphi$ equal at x_1

as in Lemma 5, $\exists P_{x_1}(x)$ opening $M_0 > 1$. (P_{x_1} concave)

$P_{x_1}(x) \le L_1 - \varphi \le V$ equal at x_1

$P_{x_1}(x) < -\varphi < 0$ $\partial B_{4\sqrt{n}}$ & $P_{x_1}(x_1) = V(x_1) \ge \underset{=P(x)}{P(x_1)}$ & $\{P_{x_1} - P \ge 0\}$ is convex

$\Rightarrow$ $P_{x_1} - P < 0$ $\bar{B}_{4\sqrt{n}}^c$

$\Rightarrow$ $P_{x_1} \le V$ in Ω

$\tilde{P} = [P_{x_1} - \tilde{L}(x)]\delta n \le u$ in Ω equal at x_1

$\Rightarrow$: $x_1 \in G_M(u.\Omega) \cap Q_1$ $(M = \delta n M_0)$ □

Lemma 7. Condition As Lemma 5. f 0-extension to $B_{6\sqrt{n}}$. (Still denote f). $k = 0.1.2\cdots$

$A := A_{M^{k+1}}(u.\Omega) \cap Q_1$ $B = A_{M^k}(u.\Omega) \cap Q_1 \cup \{x | M(f^n)(x) \ge (CM^k)^n\}$

$\Rightarrow$: $|A| \le \sigma |B|$

$M = \max(M$ in Lemma 5. 6). $\mathcal{M}(g)$ is the maximal function of g $(g \in L^1_{loc}(\mathbb{R}^n))$. C is a universal constant.

$\sigma = \max\{\sigma$ in Lemma 5. 6$\}$. H-L (using cube)

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proof: Lemma 5 $\Longrightarrow$ $|A|\leqslant\sigma$ if $Q=Q_{\frac{1}{2^i}}(x_0)$ is a dyadic in Q_1 and $|A\cap Q|\geqslant\sigma|Q|$

we claim $\tilde{Q}\subseteq B$. If not, $\exists\, x_1\in\tilde{Q}\cap G_{M^{k+1}}(u.\Omega)$ & $M(f^n)(x_1)<(CM^k)^n$

$$\tilde{u}(y):=u\left(x_0+\frac{1}{2^i}y\right)\cdot\frac{2^{2i}}{M^k}\quad \text{in } \tilde{\Omega}:=2^i(\Omega-x_0).$$

$y_1:=2^i(x_1-x_0)\in Q_3$ by $x_1\in Q_{\frac{3}{2^i}}(x_0)$

$y_1\in Q_3\cap G_M(\tilde{u}.\tilde{\Omega})$ by: $P(x)\leq u(x)$ P concave and opening M^{k+1} equal at x_1

$\Longrightarrow$: $\tilde{P}(y):=P\left(x_0+\frac{1}{2^i}y\right)\cdot\frac{2^{2i}}{M^k}\leq\tilde{u}(y)$ equal at y_1

$\tilde{P}(y)$ concave and opening M.

$\tilde{u}\in S\left(f\left(x_0+\frac{1}{2^i}\cdot\right)\cdot\frac{1}{M^k}\right)$ in $B_{6\sqrt{n}}$ (By $B_{\frac{6\sqrt{n}}{2^i}}(x_0)\subseteq\Omega$)

$$\Longrightarrow \|\tilde{f}(y)\|^n_{L^n(B_{6\sqrt{n}})}=\frac{2^{in}}{M^{kn}}\|f\|^n_{L^n(B_{\frac{6\sqrt{n}}{2^i}}(x_0))}\leq\frac{2^{in}}{M^{kn}}\int_{Q_{\frac{19}{2^i}\sqrt{n}}(x_0)}f^n\,dy$$

$\subseteq Q_{\frac{19}{2^i}\sqrt{n}}(x_1)$

$$\leq\frac{2^{in}}{M^{kn}}\cdot\left(\frac{19}{2^i}\sqrt{n}\right)^n(CM^k)^n\leq\delta_0^n$$

if $C:=\delta_0/19\sqrt{n}$ is universal.

