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Pseudo-harmonic Maps from Complete Noncompact Pseudo-Hermitian Manifolds to Regular Balls

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Abstract

In this paper, we give an estimate of sub-Laplacian of Riemannian distance functions in pseudo-Hermitian geometry which plays a similar role as Laplacian comparison theorem in Riemannian geometry, and deduce a prior horizontal gradient estimate of pseudo-harmonic maps from pseudo-Hermitian manifolds to regular balls of Riemannian manifolds. As an application, Liouville theorem is established under the conditions of nonnegative pseudo-Hermitian Ricci curvature and vanishing pseudo-Hermitian torsion. Moreover, we obtain the existence of pseudo-harmonic maps from complete noncompact pseudo-Hermitian manifolds to regular balls of Riemannian manifolds.

Keywords Sub-Laplacian comparison theorem · Regular ball · Pseudo-harmonic maps · Horizontal gradient estimate · Liouville theorem · Existence theorem

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1 Introduction

Inspired by Eells–Sampson's theorem, one natural problem is to consider the existence of harmonic maps from complete noncompact Riemannian manifolds. Usually some convexity conditions on the images will lead this existence (cf. [10,17,18]). Based on elliptic theory, some existence theorems have been studied for generalized harmonic maps (cf. [7,20]).

The pseudo-harmonic map is an analogue of the harmonic map in pseudo-Hermitian geometry. Let (M, θ) be a pseudo-Hermitian manifold of real dimension $2m + 1$ and (N, h) be a Riemannian manifold. The horizontal energy of a smooth map $f : M \to N$ is defined by

$$E_H(f) = \int_M |\mathrm{d}_b f|^2 \theta \wedge (\mathrm{d}\theta)^m, \tag{1.1}$$

where $\mathrm{d}_b f$ is the horizontal part of $\mathrm{d}f$. The pseudo-harmonic map is a critical point of E_H. Hence it locally satisfies the following Euler–Lagrange equation

$$\tau_H^i(f) \triangleq \Delta_b f^i + \sum_{j,k} \Gamma_{jk}^i(f)\langle \mathrm{d}_b f^j, \mathrm{d}_b f^k\rangle = 0, \tag{1.2}$$

where Γ_{jk}^i's are Christoffel symbols of Levi-Civita connection in (N, h). Here Δ_b denotes the sub-Laplacian which is a subelliptic operator enjoying nice regularity as elliptic operators. By heat flow method, the Eells–Sampson's type theorem also holds for pseudo-harmonic maps (cf. [5,21]). The Dirichlet problem of pseudo-harmonic maps to regular balls has also been solved by Jost–Xu (cf. [15]).

This paper studies pseudo-harmonic maps from complete noncompact pseudo-Hermitian manifolds to regular balls. In order to establish some local estimates, we need sub-Laplacian comparison theorem in pseudo-Hermitian manifolds. Actually such kinds of theorems have been investigated for Sasakian manifolds in [1,3,6,16]. However, up to now, there is no satisfactory comparison theorem for a pseudo-Hermitian manifold, which is not Sasakian. For our purpose, we will give a new sub-Laplacian comparison theorem for a pseudo-Hermitian manifold. Note that the Riemannian distance associated with Webster metric has better regularity than the Carnot–Carathéodory distance, and its variational theory is well studied in Riemannian geometry. By the index comparison theorem in Riemannian geometry, we can derive the following estimate of sub-Laplacian of Riemannian distance on pseudo-Hermitian manifolds. Let $B_R(x_0)$ be the Riemannian geodesic ball of radius R centered at $x_0 \in M$.

Theorem 1.1 *Suppose (M^{2m+1}, θ) is a complete pseudo-Hermitian manifold. If for some $k, k_1 \geq 0$,*

$$R_* \geq -k, \text{ and } |A|, |divA| \leq k_1, \quad \text{on } B_R(x_0),$$

where R_* is the pseudo-Hermitian Ricci curvature and A is the pseudo-Hermitian torsion, then there exists $C_1 = C_1(m)$ such that

$$\Delta_b r \le C_1\left(\frac{1}{r} + \sqrt{1 + k + k_1 + k_1^2}\right), \quad on\ B_R(x_0) \setminus Cut(x_0), \tag{1.3}$$

where r is the Riemannian distance from x_0 and $Cut(x_0)$ is the cut locus of x_0.

The proof will be given in Sect. 3. Based on this sub-Laplacian comparison theorem, we will establish the following local prior horizontal gradient estimate of pseudo-harmonic maps by maximum principle.

Theorem 1.2 *Suppose that (M^{2m+1}, θ) is a noncompact complete pseudo-Hermitian manifold and (N, h) is a Riemannian manifold with sectional curvature $K^N \le \kappa$ for some $\kappa \ge 0$. On $B_{2R}(x_0) \subset M$ with $R > 1$,*

$$R_* \ge -k \quad and \quad |A|, |divA| \le k_1, \tag{1.4}$$

for some $k, k_1 \ge 0$. Assume that $f : B_{2R}(x_0) \subset M \to B_D(p_0) \subset N$ is pseudo-harmonic where $B_D(p_0)$ is a regular ball in N. Then the horizontal energy density $|\mathrm{d}_b f|^2$ on $B_R(x_0)$ is uniformly bounded. More precisely,

$$\max_{B_R(x_0)} |\mathrm{d}_b f|^2 \le C_3\left[C_2 + \frac{C_2}{C_2 + R^{-1}} + \frac{1}{R}\right], \tag{1.5}$$

where C_2 is given in Lemma 2.3 which depends on k, k_1 and C_3 depends on k, k_1, κ, D. In particular, if $k = 0$ and $k_1 = 0$, then $C_2 = 0$.

The proof will be given in Sect. 4. A direct application is the following Liouville theorem for pseudo-harmonic maps which is a generalization of the one for harmonic maps by Choi [9].

Theorem 1.3 *Suppose that (M, θ) is a noncompact complete Sasakian manifold with nonnegative pseudo-Hermitian Ricci curvature and (N, h) is a Riemannian manifold with sectional curvature bounded above. Then there is no nontrivial pseudo-Hermitian map from M to any regular ball of N.*

Another application of Theorem 1.2 is the global existence of pseudo-harmonic maps from complete noncompact pseudo-Hermitian manifolds to regular balls which is due to an exhaustion process combined with the Dirichlet existence of pseudo-harmonic maps.

Theorem 1.4 *Suppose that (M, θ) is a complete noncompact pseudo-Hermitian manifold and (N, h) is a Riemannian manifold with sectional curvature bounded from above. Then there is a pseudo-harmonic map from M to any regular ball $B_D(p_0)$ of N.*

The proof will be given in Sect. 5. One may doubt whether the pseudo-harmonic map given by Theorem 1.4 is trivial. We will show an example whose domain is Sasakian with negative pseudo-Hermitian Ricci curvature.

2 Basic Notions

In this section, we present some basic notions of pseudo-Hermitian geometry and pseudo-harmonic maps. For details, readers may refer to [12,25,26]. Recall that a smooth manifold M of real dimension $2m + 1$ is said to be a CR manifold if there exists a smooth rank n complex subbundle $T_{1,0}M \subset TM \otimes \mathbb{C}$ such that

$$T_{1,0}M \cap T_{0,1}M = \{0\} \tag{2.1}$$

$$[\Gamma(T_{1,0}M), \Gamma(T_{1,0}M)] \subset \Gamma(T_{1,0}M), \tag{2.2}$$

where $T_{0,1}M = \overline{T_{1,0}M}$ is the complex conjugate of $T_{1,0}M$. Equivalently, the CR structure may also be described by the real subbundle $HM = Re\,\{T_{1,0}M \oplus T_{0,1}M\}$ of TM which carries an almost complex structure $J : HM \to HM$ defined by $J(X + \overline{X}) = i(X - \overline{X})$ for any $X \in T_{1,0}M$. Since HM is naturally oriented by the almost complex structure J, then M is orientable if and only if there exists a global nowhere vanishing 1-form θ such that $HM = Ker(\theta)$. Any such section θ is referred to as a pseudo-Hermitian structure on M. The Levi form L_θ of a given pseudo-Hermitian structure θ is defined by

$$L_\theta(X, Y) = \mathrm{d}\theta(X, JY) \quad \text{for any } X, Y \in HM.$$

An orientable CR manifold (M, HM, J) is called strictly pseudo-convex if L_θ is positive definite for some θ. Such a quadruple (M, HM, J, θ) is called a pseudo-Hermitian manifold. For simplicity, we denote it by (M, θ).