Lemma 6. $\Longrightarrow$ $|G_M(\tilde{u}.\tilde{\Omega})\cap Q_1|\geqslant 1-\sigma$

By $G_M(\tilde{u}.\tilde{\Omega})\cap Q_1=2^i\left(G_{M^{k+1}}(u.\Omega)\cap Q_{\frac{1}{2^i}}(x_0)-x_0\right)$

$\Longrightarrow$, $|G_{M^{k+1}}(u,\Omega)\cap Q|\geqslant(1-\sigma)|Q|$ which yields contradiction! □

Lemma 8. Condition As Lemma 5. Then $|A_t(u.\Omega)\cap Q_1|\leqslant Ct^{-\mu}$ $\forall t>0$ for some universal $C.\ \mu$.

Moreover, $u\in S(f)(B_{6\sqrt{n}})\Rightarrow|A_t(u.\Omega)\cap Q_1|\leqslant Ct^{-\mu}$. $\forall t>0$.

proof: $M.\ \sigma.\ \delta_0$ as Lemma 7.

$\alpha_k:=A_{M^k}(u.\Omega)\cap Q_1$ $\quad\beta_k:=|\{x\mid M(f^n)\geqslant(CM^k)^n\}|$

Lemma 7 $\longrightarrow$ $\alpha_{k+1}\leq\sigma(\alpha_k+\beta_k)$

(For $k\geqslant k_0=\sup\{k\in\mathbb{Z}^+\mid k>(\frac{\delta}{\delta_0})^k\}+1$)

By Induction $\alpha_k\leq\sigma^k+\sum_{i=1}^{k-1}\sigma^{k-i}\beta_i\leq\sigma^k+ck\delta_0^k\leq(1+ck)\delta_0^k\leq 2c\,\delta^k$

$\delta_0=\max\{\sigma.\ M^{-n}\}$ $\delta=\frac{1}{2}(\delta_0+1)$

By: $\beta_k\leq(CM^k)^{-n}C(n)\|f\|^n_{L^n(B_{6\sqrt{n}})}\leq C\cdot(M^{-n})^k$ $\quad C=C(n.\lambda.\Lambda)$ $\Rightarrow\sigma^{k-i}\beta_i\leq C\delta_0^k$

Case 1. $t\leq M^{k_0}$ we choose C_1 st $C_1M^{-\mu k_0}\geqslant 1$ μ is undetermined.

($\frac{\ln t}{\ln M} \geq$

Case 2. $t > M^{k_0}$ For $k \geq k_0$, $t \in (M^k, M^{k+1}]$ $(\Rightarrow k \geq \frac{\ln t}{\ln M} - 1)$

$$|A_t(u.\Omega) \cap Q_1| \leq |A_{M^k}(u.\Omega) \cap Q_1|$$
$$\leq (2c\delta^k)$$
$$\leq 2c \cdot \delta^{-1} \cdot t^{-\frac{\ln\delta^{-1}}{\ln M}} = C \cdot t^{-\mu} \quad \mu = \frac{\ln\delta^{-1}}{\ln M} > 0.$$

Then we choose $C \geq M^{k_0\mu}$

If $u \in S(f) \Rightarrow |A_t(-u.\Omega) \cap Q_1| \leq C t^{-\mu}$ $\forall t > 0$

$\Rightarrow |A(u.\Omega) \cap Q_1| \leq 2Ct^{-\mu}$ $\forall t > 0$. □

Lemma 9. Let $u \in S(\lambda.\Lambda.0)$ in B_1 $\beta \in (0.1)$. $u \in C(\overline{B_1})$ $u|_{\partial B_1} = \varphi \in C^\beta(\partial B_1)$. Then

$$\sup_{x \in B_1} \frac{|u(x)-u(x_0)|}{|x-x_0|^{\beta/2}} \leq 2^{\beta/2} \sup_{x \in \partial B_1} \frac{|\varphi(x)-\varphi(x_0)|}{|x-x_0|^\beta}$$

proof: WLoG. $u(x_0)=0$ $x_0=0$ $B_1 = B_1((0,0\cdots,1))$ $K := \sup_{x \in \partial B_1} \frac{|\varphi(x)|}{|x|^\beta}$ $(h(x) := x_n^{\frac{\beta}{2}})$

$x_1^2 + \cdots + (x_n-1)^2 = 1 \Rightarrow |x|^2 = 2x_n$ $\Longrightarrow u(x) = \varphi(x) \leq K \cdot 2^{\frac{\beta}{2}} \cdot x_n^{\frac{\beta}{2}} = 2^{\frac{\beta}{2}} \cdot K \cdot h(x)$ on ∂B_1