For a pseudo-Hermitian manifold (M, θ), there exists a unique nowhere zero vector field ξ, called the Reeb vector field, transverse to HM and satisfying $\xi \lrcorner \theta = 1,\ \xi \lrcorner \mathrm{d}\theta = 0$. It gives a decomposition of the tangent bundle TM:

$$TM = HM \oplus \mathbb{R}\xi \tag{2.3}$$

which induces the projection $\pi_H : TM \to HM$. Set $G_\theta = \pi_H^* L_\theta$. Since L_θ is a metric on HM, it is natural to define a Riemannian metric

$$g_\theta = G_\theta + \theta \otimes \theta \tag{2.4}$$

which makes HM and $\mathbb{R}\xi$ orthogonal. The metric g_θ is called Webster metric, which is also denoted by $\langle\cdot,\cdot\rangle$ for simplicity. By requiring $J\xi = 0$, the almost complex structure J can be extended to an endomorphism of TM. Clearly, $\theta \wedge (\mathrm{d}\theta)^m$ differs a constant with the volume form of g_θ. Henceforth it is always regarded as the canonical volume form in pseudo-Hermitian geometry.

It is remarkable that (M, HM, G_θ) could also be viewed as a sub-Riemannian manifold which satisfies the strong bracket generating hypothesis (see Appendix for details). The completeness of a sub-Riemannian manifold is well settled under the Carnot–Carathéorody distance (cf. [24]). Locally, the Carnot–Carathéorody distance and the Riemannian distance associated with the Webster metric g_θ can be controlled by each other (cf. [19]), which leads that the former completeness is equivalent with the latter. In this paper, a pseudo-Hermitian manifold (M,θ) is called complete if it is complete about the Webster metric g_θ.

On a pseudo-Hermitian manifold, there exists a canonical connection ∇, which is called Tanaka–Webster connection (cf. [12]), preserving the horizontal distribution, almost complete structure and Webster metric. Moreover, its torsion T_∇ satisfies

$$T_\nabla(X,Y)=2d\theta(X,Y)\xi \quad \text{and} \quad T_\nabla(\xi,JX)+JT_\nabla(\xi,X)=0. \tag{2.5}$$

The pseudo-Hermitian torsion, denoted by τ, is a symmetric and traceless tensor defined by $\tau(X)=T_\nabla(\xi,X)$ for any $X\in TM$ (cf. [12]). Set

$$A(X,Y)=g_\theta(\tau(X),Y), \quad \text{for any } X,Y\in TM.$$

A pseudo-Hermitian manifold is Sasakian if $\tau\equiv 0$. Sasakian geometry plays important roles in Kähler geometry and Einstein metrics (cf. [4]).

Suppose that (M,θ) is a pseudo-Hermitian manifold of real dimension $2m+1$. Let R be the curvature tensor of the Tanaka–Webster connection. Set

$$R(X,Y,Z,W)=\langle R(Z,W)Y,X\rangle, \quad \text{for any } X,Y,Z,W\in TM.$$

Let $\{\eta_\alpha\}_{\alpha=1}^m$ be a local unitary frame of $T_{1,0}M$ and R_{ABCD} be the components of R under the frame $\{\eta_0=\xi,\eta_\alpha,\eta_{\bar\alpha}\}$. Webster [26] derived the first Bianchi identity, i.e.,

$$R_{\bar\alpha\beta\lambda\bar\mu}=R_{\bar\alpha\lambda\beta\bar\mu}.$$

The other components of R can be expressed by the pseudo-Hermitian torsion and its derivative. For example,

$$R_{\bar\alpha\beta\lambda\mu}=2i(A_{\beta\mu}\delta_{\bar\alpha\lambda}-A_{\beta\lambda}\delta_{\bar\alpha\mu}), \quad R_{\bar\alpha\beta0\mu}=-A_{\beta\mu,\bar\alpha}, \quad R_{\bar\alpha\beta0\bar\mu}=A_{\bar\alpha\bar\mu,\beta},$$

where $A_{\beta\mu,\bar\alpha}$, $A_{\bar\alpha\bar\mu,\beta}$ are the components of ∇A. Tanaka [25] defined the pseudo-Hermitian Ricci tensor R_* by

$$R_*X=-i\sum_{\lambda=1}^m R(\eta_\lambda,\eta_{\bar\lambda})JX \quad \text{for any } X\in T_{1,0}M. \tag{2.6}$$

The pseudo-Hermitian scalar curvature is given by

$$s=\frac{1}{2}trace_{G_\theta}R_*. \tag{2.7}$$

In this paper, we will use Einstein summation convention when there is a repeated index. Denote $R_{\lambda\bar\mu}=R_{\bar\alpha\alpha\lambda\bar\mu}$. Hence by the first Bianchi identity, $R_*\eta_\alpha=R_{\alpha\bar\beta}\eta_\beta$ and $s=R_{\alpha\bar\alpha}$.

Assume that (N,h) is a Riemannian manifold. Let $\{\sigma^i\}$ be an local orthonormal frame of T^*N. Denote the Levi-Civita connection and the Riemannian curvature of

(N, h) by ∇^N and R^N, respectively. Suppose that $f : M \to N$ is a smooth map. The pullback connection on the pullback bundle $f^*(TN)$ and the Tanaka–Webster connection induce a connection on $TM \otimes f^*(TN)$, also denoted by ∇.

Definition 2.1 A smooth map $f : M \to N$ is called pseudo-harmonic if the tensor field

$$\tau_H(f) \triangleq \operatorname{trace}_{G_\theta} \nabla_b d_b f \equiv 0,$$

where $\nabla_b \mathrm{d}_b f$ is the restriction of $\nabla \mathrm{d} f$ onto $HM \times HM$.

Actually, pseudo-harmonic maps are the Dirichlet critical points of the horizontal energy (cf. [2,12])

$$E_H(f) = \frac{1}{2}\int_M |\mathrm{d}_b f|^2 \theta \wedge (\mathrm{d}\theta)^m, \tag{2.8}$$

where $\mathrm{d}_b f$ is the horizontal restriction of $\mathrm{d}f$. The sub-Laplacian $\Delta_b u$ of a smooth function u is defined by

$$\Delta_b u = \operatorname{trace}_{G_\theta} \nabla_b \mathrm{d}_b u, \tag{2.9}$$

which is viewed as the special case of τ_H acting on functions.