$\mathcal{M}^+(D^2 h) < 0$ in B_1 $\Rightarrow u - 2^{\frac{\beta}{2}} K h \in \underline{S}(\lambda.\Lambda.0)$

$\xrightarrow{ABP} u - 2^{\frac{\beta}{2}} K h \leq 0$ B_1 $\Leftrightarrow u \leq 2^{\frac{\beta}{2}} K h \leq 2^{\frac{\beta}{2}} K |x|^{\frac{\beta}{2}}$

$-u$ substitute u $\Longrightarrow |u| \leq 2^{\frac{\beta}{2}} K |x|^{\frac{\beta}{2}}$ which yields the result. □

Lemma 10. $u \in S(\lambda.\Lambda.0)$ in B_1 $\beta \in (0.1)$. $u \in C(\overline{B_1})$ $u|_{\partial B_1} = \varphi$ $\varphi \in C^\beta(\partial B_1) \Rightarrow \varphi \in C^\nu(\overline{B_1})$ &

$$\|u\|_{C^\nu(\overline{B_1})} \leq C \cdot \|\varphi\|_{C^\beta(\partial B_1)}$$

$\nu = \min(\alpha. \frac{\beta}{2})$ α is a universal constant, $\alpha \in (0.1)$.

proof:

$\forall x \neq y \in B_1$ $d_x := dist(x.\partial B_1)$ $d_y := dist(y.\partial B_1)$ WLoG $d_y \leq d_x$ $x_0, y_0 \in \partial B_1$

$= d(x.x_0)$ $= d(y.y_0)$

① $d_x \geq 2|x-y| \Leftrightarrow y \in B_{\frac{d_x}{2}}(x) \subseteq B_{d_x}(x) \subseteq B_1$

$$\|u\|_{C^\alpha(B_{\frac{d_x}{2}})} = \|u-u(x_0)\|_{C^\alpha(B_{\frac{d_x}{2}}(x))} \leq C(\|u-u(x_0)\|_{L^\infty(B_{d_x})})$$

$$\Rightarrow \frac{d_x^\nu}{|x-y|^\nu} |u(x)-u(y)| \leq \frac{d_x^\alpha}{|x-y|^\alpha} |u(x)-u(y)| \leq C \cdot (\|u-u(x_0)\|_{L^\infty(B_1)})$$
$$\leq C \cdot 2^{\frac{\beta}{2}} \cdot \|\varphi\|_{C^\beta(\partial B_1)} d_x^{\frac{\beta}{2}}$$

$$\Rightarrow \frac{|u(x)-u(y)|}{|x-y|^\nu} \leq C \cdot \|\varphi\|_{C^\beta(\partial B_1)}$$

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② $d_x < 2|x-y|$ $\quad |u(x)-u(y)| \le |u(x)-u(x_0)| + |u(x_0)-u(y_0)| + |u(y)-u(y_0)|$

$$\le C\left(d_x^{\frac{\beta}{2}} + |x_0-y_0|^{\frac{\beta}{2}} + d_y^{\frac{\beta}{2}}\right) \|\varphi\|_{C^\alpha(\partial B_1)}$$

$$\le C\cdot |x-y|^{\frac{\beta}{2}} \|\varphi\|_{C^\alpha(\partial B_1)}$$

$$\le C\cdot |x-y|^{\gamma} \|\varphi\|_{C^\alpha(\partial B_1)}$$

By $|x_0-y_0| \le d_x + d_y + |x-y| \le 5|x-y|$. □

Lemma 11. Let $\varepsilon \in (0,1)$. $F(D^2u, x) = f(x)$ $(B_{8\sqrt{n}})$ $\|u\|_{C(8\sqrt{n})} \le 1$. $\|\beta\|_{L^n(B_{7\sqrt{n}})} = \|\beta(\cdot,0)\|_{L^n(B_{7\sqrt{n}})} \le \varepsilon$ & $F(D^2w, 0)=0$ has $C^{1,1}$ interior estimates (with constant C_e). Then there exist $h \in C^2(\overline{B_{6\sqrt{n}}})$ and $\varphi \in C(B_{6\sqrt{n}})$ such that $\|h\|_{C^{1,1}(\overline{B_{6\sqrt{n}}})} \le C(n)C_e$. $u-h \in S(\frac{\lambda}{n}, \Lambda, \varphi)$ in $B_{6\sqrt{n}}$ and

$$\|u-h\|_{L^\infty(B_{6\sqrt{n}})} + \|\varphi\|_{L^n(B_{6\sqrt{n}})} \le C(\varepsilon^\gamma + \|f\|_{L^n(B_{8\sqrt{n}})})$$