Lemma 2.2 (*CR Bochner Formulas, cf.* [5,14,22]) *For any smooth map $f : M \to N$, we have*

$$\begin{aligned}
\frac{1}{2}\Delta_b |\mathrm{d}_b f|^2 &= |\nabla_b \mathrm{d}_b f|^2 + \langle \nabla_b \tau_H(f), \mathrm{d}_b f\rangle + 4i\big(f^i_{\bar\alpha} f^i_{0\alpha} - f^i_\alpha f^i_{0\bar\alpha}\big)\\
&\quad + 2R_{\alpha\bar\beta} f^i_{\bar\alpha} f^i_\beta - 2i(m-2)\big(f^i_\alpha f^i_\beta A_{\bar\alpha\bar\beta} - f^i_{\bar\alpha} f^i_{\bar\beta} A_{\alpha\beta}\big)\\
&\quad + 2\big(f^i_{\bar\alpha} f^j_\beta f^k_{\bar\beta} f^l_\alpha R^N_{ijkl} + f^i_\alpha f^j_\beta f^k_{\bar\beta} f^l_{\bar\alpha} R^N_{ijkl}\big)
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\frac{1}{2}\Delta_b |f_0|^2 &= |\nabla_b f_0|^2 + \langle \nabla_\xi \tau_H(f), f_0\rangle + 2 f^i_0 f^j_\alpha f^k_{\bar\alpha} f^l_0 R^N_{ijkl}\\
&\quad + 2\big(f^i_0 f^i_\beta A_{\bar\beta\bar\alpha,\alpha} + f^i_0 f^i_{\bar\beta} A_{\beta\alpha,\bar\alpha} + f^i_0 f^i_{\bar\beta\bar\alpha} A_{\beta\alpha} + f^i_0 f^i_{\beta\alpha} A_{\bar\beta\bar\alpha}\big),
\end{aligned} \tag{2.11}$$

where f^i_A and f^i_{AB} are the components of df and $\nabla \mathrm{d}f$, respectively, under the orthonormal coframe $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$ of T^*M and an orthonormal frame $\{\sigma_i\}$ of T^*N, and $f_0 = \mathrm{d}f(\xi)$.

Let $\pi_{(1,1)}\nabla_b \mathrm{d}_b f$ be the $(1,1)$-part of $\nabla_b \mathrm{d}_b f$ and

$$\pi^\perp_{(1,1)}\nabla_b \mathrm{d}_b f = \nabla_b \mathrm{d}_b f - \pi_{(1,1)}\nabla_b \mathrm{d}_b f$$

which is orthogonal to $\pi_{(1,1)}\nabla_b \mathrm{d}_b f$. The commutation relation (cf. [5,22])

$$f^i_{\alpha\bar\beta} - f^i_{\bar\beta\alpha} = 2i f^i_0 \delta_{\alpha\bar\beta} \tag{2.12}$$

shows that

$$
\begin{aligned}
|\pi_{(1,1)}\nabla_b \mathrm{d}_b f|^2 &\geq 2\sum_{\alpha=1}^{m} f^i_{\alpha\bar{\alpha}} f^i_{\bar{\alpha}\alpha} \\
&= \frac{1}{2}\sum_{\alpha=1}^{m}\left[|f^i_{\alpha\bar{\alpha}} + f^i_{\bar{\alpha}\alpha}|^2 + |f^i_{\alpha\bar{\alpha}} - f^i_{\bar{\alpha}\alpha}|^2\right] \\
&\geq \frac{1}{2}\sum_{\alpha=1}^{m} |f^i_{\alpha\bar{\alpha}} - f^i_{\bar{\alpha}\alpha}|^2 \\
&= 2m|f_0|^2. \qquad (2.13)
\end{aligned}
$$

Combining with Lemma 2.2, we have the following lemma.

Lemma 2.3 *Suppose that (M^{2m+1},θ) is a pseudo-Hermitian manifold with*

$$
R_* \geq -k, \text{ and } |A|, |div\ A| \leq k_1 \qquad (2.14)
$$

and (N,h) is a Riemannian manifold with sectional curvature

$$
K^N \leq \kappa \qquad (2.15)
$$

for $k, k_1, \kappa \geq 0$. Then there exists $C_2 = C_2(k, k_1)$ such that for any pseudo-harmonic map $f : M \to N$, we have

$$
\begin{aligned}
\Delta_b|\mathrm{d}_b f|^2 \geq &(2-\epsilon)|\nabla_b \mathrm{d}_b f|^2 + 2m\epsilon|f_0|^2 + \epsilon|\pi^{\perp}_{(1,1)}\nabla_b \mathrm{d}_b f|^2 \\
&- \epsilon_1|\nabla_b f_0|^2 - (C_2 + 16\epsilon_1^{-1})|\mathrm{d}_b f|^2 - 2\kappa|\mathrm{d}_b f|^4 \qquad (2.16)
\end{aligned}
$$

and

$$
\Delta_b|f_0|^2 \geq 2|\nabla_b f_0|^2 - 2\kappa|f_0|^2|d_b f|^2 - C_2|\pi^{\perp}_{(1,1)}\nabla_b \mathrm{d}_b f|^2 - C_2|f_0|^2 - C_2|\mathrm{d}_b f|^2, \qquad (2.17)
$$

where ϵ and ϵ_1 are any positive number. In particular, if $k = 0$ and $k_1 = 0$, then $C_2 = 0$.

Proof For (2.16), due to (2.13), Cauchy inequality and the identity

$$
i(f^i_{\bar{\alpha}} f^i_{0\alpha} - f^i_{\alpha} f^i_{0\bar{\alpha}}) = -\langle \nabla_b f_0, \mathrm{d}_b f \circ J\rangle,
$$

it suffice to prove that

$$
f^i_{\bar{\alpha}} f^j_{\beta} f^k_{\bar{\beta}} f^l_{\alpha} R^N_{ijkl} + f^i_{\alpha} f^j_{\beta} f^k_{\bar{\beta}} f^l_{\bar{\alpha}} R^N_{ijkl} \geq -\frac{1}{2}\kappa|\mathrm{d}_b f|^4. \qquad (2.18)
$$

Set

$$df(\eta_\alpha) = t_\alpha + it'_\alpha.$$

Hence due to sectional curvature $K^N \leq \kappa$, a direct calculation shows that

$$\begin{aligned}
& f^i_{\bar\alpha} f^j_\beta f^k_{\bar\beta} f^l_\alpha R^N_{ijkl} + f^i_\alpha f^j_\beta f^k_{\bar\beta} f^l_{\bar\alpha} R^N_{ijkl} \\
&= 2\big(\langle R^N(t_\beta, t_\alpha)t_\beta, t_\alpha\rangle + \langle R^N(t_\beta, t'_\alpha)t_\beta, t'_\alpha\rangle \\
&\quad + \langle R^N(t'_\beta, t_\alpha)t'_\beta, t_\alpha\rangle + \langle R^N(t'_\beta, t'_\alpha)t'_\beta, t'_\alpha\rangle\big) \\
&\geq -2\kappa \sum_{\alpha,\beta=1}^m (|t_\alpha|^2|t_\beta|^2 + |t'_\alpha|^2|t_\beta|^2 + |t_\alpha|^2|t'_\beta|^2 + |t'_\alpha|^2|t'_\beta|^2) \\
&= -2\kappa \left(\sum_{\alpha=1}^m (|t_\alpha|^2 + |t'_\alpha|^2)\right)\left(\sum_{\beta=1}^m (|t_\beta|^2 + |t'_\beta|^2)\right)
\end{aligned}$$

which, combining with

$$|\mathrm{d}_b f|^2 = 2\sum_{\alpha=1}^m \langle df(\eta_\alpha), \mathrm{d}f(\eta_{\bar\alpha})\rangle = 2\sum_{\alpha=1}^m \langle t_\alpha + it'_\alpha, t_\alpha - it'_\alpha\rangle = 2\sum_{\alpha=1}^m (|t_\alpha|^2 + |t'_\alpha|^2),$$

yields (2.18).

Similarly, (2.17) follows from the following process:

$$\begin{aligned}
f^i_0 f^j_\alpha f^k_{\bar\alpha} f^l_0 R^N_{ijkl} &= \langle R^N(t_\alpha - it'_\alpha, f_0)(t_\alpha + it'_\alpha), f_0\rangle \\
&= \langle R^N(t_\alpha, f_0)t_\alpha, f_0\rangle + \langle R^N(t'_\alpha, f_0)t'_\alpha, f_0\rangle \\
&\geq -\kappa|f_0|^2\left(\sum_{\alpha=1}^m (|t_\alpha|^2 + |t'_\alpha|^2)\right) \\
&= -\frac{1}{2}\kappa|f_0|^2|\mathrm{d}_b f|^2.
\end{aligned}$$

□

At the end of this Section, we briefly recall Folland–Stein space. Let (M, θ) be a pseudo-Hermitian manifold and $\Omega \Subset M$. For any $k \in \mathbb{N}$ and $p > 1$, the Folland–Stein space $S^p_k(\Omega)$ is given by

$$S^p_k(\Omega) = \left\{u \in L^p(\Omega) \,\middle|\, \nabla^l_b u \in L^p(\Omega), l = 0, 1, \ldots, k\right\},$$

where $\nabla^l_b u$ is the horizontal restriction of $\nabla^l u$ and its S^p_k-norm is defined by

$$||u||_{S^p_k(\Omega)} = \sum_{l=0}^k ||\nabla^l_b u||_{L^p(\Omega)}, \tag{2.19}$$

which is equivalent to the local Folland–Stein norm in [12] (see Appendix for details). Under this generalized Sobolev space, the interior regularity theorem of subelliptic equations will behave as elliptic ones.

Theorem 2.4 *Suppose that (M,θ) is a pseudo-Hermitian manifold and $\Omega \Subset M$. Assume that $u, v \in L^1_{loc}(\Omega)$ and $\Delta_b u = v$ in the distribution sense. For any $\chi \in C_0^\infty(\Omega)$, if $v \in S^p_k(\Omega)$ with $p > 1$ and $k \in \mathbb{N}$, then $\chi u \in S^p_{k+2}(\Omega)$ and*

$$||\chi u||_{S^p_{k+2}(\Omega)} \leq C_\chi \left(||u||_{L^p(\Omega)} + ||v||_{S^p_k(\Omega)} \right), \tag{2.20}$$

where C_χ only depends on χ.

The proof is based on partition of unity and the corresponding version on coordinate neighborhoods (cf. [12, Theorem 3.17], [23, Theorem 16]). For completeness, we will give the details in Appendix. A direct calculation shows that for any $\sigma \in \Gamma(\otimes^k T^*M)$ and $X_1, \cdots, X_k, X, Y \in \Gamma(HM)$, we have

$$\begin{aligned}
&(\nabla^2\sigma)(X_1, \cdots, X_k; X, Y) - (\nabla^2\sigma)(X_1, \cdots, X_k; Y, X) \\
&= \sum_{i=1}^k \sigma(X_1, \cdots, R(X,Y)X_i, \cdots, X_k) + \left(\nabla_{T_\nabla(X,Y)}\sigma\right)(X_1, \cdots, X_k).
\end{aligned}$$

By taking $\sigma = \nabla^k_b u$ and $X = \eta_\alpha, Y = \eta_{\bar\beta}$, we obtain that

$$\begin{aligned}
2i\delta_{\alpha\bar\beta}\nabla_\xi \nabla^k_b u(X_1, \cdots, X_k) \ &= (\nabla^{k+2}_b u)(X_1, \cdots, X_k; \eta_\alpha, \eta_{\bar\beta}) \\
&\quad -(\nabla^{k+2}_b u)(X_1, \cdots, X_k; \eta_{\bar\beta}, \eta_\alpha) \\
&\quad -\sum_{i=1}^k \nabla^k_b u\big(X_1, \cdots, R(\eta_\alpha, \eta_{\bar\beta})X_i, \cdots, X_k\big)
\end{aligned}$$

which implies that Reeb covariant derivatives can be controlled by horizontal covariant derivatives. Hence the Folland–Stein space may be embedded into some classical Sobolev space which is a generalization of [13, Theorem 19.1].

Theorem 2.5 *Suppose that (M,θ) is a pseudo-Hermitian manifold and $\Omega \Subset M$. Then for any $k \in \mathbb{N}$ and $p > 1$,*

$$S^p_k(\Omega) \subset L^p_{k/2}(\Omega),$$

where $L^p_{k/2}(\Omega)$ is the classical Sobolev space. Moreover, for any $r \in \mathbb{N}$ and $p > \dim M$, there exists $k \in \mathbb{N}$ such that

$$S^p_k(\Omega) \subset C^{r,\alpha}(\Omega).$$

3 Sub-Laplacian Comparison Theorem

In this section, we will deduce Theorem 1.1 which plays a similar role as Laplacian comparison theorem in Riemannian geometry.

Suppose that (M^{2m+1}, θ) is a complete noncompact pseudo-Hermitian manifold. Let r be the Riemannian distance with respect to Webster metric g_θ from a reference point $x_0 \in M$. We formulate all Riemannian symbols with "ˆ" to distinguish with ones in pseudo-Hermitian geometry, such as Levi-Civita connection $\hat{\nabla}$ and Riemannian curvature tensor $\hat{R}$. Lemma 1.3 in [12] shows the relation of Tanaka–Webster connection and Levi-Civita connection associated with Webster metric:

$$\hat{\nabla} = \nabla - (\mathrm{d}\theta + A) \otimes \xi + \tau \otimes \theta + 2\theta \odot J, \tag{3.1}$$

where $2\theta \odot J = \theta \otimes J + J \otimes \theta$. Hence the sub-Laplacian of r can also be calculated by Levi-Civita connection as follows:

$$\Delta_b r = \mathrm{trace}_{G_\theta} \widehat{Hess}(r)\big|_{HM \times HM}, \tag{3.2}$$

where $\widehat{Hess}$ is the Riemannian Hessian.

Let us recall the Index Lemma in Riemannian geometry (cf. [11, p. 212]).

Lemma 3.1 (Index Lemma) *Let $\gamma : [0, a] \to M$ be a Riemannian geodesic without conjugate points to $\gamma(0)$ in $(0, a]$ and X be a Jacobi field along γ with $X \perp \dot{\gamma}$ and $X(0) = 0$. If $V \in \Gamma(TM)\big|_\gamma$ with $V(0) = 0$, $V(a) = X(a)$ and $V \perp \dot{\gamma}$. Then*

$$I_a(X, X) \le I_a(V, V), \tag{3.3}$$

where

$$I_a(V, V) = \int_0^a \left(\left|\hat{\nabla}_{\dot{\gamma}} V\right|^2 - \langle \hat{R}(V, \dot{\gamma})\dot{\gamma}, V\rangle \right) \mathrm{d}t.$$

Now let $\gamma : [0, a] \to M$ be such a geodesic and $\{e_B(a)\}_{B=1}^{2m}$ be an orthonormal basis of $HM\big|_{\gamma(a)}$. Set

$$e_B^\perp(a) = e_B(a) - \langle e_B(a), \nabla r\rangle \nabla r \in TM\big|_{\gamma(a)}$$

which is perpendicular to $\dot{\gamma}(a) = \nabla r\big|_{\gamma(a)}$. Since $\widehat{Hess}(r)(\nabla r, \cdot) = 0$, then

$$\Delta_b r\big|_{\gamma(a)} = \sum_{B=1}^{2m} \widehat{Hess}(r)(e_B(a), e_B(a)) = \sum_{B=1}^{2m} \widehat{Hess}(r)(e_B^\perp(a), e_B^\perp(a)). \tag{3.4}$$