$\gamma \in (0,1)$ is universal, $C = C(n, \lambda, \Lambda, C_e)$

proof:

Let $h \in C^2(B_{7\sqrt{n}})$ be the solution of (By Fact 3)

$$\begin{cases} F(D^2h, 0) = 0 & B_{7\sqrt{n}} \\ h = u & \partial B_{7\sqrt{n}} \end{cases}$$

$\xrightarrow[\text{+ covering argument}]{\text{Fact 3}}$ $\|h\|_{C^{1,1}(\overline{B_{6\sqrt{n}}})} \le C_e \cdot C(n) \|u\|_{L^\infty(B_{7\sqrt{n}})} \le C_e \cdot C(n)$.

$$u-h \in S\left(\frac{\lambda}{n}, \Lambda, f - F(D^2h, x)\right) \qquad \varphi := f(x) - F(D^2h(x), x) \quad (B_{6\sqrt{n}})$$

$\xrightarrow{\text{Interior } C^\alpha \text{ Holder}}$ $\|u\|_{C^\alpha(\overline{B_{7\sqrt{n}}})} \le C(1 + \|f\|_{L^n(B_{8\sqrt{n}})})$ (1)

$\xrightarrow{\text{Lemma 10}}$ $\|h\|_{C^{\frac{\alpha}{2}}(\overline{B_{7\sqrt{n}}})} \le C\cdot \|u\|_{C^\alpha(\overline{B_{7\sqrt{n}}})} \le C(1 + \|f\|_{L^n(B_{8\sqrt{n}})})$ (2)

choose undetermined $\delta \in (0,1)$. $\forall x_0 \in B_{7\sqrt{n}-\delta}$. $x_1 \in \partial B_\delta(x_0) \subseteq \overline{B_{7\sqrt{n}}}$.

$$\delta^2 \|D^2h(x_0)\| = \delta^2 \|D^2(h - h(x_0))\| \overset{C^{1,1}}{\le} C\|h - h(x_0)\|_{L^\infty(B_\delta(x_0))} \overset{(2)}{\le} C\,\delta^{\frac{\alpha}{2}} \cdot \|h\|_{C^{\frac{\alpha}{2}}(B_{7\sqrt{n}})} \le C\cdot \delta^{\frac{\alpha}{2}} (1 + \|f\|_{L^n(B_{8\sqrt{n}})})$$

$$\Longrightarrow \|D^2h(x_0)\| \le C\delta^{\frac{\alpha}{2}-2} (1 + \|f\|_{L^n(B_{8\sqrt{n}})}) \qquad (3)$$

$\xrightarrow{\text{By def of } \beta}$ $|F(D^2h(x_0), x_0)| \le \beta(x_0)\cdot C\cdot \delta^{\frac{\alpha}{2}-2}(1 + \|f\|_{L^n(B_{8\sqrt{n}})})$ (4)

$+ \|\varphi\|_{L^n(B_{7\sqrt{n}-\delta})}$

$$\|u-h\|_{L^\infty(B_{7\sqrt{n}-\delta})} \le \|u-h\|_{L^\infty(\partial B_{7\sqrt{n}-\delta})} + C\cdot \|\varphi\|_{L^n(B_{7\sqrt{n}-\delta})}$$

$$\|u-h\|_{L^\infty(\partial B_{7\sqrt{n}-\delta})} \overset{u-h=0\ \partial B_{7\sqrt{n}}}{\leq} C\cdot \delta^{\frac{\alpha}{2}} \|u-h\|_{C^{\frac{\alpha}{2}}(\bar{B}_{7\sqrt{n}})}$$

$$\overset{(1),(2)}{\leq} C\,\delta^{\frac{\alpha}{2}}(1+\|f\|_{L^n(B_{8\sqrt{n}})})$$

$$\Rightarrow: \|u-h\|_{L^\infty(B_{7\sqrt{n}-\delta})} + \|\varphi\|_{L^n(B_{7\sqrt{n}-\delta})} \leq C\delta^{\frac{\alpha}{2}}(1+\|f\|_{L^n(B_{8\sqrt{n}})}) + C\|\varphi\|_{L^n(B_{7\sqrt{n}-\delta})}$$