Using the Riemannian exponential map, we could extend $e_B^{\perp}(a)$ as a Jacobi field U_B along γ with

$$U_B(0) = 0, U_B(a) = e_B^{\perp}(a), [U_B, \dot{\gamma}] = 0.$$

Hence we find

$$\begin{aligned}
\widehat{Hess}(r)(e_B^{\perp}(a), e_B^{\perp}(a)) &= \widehat{Hess}(r)(U_B(a), U_B(a)) \\
&= \langle U_B, \hat{\nabla}_{U_B} \nabla r\rangle\big|_{\gamma(a)} \\
&= \langle U_B, \hat{\nabla}_{\dot{\gamma}} U_B\rangle\big|_{\gamma(a)} = \int_0^a \frac{\mathrm{d}}{\mathrm{d}t} \langle U_B, \hat{\nabla}_{\dot{\gamma}} U_B\rangle \mathrm{d}t \\
&= \int_0^a \left(\left|\hat{\nabla}_{\dot{\gamma}} U_B\right|^2 + \langle U_B, \hat{\nabla}_{\dot{\gamma}} \hat{\nabla}_{\dot{\gamma}} U_B\rangle \right) \mathrm{d}t \\
&= I_a(U_B, U_B),
\end{aligned}$$

where the last equation is due to the Jacobi equation. Hence

$$\Delta_b r\big|_{\gamma(a)} = \sum_{B=1}^{2m} I_a(U_B, U_B). \tag{3.5}$$

Lemma 3.2 *Let $e_B(t)$ be the parallel extension of $e_B(a)$ along γ with respect to Tanaka–Webster connection. Suppose the curvature along γ satisfies*

$$\sum_{B=1}^{2m} \langle \hat{R}(e_B, \nabla r)\nabla r, e_B\rangle \geq -\hat{k} \tag{3.6}$$

and the pseudo-Hermitian torsion is bounded, i.e.,

$$|A| \leq k_1, \tag{3.7}$$

for some for $\hat{k}, k_1 \geq 0$. Then there is a constant $C_4 = C_4(m)$ such that

$$\Delta_b r\big|_{\gamma(a)} \leq C_4 \left(\frac{1}{a} + \sqrt{1 + k_1 + k_1^2 + \hat{k}} \right). \tag{3.8}$$

Proof Due to (3.1), we have

$$\begin{aligned}
\hat{\nabla}_{\dot{\gamma}} e_B &= -[\mathrm{d}\theta(\dot{\gamma}, e_B) + A(\dot{\gamma}, e_B)]\xi + \theta(\dot{\gamma}) J e_B \\
&= -[g_\theta(J\dot{\gamma}, e_B) + A(\dot{\gamma}, e_B)]\xi + \theta(\dot{\gamma}) J e_B
\end{aligned}$$

which implies that

$$\sum_{B=1}^{2m}\left|\hat{\nabla}_{\dot{\gamma}}e_B\right|^2 = 2m\,|\theta(\dot{\gamma})|^2 + \sum_{B=1}^{2m}\Big[|g_\theta(J\dot{\gamma},e_B)|^2 + 2g_\theta(J\dot{\gamma},e_B)A(\dot{\gamma},e_B) + |A(\dot{\gamma},e_B)|^2\Big]$$
$$\leq 2m + 2A(\dot{\gamma},J\dot{\gamma}) + \sum_{B=1}^{m}|A(\dot{\gamma},e_B)|^2 \leq 2m + 2k_1 + k_1^2.$$

Set

$$e'_B(t) = e_B(t) - \langle e_B(t),\nabla r\rangle\nabla r \perp \dot{\gamma}, \qquad V_B(t) = \frac{s_\kappa(t)}{s_\kappa(a)}e'_B(t),$$

where

$$s_\kappa(t) = \frac{1}{\sqrt{\kappa}}\sinh(\sqrt{\kappa}t) \quad \text{ and } \quad \kappa = \frac{1}{4m}(4m + 4k_1 + 2k_1^2 + \hat{k}).$$

Hence $V_B(0) = 0$, $V_B(a) = e'_B(a)$, $V_B \perp \dot{\gamma}$ and

$$\sum_{B=1}^{2m}\left|\hat{\nabla}_{\dot{\gamma}}V_B\right|^2 = \sum_{B=1}^{2m}\left|\frac{\dot{s}_\kappa(t)}{s_\kappa(a)}e'_B + \frac{s_\kappa(t)}{s_\kappa(a)}\hat{\nabla}_{\dot{\gamma}}e'_B\right|^2$$
$$\leq \frac{3}{2}\sum_{B=1}^{2m}\left|\frac{\dot{s}_\kappa(t)}{s_\kappa(a)}e'_B\right|^2 + 3\sum_{B=1}^{2m}\left|\frac{s_\kappa(t)}{s_\kappa(a)}\hat{\nabla}_{\dot{\gamma}}e'_B\right|^2$$
$$\leq 4m\left|\frac{\dot{s}_\kappa(t)}{s_\kappa(a)}\right|^2 + (4m + 4k_1 + 2k_1^2)\left|\frac{s_\kappa(t)}{s_\kappa(a)}\right|^2$$

due to Cauchy inequality. By the curvature assumption, the Index lemma and (3.5), we have

$$\Delta_b r\big|_{\gamma(a)} \leq \sum_{B=1}^{2m} I_a(V_B,V_B) = \sum_{B=1}^{2m}\int_0^a\left(\left|\hat{\nabla}_{\dot{\gamma}}V_B\right|^2 - \langle\hat{R}(V_B,\nabla r)\nabla r, V_B\rangle\right)\mathrm{dt}$$
$$= \int_0^a\left(4m\left|\frac{\dot{s}_\kappa(t)}{s_\kappa(a)}\right|^2 + (4m + 4k_1 + 2k_1^2 + \hat{k})\left|\frac{s_\kappa(t)}{s_\kappa(a)}\right|^2\right)\mathrm{dt}$$
$$\leq \frac{4m}{|s_\kappa(a)|^2}\int_0^a\left(|\dot{s}_\kappa(t)|^2 + \kappa|s_\kappa(t)|^2\right)\mathrm{dt}$$
$$= 4m\sqrt{\kappa}\,\coth\sqrt{\kappa}a$$
$$\leq 4m\left(\frac{1}{a} + \sqrt{\kappa}\right)$$

which finishes the proof. □

To prove Theorem 1.1, it suffices to demonstrate (3.6). It can be expressed by pseudo-Hermitian data due to the relationship between the Riemannian curvature tensor $\hat{R}$ and the curvature tensor R associated with Tanaka–Webster connection ∇ (cf. Theorem 1.6 in [12]):

$$\begin{aligned}\hat{R}(X,Y)Z = R(X,Y)Z &+ (LX\wedge LY)Z + 2\mathrm{d}\theta(X,Y)JZ\\ &- g_\theta(S(X,Y),Z)\xi + \theta(Z)S(X,Y)\\ &- 2g_\theta(\theta\wedge\mathcal{O}(X,Y),Z)\xi + 2\theta(Z)(\theta\wedge\mathcal{O})(X,Y),\end{aligned}\tag{3.9}$$

where

$$\begin{aligned}S(X,Y) =&(\nabla_X\tau)Y-(\nabla_Y\tau)X\\ \mathcal{O} =&\tau^2+2J\tau-I\\ L =&\tau+J.\end{aligned}$$

Here I is the identity, that is $I(X)=X$. Note that the left side of (3.6) is independent of the choice of horizontal orthonormal frame of $\{e_B\}_{B=1}^{2m}$. Let $\{e_B\}_{B=1}^{2m}$ be a local real orthonormal basis of HM with $e_{\alpha+m}=Je_\alpha$ for $\alpha=1,\dots m$. Denote $\eta_\alpha=\frac{1}{\sqrt{2}}(e_\alpha-iJe_\alpha)$.