$$\overset{(4)}{\leq} C\delta^{\frac{\alpha}{2}}(1+\|f\|_{L^n(B_{8\sqrt{n}})}) + C\cdot(\|f\|_{L^n(B_{7\sqrt{n}-\delta})} + \|\beta\|_{L^n(B_{7\sqrt{n}-\delta})}\delta^{\frac{\alpha}{2}-2}(1+\|f\|_{L^n(B_{8\sqrt{n}})}))$$

$$\leq C(\delta^{\frac{\alpha}{2}} + \|\beta\|_{L^n(B_{7\sqrt{n}-\delta})}\delta^{\frac{\alpha}{2}-2})(1+\|f\|_{L^n(B_{8\sqrt{n}})}) + C\|f\|_{L^n(B_{7\sqrt{n}-\delta})}$$

choose $\delta = \varepsilon^{\frac{1}{2}}$ $\nu = \frac{\alpha}{4}$ $\Longrightarrow \leq C\cdot(\varepsilon^{\nu} + \|f\|_{L^n(B_{8\sqrt{n}})})$ □

Lemma 12. let $\varepsilon \in (0,1)$. $B_{8\sqrt{n}} \subseteq \Omega$ Ω is a bounded domain. $u \in C(\Omega)$ $F(D^2u, x) = f(x)$ in $B_{8\sqrt{n}}$ such that $\|u\|_{L^\infty(B_{7\sqrt{n}})} \leq 1$ & $-|x|^2 \leq u(x) \leq |x|^2$ $\Omega \setminus B_{6\sqrt{n}}$.

If $\|f\|_{L^n(B_{8\sqrt{n}})} \leq \varepsilon$ & $\|\beta\|_{L^n(B_{7\sqrt{n}})} \leq \varepsilon$ $F(D^2w, 0) = 0$ has $C^{1,1}$ interior estimates. (C_e)

Then

$$|G_M(u,\Omega) \cap Q_1| \geq 1 - \varepsilon_0$$

$\varepsilon = \varepsilon(n, \lambda, \Lambda, C_e, \varepsilon_0)$ $M = M(n, C_e)$

proof: $h(x) = \begin{cases} h & \text{in } B_{7\sqrt{n}} \quad (\text{in Lemma 11}) \\ u & \text{in } B_{7\sqrt{n}}^c \end{cases}$

$\Rightarrow \|h\|_{L^\infty(B_{7\sqrt{n}})} \leq 1$ $\|u-h\|_{L^\infty(\Omega)} \leq 2.$ $\Rightarrow -2-|x|^2 \leq h(x) \leq 2+|x|^2$ in $\cancel{\Omega} B_{6\sqrt{n}}$.

Claim: $\exists N = N(n, C_e)$ such that $Q_1 \subseteq G_N(h, \Omega)$.

By: $\forall x_1 \in Q_1$

$$h(x) = h(x_1) + \nabla h(x_1)(x-x_1) + (x-x_1)^T \nabla^2 h(\xi)(x-x_1) \quad \text{in } \overline{B_{6\sqrt{n}}}$$

$$\leq h(x_1) + \nabla h(x_1)(x-x_1) + 2(C(n)C_e)|x-x_1|^2$$

$$(\geq h(x_1) + \nabla h(x_1)(x-x_1) + (-2)(C_nC_e)(x-x_1)^2)$$

here: $\|\nabla^2 h(x)\|_{L^\infty(B_{6\sqrt{n}})} + \|\nabla h\|_{L^\infty(B_{6\sqrt{n}})} \leq C(n)C_e$ (As Lemma 11) (WLOG $\underbrace{\ }_{\text{Assume}}$ $C(n)C_e \geq 4$) $C := C(n)C_e$

$\forall x \in B_{6\sqrt{n}}^c$, we have:

$$h(x_1) + \nabla h(x_1)(x-x_1) + 2C|x-x_1|^2 \geq 2\cdot\frac{3}{4}C|x|^2 - C(|x|+\sqrt{n}) - 1$$

$$\geq \frac{1}{4}C|x|^2 + [C|x|^2 - C(|x|+\sqrt{n})] + \frac{1}{4}C|x|^2 - 1$$

$$\geq |x|^2 + 2$$

also $h(x_1) + \nabla h(x_1)(x-x_1) - 2C|x-x_1|^2 \leq -|x|^2 - 2$ in $B_{6\sqrt{n}}^c$.