Lemma 3.3 *For $X,Y\in TM$, we have*

$$\begin{aligned}\sum_{B=1}^{2m}\langle\hat{R}(e_B,X)Y,e_B\rangle = &\sum_{B=1}^{2m}\langle R(e_B,X)Y,e_B\rangle - 3\langle\pi_H X,\pi_H Y\rangle\\ &+\langle\tau X,\tau Y\rangle + (2m-|\tau|^2)\theta(X)\theta(Y)\\ &+ div\,\tau(X)\theta(Y).\end{aligned}\tag{3.10}$$

Proof By (3.9) and $e_B\in HM$, we have

$$\begin{aligned}&\sum_{B=1}^{2m}\langle\hat{R}(e_B,X)Y,e_B\rangle\\ &=\sum_{B=1}^{2m}\langle R(e_B,X)Y,e_B\rangle + \sum_{B=1}^{2m}\langle(Le_B\wedge LX)Y,e_B\rangle\\ &\quad+\sum_{B=1}^{2m}2\mathrm{d}\theta(e_B,X)\langle JY,e_B\rangle\\ &\quad+\sum_{B=1}^{2m}\theta(Y)\langle S(e_B,X),e_B\rangle + \sum_{B=1}^{2m}2\theta(Y)\langle(\theta\wedge\mathcal{O})(e_B,X),e_B\rangle.\end{aligned}\tag{3.11}$$

Now we see each term in the right side except the first one. Note that

$$\sum_{B=1}^{2m}\langle (Le_B \wedge LX)Y, e_B\rangle = \sum_{B=1}^{2m}\langle Le_B, Y\rangle\langle LX, e_B\rangle - \langle LX, Y\rangle\langle Le_B, e_B\rangle. \tag{3.12}$$

On one hand, since LX is horizontal and

$$\langle Le_B, Y\rangle = \langle e_B, \tau Y\rangle - \langle e_B, JY\rangle,$$

then we find

$$\begin{aligned}\sum_{B=1}^{2m}\langle Le_B, Y\rangle\langle LX, e_B\rangle &= \langle LX, \tau Y\rangle - \langle LX, JY\rangle \\ &= \langle \tau X, \tau Y\rangle + \langle JX, \tau Y\rangle - \langle \tau X, JY\rangle - \langle JX, JY\rangle \\ &= \langle \tau X, \tau Y\rangle - \langle \pi_H X, \pi_H Y\rangle. \end{aligned} \tag{3.13}$$

Here the last equation is due to $\tau J + J\tau = 0$ by (2.5). On the other hand,

$$\langle Le_B, e_B\rangle = trace_{G_\theta}\tau + trace_{G_\theta} J = 0. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12), the result is

$$\sum_{B=1}^{2m}\langle (Le_B \wedge LX)Y, e_B\rangle = \langle \tau X, \tau Y\rangle - \langle \pi_H X, \pi_H Y\rangle. \tag{3.15}$$

For the third term in (3.11), we have

$$\sum_{B=1}^{2m} 2\mathrm{d}\theta(e_B, X)\langle JY, e_B\rangle = \sum_{B=1}^{2m} 2\langle Je_B, X\rangle\langle JY, e_B\rangle = -2\langle \pi_H X, \pi_H Y\rangle. \tag{3.16}$$

For the fourth term in (3.11), by the formula of S, we have

$$\begin{aligned}\sum_{B=1}^{2m}\langle S(e_B, X), e_B\rangle &= \sum_{B=1}^{2m}\langle (\nabla_{e_B}\tau)X, e_B\rangle \\ &\quad - \sum_{B=1}^{2m}\langle (\nabla_X\tau)e_B, e_B\rangle = \operatorname{div}\tau(X)\end{aligned} \tag{3.17}$$

since τ is traceless. For the fifth term, by the definition of $\mathcal{O}$, we have

$$\begin{aligned}\sum_{B=1}^{2m} 2\langle(\theta \wedge \mathcal{O})(e_B, X), e_B\rangle &= \sum_{B=1}^{2m} -\langle\theta(X)\mathcal{O}(e_B), e_B\rangle \\ &= \sum_{B=1}^{2m} -\theta(X)\langle(\tau^2 + 2J\tau - I)(e_B), e_B\rangle \\ &= \theta(X)(2m - |\tau|^2)\end{aligned} \tag{3.18}$$

due to

$$\begin{aligned}-\sum_{B=1}^{2m}\langle J\tau(e_B), e_B\rangle &= \sum_{B=1}^{2m}\langle\tau J e_B, e_B\rangle \\ &= \sum_{\alpha=1}^{m}\langle\tau J e_\alpha, e_\alpha\rangle + \langle\tau J^2 e_\alpha, J e_\alpha\rangle = 0.\end{aligned}$$

By substituting (3.15), (3.16), (3.17) and (3.18) to (3.11), we get (3.10). □

Tanaka [25] obtained the following version of first Bianchi identity of R:

$$\mathcal{S}\left(R(X, Y)Z\right) = 2\mathcal{S}\left(\mathrm{d}\theta(X, Y)\tau(Z)\right), \tag{3.19}$$

where $\mathcal{S}$ stands for the cyclic sum with respect to $X, Y, Z \in HM$. One can prove it by applying Riemannian first Bianchi identity to (3.9).

Lemma 3.4 *For any $X, Y \in TM$, we have*

$$\langle R_* X, Y\rangle = \sum_{B=1}^{2m}\langle R(e_B, \pi_H X)\pi_H Y, e_B\rangle - 2(m-1)A(X, JY). \tag{3.20}$$

Proof Since JX is horizontal, we can use the first Bianchi identity (3.19) and obtain

$$\begin{aligned}&-i\sum_{\alpha=1}^{m} R(\eta_\alpha, \eta_{\bar\alpha})JX - i\sum_{\alpha=1}^{m} R(\eta_{\bar\alpha}, JX)\eta_\alpha - i\sum_{\alpha=1}^{m} R(JX, \eta_\alpha)\eta_{\bar\alpha} \\ &= -i\sum_{\alpha=1}^{m} 2\mathrm{d}\theta(\eta_\alpha, \eta_{\bar\alpha})\tau JX \\ &\quad - i\sum_{\alpha=1}^{m} 2\mathrm{d}\theta(\eta_{\bar\alpha}, JX)\tau\eta_\alpha - i\sum_{\alpha=1}^{m} 2\mathrm{d}\theta(JX, \eta_\alpha)\tau\eta_{\bar\alpha}\end{aligned}$$

$$
\begin{aligned}
&= 2m\tau JX - 2\sum_{\alpha=1}^{m} \tau J\left(\langle \eta_{\bar\alpha}, X\rangle \eta_\alpha + \langle \eta_\alpha, X\rangle \eta_{\bar\alpha}\right)\\
&= 2(m-1)\tau JX. \qquad (3.21)
\end{aligned}
$$

On the other hand, note that

$$
\begin{aligned}
& i\sum_{\alpha=1}^{m} R(\eta_{\bar\alpha}, JX)\eta_\alpha + i\sum_{\alpha=1}^{m} R(JX, \eta_\alpha)\eta_{\bar\alpha}\\
&\quad = -i\sum_{\alpha=1}^{m} R(JX, \eta_{\bar\alpha})\eta_\alpha + i\sum_{\alpha=1}^{m} R(JX, \eta_\alpha)\eta_{\bar\alpha}\\
&\quad = -J\left(\sum_{\alpha=1}^{m} R(JX, \eta_{\bar\alpha})\eta_\alpha + R(JX, \eta_\alpha)\eta_{\bar\alpha}\right)\\
&\quad = -J\left(\sum_{B=1}^{2m} R(JX, e_B)e_B\right). \qquad (3.22)
\end{aligned}
$$

Substituting (3.22) into (3.21), we obtain

$$
\langle R_* X, Y\rangle = \sum_{B=1}^{2m} \langle R(e_B, JX)JY, e_B\rangle + 2(m-1)A(JX, Y).
$$

By replacing X, Y by JX, JY, the proof is finished. □

For any $Y \in HM$, using (3.9), we have

$$
\sum_{B=1}^{2m} \langle \hat{R}(e_B, \xi)Y, e_B\rangle = \sum_{B=1}^{2m} \langle R(e_B, \xi)Y, e_B\rangle
$$

and

$$
\sum_{B=1}^{2m} \langle \hat{R}(e_B, Y)\xi, e_B\rangle = \sum_{B=1}^{2m} \langle S(e_B, Y), e_B\rangle = \operatorname{div} \tau(Y).
$$

Applying the symmetric property of Riemannian curvature, we get

$$
\sum_{B=1}^{2m} \langle R(e_B, \xi)Y, e_B\rangle = \operatorname{div} \tau(Y). \qquad (3.23)
$$

Combing Lemmas 3.3, 3.4 and (3.23), we obtain the following lemma.