$\Rightarrow$: $x_1 \in G_N(h, \Omega)$ $N = 2C(n)C_e$.

$w := \alpha(u-h)$ α is undetemined $w \in S(\frac{\lambda}{n}, \Lambda, \alpha\varphi)$

By Lemma 11.

$\|u-h\|_{L^\infty(B_{6\sqrt{n}})} \le 2C\varepsilon^\nu \qquad \|\varphi\|_{L^n(B_{6\sqrt{n}})} \le 2C\varepsilon^\nu$

$\Rightarrow \|w\|_{L^\infty(B_{6\sqrt{n}})} \le \alpha\, 2C\varepsilon^\nu \qquad \|\alpha\varphi\|_{L^n(B_{6\sqrt{n}})} \le \alpha\cdot 2C\varepsilon^\nu$

for using Lemma 5, we need $\alpha 2C\varepsilon^\nu \le 1$ & $\alpha 2C\varepsilon^\nu \le \delta_0$ we can choose $\alpha = \frac{\min(1,\delta_0)}{2C\varepsilon^\nu}$.

$\overset{\text{Lemma 8}}{\Longrightarrow} \quad |A_t(w,\Omega)\cap Q_1| \le Ct^{-\mu} \iff |G_t(w,\Omega)\cap Q_1| \ge 1-Ct^{-\mu}$

$\iff |G_{\frac{t}{\alpha}}(u-h,\Omega)\cap Q_1| \ge 1-Ct^{-\mu}$

$\iff |G_t(u-h,\Omega)\cap Q_1| \ge 1-C\cdot\varepsilon^{\nu\mu}t^{-\mu} \ge 1-\varepsilon_0$

choose ε st last inequality. (all $C=C(n,\lambda,\Lambda,C_e)$) $(t=N)$

By $G_N(u-h,\Omega)\cap Q_1 \cap G_N(h,\Omega) \subseteq G_{2N}(u,\Omega)\cap Q_1$

$\Longrightarrow |G_{2N}(u,\Omega)\cap Q_1| \ge 1-\varepsilon_0$ □

Fact: $G_t(f,\Omega)\cap G_s(g,\Omega) \subseteq G_{t+s}(f+g,\Omega)$ (By definition) □

Lemma 13. Let $\varepsilon_0\in(0,1)$. Ω be a bounded domain st $B_{8\sqrt{n}}\subseteq\Omega$, $u\in C(\Omega)$. $F(D^2u,x)=f(x)$ $(B_{8\sqrt{n}})$
Assume $\|f\|_{L^n(B_{8\sqrt{n}})}\le\varepsilon$. $\|\beta\|_{L^n(B_{7\sqrt{n}})}\le\varepsilon$ $F(D^2w,0)=0$ has $C^{1,1}$ interior estimates (with C_e) Then

$$G_1(u,\Omega)\cap Q_3\neq\emptyset \Longrightarrow |G_M(u,\Omega)\cap Q_1| \ge 1-\varepsilon_0$$

$M=M(n,C_e)$ $\varepsilon=\varepsilon(\lambda,n,\Lambda,C_e,\varepsilon_0)$.

proof: $x_1\in G_1(u,\Omega)\cap Q_3$

$\Longrightarrow \exists L$ affine $\quad -\frac{1}{2}|x-x_1|^2 \le u-L(x) \le \frac{1}{2}|x-x_1|^2$ in Ω

$$v := \frac{u-L(x)}{C(n)}$$

st $\|v\|_{L^\infty(B_{8\sqrt{n}})} \le 1$ & $\quad (C(n)=(11\sqrt{n})^2)$

$-|x|^2 \le v(x) \le |x|^2$ in $\Omega\setminus B_{8\sqrt{n}}$

$G(M,x) := C(n)^{-1}F(C(n)M,x) \Longrightarrow G(D^2v,x)=C(n)^{-1}f(x)$ in $B_{8\sqrt{n}}$.