Lemma 3.5 *For any $X, Y \in TM$, we have*

$$\begin{aligned}\sum_{B=1}^{2m} \langle \hat{R}(e_B, X)Y, e_B\rangle &= \langle R_* X, Y\rangle + 2(m-1)A(X, JY) \\ &\quad + \langle \tau X, \tau Y\rangle - 3\langle \pi_H X, \pi_H Y\rangle \\ &\quad + (2m - |\tau|^2)\theta(X)\theta(Y) \\ &\quad + \operatorname{div} \tau(X)\theta(Y) + \operatorname{div} \tau(Y)\theta(X). \end{aligned} \tag{3.24}$$

Hence Theorem 1.1 can be obtained by Lemmas 3.2 and 3.5.

4 Horizontal Gradient Estimates

Suppose that (M^{2m+1}, θ) is a complete noncompact pseudo-Hermitian manifold. Let r be the Riemannian distance function from $x_0 \in M$ associated with the Webster metric g_θ and B_R be the geodesic ball of radius R centered at x_0. Assume that

$$R_* \geq -k, \text{ and } |A|, |\mathrm{div} A| \leq k_1, \quad \text{on } B_{2R}$$

for some $R \geq 1$. Choose a cut-off function $\varphi \in C^\infty([0, \infty))$ such that

$$\varphi\big|_{[0,1]} = 1, \quad \varphi\big|_{[2,\infty)} = 0, \quad -C_5'|\varphi|^{\frac{1}{2}} \leq \varphi' \leq 0,$$

where C_5' is a universal constant. By defining $\chi(r) = \varphi(\frac{r}{R})$ and using Theorem 1.1, we find that

$$\frac{|\nabla_b \chi|^2}{\chi} \leq \frac{C_5}{R^2}, \quad \Delta_b \chi \geq -\frac{C_5}{R}, \qquad \text{on } B_{2R} \setminus Cut(x_0), \tag{4.1}$$

where $C_5 = C_5(m, k, k_1)$.

Suppose that (N, h) is a Riemannian manifold with sectional curvature

$$K^N \leq \kappa$$

for some $\kappa \geq 0$. Denote the Riemannian distance function from $p_0 \in N$ by ρ. Let $B_D = B_D(p_0)$ be a regular ball of radius D around p_0, that is $D < \frac{\pi}{2\sqrt{\kappa}}$ and B_D lies inside the cut locus of p_0 where $\frac{\pi}{2\sqrt{\kappa}} = +\infty$ if $\kappa = 0$. Set

$$\phi(t) = \begin{cases} \frac{1-\cos(\sqrt{\kappa}t)}{\kappa}, & \kappa > 0 \\ \frac{t^2}{2}, & \kappa = 0 \end{cases},$$

and

$$\psi(q) = \phi \circ \rho(q).$$

Obviously, ϕ is an increasing function and ψ is at least C^2 in the cut locus of p_0. Moreover, Hessian comparison theorem shows that

$$\text{Hess}\ \psi \geq \cos(\sqrt{\kappa}\rho) \cdot h. \tag{4.2}$$

Lemma 4.1 *For any* $0 < D < \frac{\pi}{2\sqrt{\kappa}}$, *there exist* $\nu \in [1, 2)$, $b > \phi(D)$ *and* $\delta > 0$ *only depending on D such that*

$$\nu \frac{\cos(\sqrt{\kappa}t)}{b - \phi(t)} - 2\kappa > \delta, \quad \forall t \in [0, D]. \tag{4.3}$$

Proof For the case $\kappa > 0$, it suffices to find $\nu \in [1, 2)$ and $b > \phi(D)$ such that

$$\phi(D) < b < \inf_{s\in[0,\phi(D)]} \left(\frac{\nu}{2\kappa} + \left(1 - \frac{\nu}{2}\right) s\right), \tag{4.4}$$

which is obvious due to $\phi(D) < \frac{1}{\kappa}$.

The case $\kappa = 0$ is obvious by choosing $\nu = 1$. □

Assume that $f : B_{2R}(x_0) \subset M \to B_D(p_0)$ is a pseudo-harmonic map. By (4.2), we have the following estimate:

Lemma 4.2 *Let* ν, b, δ *be given in Lemma* 4.1. *Then*

$$\nu \frac{\Delta_b \psi \circ f}{b - \psi \circ f} - 2\kappa |\mathrm{d}_b f|^2 \geq \delta |\mathrm{d}_b f|^2. \tag{4.5}$$

To estimate $|\mathrm{d}_b f|^2$, we consider the following auxiliary function

$$\Phi_{\mu\chi} = |\mathrm{d}_b f|^2 + \mu\chi |f_0|^2,$$

where μ will be determined later.

Lemma 4.3 *Suppose* μ *and* ϵ *satisfy*

$$C_2\mu \leq \epsilon \leq 1.$$

If $\chi(x) \neq 0$ *and* $\Phi_{\mu\chi}(x) \neq 0$, *then at x, we have*

$$\begin{aligned} \Delta_b \Phi_{\mu\chi} \geq & \frac{1-\epsilon}{2} \frac{|\nabla_b \Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}} - 2\kappa |\mathrm{d}_b f|^2 \Phi_{\mu\chi} \\ & + \left(2m\epsilon - C_2\mu\chi - 4\epsilon^{-1}\mu\chi^{-1}|\nabla_b \chi|^2 + \mu\Delta_b\chi\right)|f_0|^2 \\ & - \left[C_2 + C_2\mu\chi + 16(\epsilon\mu\chi)^{-1}\right] |\mathrm{d}_b f|^2. \end{aligned} \tag{4.6}$$

Proof Using (2.16) and (2.17) with $\epsilon_1 = \epsilon\mu\chi$, we have

$$\begin{aligned}\Delta_b\Phi_{\mu\chi} =&\Delta_b(|\mathrm{d}_b f|^2 + \mu\chi|f_0|^2)\\ \geq&(2-\epsilon)(|\nabla_b \mathrm{d}_b f|^2 + \mu\chi|\nabla_b f_0|^2)\\ &+ 4\mu\langle\nabla_b\chi\otimes f_0, \nabla_b f_0\rangle - 2\kappa\Phi_{\mu\chi}|\mathrm{d}_b f|^2\\ &+ [2m\epsilon - C_2\mu\chi + \mu\Delta_b\chi]\,|f_0|^2\\ &- \Big[C_2 + C_2\mu\chi + 16(\epsilon\mu\chi)^{-1}\Big]\,|\mathrm{d}_b f|^2. \end{aligned}\tag{4.7}$$