$\Rightarrow: \exists M_0=M_0(n,C_e), |G_{M_0}(v,\Omega)\cap Q_1| \ge 1-\varepsilon_0$

$\Longleftrightarrow: \quad |G_{C(n)M_0}(u,\Omega)\cap Q_1| \ge 1-\varepsilon_0$

$M := M_0\cdot C(n)$ □

Lemma 14. Let $\varepsilon_0 \in (0,1)$. $F(D^2u, x) = f(x)$ in $B_{8\sqrt{n}}$. Assume that $\|u\|_{L^\infty(B_{8\sqrt{n}})} \le 1$ $\|f\|_{L^n(B_{8\sqrt{n}})} \le \varepsilon$

$$\left(⨍_{B_r(x_0)} \beta(x, x_0)^n\right)^{\frac{1}{n}} \le \varepsilon \quad \forall B_r(x_0) \subseteq B_{8\sqrt{n}}.$$

$F(D^2w, x_0) = 0$ has $C^{1,1}$ interior estimates. f 0-extension outside $B_{8\sqrt{n}}$ $k = 0, 1, 2, \cdots$

$$A = A_{M^{k+1}}(u, B_{8\sqrt{n}}) \cap Q_1$$
$$B = A_{M^k}(u, B_{8\sqrt{n}}) \cap Q_1 \cup \{x \in Q_1 : m(f^n)(x) \ge (CM^k)^n\}$$

Then
$$|A| \le \varepsilon_0 |B|$$

$M = M(n, \varepsilon_0) (>1)$. C, ε depend on n, λ, Λ, C_0, ε_0

proof: By Lemma 12 ($\Omega = B_{8\sqrt{n}}$) $\Longrightarrow$ $|G_{M^{k+1}}(u, B_{8\sqrt{n}}) \cap Q_1| \ge |G_M(u, B_{8\sqrt{n}}) \cap Q_1| \ge 1 - \varepsilon_0$

$\Longleftrightarrow |A| \le \varepsilon_0$

$\forall$ Q is a dyadic cube $Q = Q_{\frac{1}{2^i}}(x_0)$ $|Q \cap A| > \varepsilon_0 |Q|$ $\longrightarrow$ $\tilde{Q} \subseteq B$

If not. $\exists x_1 \in \tilde{Q}$ $x_1 \in G_{M^k}(u, B_{8\sqrt{n}}) \cap \tilde{Q}$

$m(f^n)(x_1) < (CM^k)^n$

$$\tilde{u}(y) := u\left(x_0 + \tfrac{1}{2^i} y\right) \frac{2^{2i}}{M^k} \quad \text{in } 2^i(B_{8\sqrt{n}} - x_0) := \tilde{\Omega}$$
$$\tilde{f}(y) := \frac{1}{M^k} f\left(x_0 + \tfrac{1}{2^i} y\right)$$

$$G(D^2w, y) := \frac{1}{M^k} F\left(M^k D^2 w, x_0 + \tfrac{1}{2^i} y\right) \quad (G(D^2w, 0) = 0 \text{ has } C^{1,1} \text{ interior estimates})$$
$$\Longrightarrow G(D^2\tilde{u}, y) = \tilde{f}(y) \quad (\tilde{\Omega}).$$

$$\beta_G(y) = \beta_G(y, 0) = \sup_{A \in S \setminus \{0\}} \frac{|F(M^kA, x_0 + \frac{1}{2^i}y) - F(M^kA, x_0)|}{\|A\|} \frac{1}{M^k}$$
$$= \beta_F(x, x_0) \qquad x = x_0 + \tfrac{1}{2^i} y.$$

$B_{\frac{7\sqrt{n}}{2^i}}(x_0) \subseteq B_{8\sqrt{n}} \Longrightarrow B_{7\sqrt{n}} \subseteq \tilde{\Omega}$

$$\|\beta_G\|_{L^n(B_{7\sqrt{n}})} = \left(\int_{B_{7\sqrt{n}}} \beta_F\left(x_0 + \tfrac{1}{2^i} y, x_0\right)^n dy\right)^{\frac{1}{n}}$$
$$= \left(\omega(1) \cdot \left(\frac{7\sqrt{n}}{2^i}\right)^n \cdot ⨍_{B_{\frac{7\sqrt{n}}{2^i}}(x_0)} \beta_F(x, x_0)\, dx \cdot 2^{in}\right)^{\frac{1}{n}}$$
$$\le \omega(1)^{\frac{1}{n}} (7\sqrt{n}) \cdot \varepsilon \qquad \omega(1) := |B(1)|.$$

$$\|\tilde{f}(y)\|^n_{L^n(B_{8\sqrt{n}})} = \frac{1}{M^{kn}} \int_{B_{\frac{8\sqrt{n}}{2^\nu}}(x_0)} f(x)\,dx \cdot 2^{\nu n}$$

$$\leq \frac{2^{\nu n}}{M^{kn}} \cdot \left(\frac{19\sqrt{n}}{2^\nu}\right)^n \int_{Q_{\frac{M\sqrt{n}}{2^\nu}}(x_0)} f(x)\,dx$$

$$\leq C^n (19\sqrt{n})^n. \leq \varepsilon$$

$C = \varepsilon^{\frac{1}{n}} / 19\sqrt{n} = C(n, \lambda, \Lambda, C_0, \varepsilon_0)$

$G_1(\tilde{u}, \tilde{\Omega}) \cap Q_3 \neq \phi \Leftarrow (G_{M^k}(u, \Omega) \cap \tilde{Q} \neq \phi)$

$\overset{\text{Lemma 13}}{\Longrightarrow} \quad |G_M(\tilde{u}, \tilde{\Omega}) \cap Q_1| \geq 1 - \varepsilon_0$

$\Leftrightarrow \begin{cases} |G_{M^{k+1}}(u, \Omega) \cap Q| \geq (1-\varepsilon_0)|Q| \\ |A_{M^{k+1}}(u, \Omega) \cap Q| > \varepsilon_0 |Q| \end{cases} \Rightarrow$ contradiction!

□

proof of proposition 2.: ε. C ∞ in Lemma 14.

choose $\varepsilon_0 M^p = \frac{1}{2}$

Define $\alpha_k := |A_{M^k}(B_{8\sqrt{n}}) \cap Q_1| \quad \beta_k = |\{x \in Q_1 : m(f^n)(x) \geq (CM^k)^n\}|$

$$\alpha_{k+1} \leq \varepsilon_0(\alpha_k + \beta_k) \leq \varepsilon_0(\varepsilon_0(\alpha_{k-1} + \beta_{k-1}) + \beta_k)$$

$$\cdots \leq \varepsilon_0^{k+1} + \varepsilon_0\beta_k + \varepsilon^2\beta_{k-1} + \cdots + \varepsilon_0^k\beta_1 + \varepsilon_0^{k+1}\beta_0$$

$$\Rightarrow \alpha_k \leq \varepsilon_0^k + \sum_{\nu=0}^{k-1} \varepsilon_0^{k-\nu}\beta_\nu$$

$p > n$

$$\|m(f^n)\|_{L^{\frac{p}{n}}(Q_1)} \leq C(n,p)\,\|f\|^n_{L^p(Q_1)} \leq C \; (= C(n, \lambda, \Lambda, C_0, p))$$

$$\overset{\text{Lemma 3}}{\Longrightarrow} \sum_{k\geq 1} (M^n)^{\frac{p}{n}k} \beta_k \leq C$$

For $\|\Theta\|_{L^p(B_{\frac{1}{2}})} < +\infty \qquad d_\Theta(t) := |\{t \in B_{\frac{1}{2}} \mid \Theta(x) > t\}| \leq |A_t(u, B_{\frac{1}{2}})|$

$$\leq |A_t(u, B_{8\sqrt{n}}) \cap Q_1|$$

$$\sum_{k\geq 1} M^{kp} d_\Theta(M^k) \leq \sum_{k\geq 1} M^{kp} |A_{M^k}(u, B_{8\sqrt{n}}) \cap Q_1|$$

$$\leq \sum_{k\geq 1} M^{kp} \left(\varepsilon_0^k + \sum_{\nu=0}^{k-1} \varepsilon_0^{k-\nu}\beta_\nu\right)$$

$$\leq 1 + \varepsilon_0 M^p \sum_{k\geq 1}\sum_{\nu=0}^{k-1} M^{k-1} = 1 + \varepsilon M^p \sum_{k=0}^{\infty}\sum_{\nu=0}^{k} (M^p\varepsilon_0)^{k-\nu} M^{p\nu}\beta_\nu$$

$$= 1 + \sum_{k\geq 1} (\varepsilon M^p)^k \sum_{\nu\geq 0} M^{p\nu}\beta_\nu < +\infty$$

□