By Cauchy inequality, we have the following estimate

$$\begin{aligned}|\nabla_b\Phi_{\mu\chi}|^2 &= |\nabla_b(|\mathrm{d}_b f|^2 + \mu\chi|f_0|^2)|^2\\ &= |\nabla_b\langle \mathrm{d}_b f + \sqrt{\mu\chi} f_0\otimes\theta, \mathrm{d}_b f + \sqrt{\mu\chi} f_0\otimes\theta\rangle|^2\\ &= 4\,\Big|\Big\langle \mathrm{d}_b f + \sqrt{\mu\chi} f_0\otimes\theta, \nabla_b \mathrm{d}_b f\\ &\quad +\sqrt{\mu\chi}\nabla_b f_0\otimes\theta + \sqrt{\mu}\frac{\nabla_b\chi}{2\sqrt{\chi}}\otimes f_0\otimes\theta\Big\rangle\Big|^2\\ &\leq 4\Big|\mathrm{d}_b f + \sqrt{\mu\chi} f_0\otimes\theta\Big|^2\cdot\Big|\nabla_b \mathrm{d}_b f + \sqrt{\mu\chi}\nabla_b f_0\otimes\theta\\ &\quad +\sqrt{\mu}\frac{\nabla_b\chi}{2\sqrt{\chi}}\otimes f_0\otimes\theta\Big|^2\\ &= 4\Phi_{\mu\chi}\Big(|\nabla_b \mathrm{d}_b f|^2 + \mu\chi|\nabla_b f_0|^2 + \frac{\mu|\nabla_b\chi|^2}{4\chi}|f_0|^2\\ &\quad +\mu\langle\nabla_b f_0, \nabla_b\chi\otimes f_0\rangle\Big)\end{aligned}$$

which, using Cauchy inequality again, implies that

$$\begin{aligned}&(2-\epsilon)(|\nabla_b \mathrm{d}_b f|^2 + \mu\chi|\nabla_b f_0|^2) + 4\mu\langle\nabla_b\chi\otimes f_0, \nabla_b f_0\rangle\\ &= (2-2\epsilon)\Big(|\nabla_b \mathrm{d}_b f|^2 + \mu\chi|\nabla_b f_0|^2\Big)\\ &\quad + \epsilon\mu\chi|\nabla_b f_0|^2 + 4\mu\langle\nabla_b\chi\otimes f_0, \nabla_b f_0\rangle\\ &\geq \frac{1-\epsilon}{2}\frac{|\nabla_b\Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}} - \frac{1-\epsilon}{2}\frac{\mu|\nabla_b\chi|^2}{\chi}|f_0|^2\\ &\quad + (2+2\epsilon)\mu\langle\nabla_b\chi\otimes f_0, \nabla_b f_0\rangle + \epsilon\mu\chi|\nabla_b f_0|^2\\ &\geq \frac{1-\epsilon}{2}\frac{|\nabla_b\Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}} - \left(\frac{1-\epsilon}{2} + \frac{(1+\epsilon)^2}{\epsilon}\right)\mu\frac{|\nabla_b\chi|^2}{\chi}|f_0|^2\\ &\geq \frac{1-\epsilon}{2}\frac{|\nabla_b\Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}} - 4\epsilon^{-1}\mu\frac{|\nabla_b\chi|^2}{\chi}|f_0|^2\end{aligned}\tag{4.8}$$

due to $\epsilon \leq 1$ and

$$\frac{1-\epsilon}{2} + \frac{(1+\epsilon)^2}{\epsilon} \leq \frac{1-\epsilon}{\epsilon} + \frac{(1+\epsilon)^2}{\epsilon} = 2\epsilon^{-1} + \epsilon + 1 \leq 4\epsilon^{-1}.$$

Submitting (4.8) to (4.7), we finished the proof. □

Proof of Theorem 1.2 Set

$$F_{\mu\chi} = \frac{\Phi_{\mu\chi}}{(b-\psi\circ f)^{\nu}},$$

where $\nu \in [1,2)$ and b are determined in Lemma 4.1. The ϵ in Lemma 4.3 is chosen as

$$\epsilon = \frac{1}{\nu} - \frac{1}{2} \leq 1 \tag{4.9}$$

and μ satisfy

$$C_2\mu \leq \epsilon. \tag{4.10}$$

Let x be a maximum point of $\chi F_{\mu\chi}$ on B_{2R} which is nonzero. Assume that r is smooth at x. Otherwise we can modify the distance function r as [8]. Hence at x, we have

$$0 = \nabla_b \ln(\chi F_{\mu\chi}) = \frac{\nabla_b\chi}{\chi} + \frac{\nabla_b\Phi_{\mu\chi}}{\Phi_{\mu\chi}} + \nu\frac{\nabla_b(\psi\circ f)}{b-\psi\circ f}, \tag{4.11}$$

$$\begin{aligned} 0 \geq \Delta_b \ln(\chi F_{\mu\chi}) = &\frac{\Delta_b\chi}{\chi} - \frac{|\nabla_b\chi|^2}{\chi^2} + \frac{\Delta_b\Phi_{\mu\chi}}{\Phi_{\mu\chi}} - \frac{|\nabla_b\Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}^2} \\ &+ \nu\frac{\Delta_b(\psi\circ f)}{b-\psi\circ f} + \nu\frac{|\nabla_b(\psi\circ f)|^2}{(b-\psi\circ f)^2}. \end{aligned} \tag{4.12}$$

By (4.6), (4.12) becomes

$$\begin{aligned} 0 \geq &\frac{\Delta_b\chi}{\chi} - \frac{|\nabla_b\chi|^2}{\chi^2} - \frac{1+\epsilon}{2}\frac{|\nabla_b\Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}^2} - 2\kappa|\mathrm{d}_b f|^2 \\ &+ \nu\frac{\Delta_b(\psi\circ f)}{b-\psi\circ f} + \nu\frac{|\nabla_b(\psi\circ f)|^2}{(b-\psi\circ f)^2} \\ &+ \left(2m\epsilon - C_2\mu\chi + \mu\Delta_b\chi - 4\epsilon^{-1}\mu\frac{|\nabla_b\chi|^2}{\chi}\right)\frac{|f_0|^2}{\Phi_{\mu\chi}} \\ &- \left[C_2 + C_2\mu\chi + 16(\epsilon\mu\chi)^{-1}\right]\frac{|\mathrm{d}_b f|^2}{\Phi_{\mu\chi}}. \end{aligned} \tag{4.13}$$

Using (4.11) and Cauchy inequality, we have at x

$$\begin{aligned} -\frac{1+\epsilon}{2}\frac{|\nabla_b\Phi_{\mu\chi}|^2}{\Phi_{\mu\chi}^2} \geq & -\frac{1+\epsilon}{2}(1+\epsilon_2^{-1})\frac{|\nabla_b\chi|^2}{\chi^2} \\ & -\frac{1+\epsilon}{2}(1+\epsilon_2)\nu^2\frac{|\nabla_b(\psi\circ f)|^2}{(b-\psi\circ f)^2}. \end{aligned} \tag{4.14}$$

Due to the choice (4.9) of ϵ, we can take

$$\epsilon_2 = \frac{2}{\nu(1+\epsilon)} - 1 = \frac{2-\nu}{2+\nu} > 0$$

and then

$$\frac{1+\epsilon}{2}(1+\epsilon_2)\nu^2 = \nu, \qquad \frac{1+\epsilon}{2}(1+\epsilon_2^{-1}) = \frac{2+\nu}{\nu(2-\nu)}. \tag{4.15}$$

Substituting (4.14), (4.2) to (4.13), we have at x

$$\begin{aligned} 0 \geq & \frac{\Delta_b\chi}{\chi} - \left(1+\frac{2+\nu}{\nu(2-\nu)}\right)\frac{|\nabla_b\chi|^2}{\chi^2} + \nu\frac{\Delta_b\psi\circ f}{b-\psi\circ f} - 2\kappa|\mathrm{d}_b f|^2 \\ & + \left(2m\epsilon - C_2\mu\chi + \mu\Delta_b\chi - 4\epsilon^{-1}\mu\frac{|\nabla_b\chi|^2}{\chi}\right)\frac{|f_0|^2}{\Phi_{\mu\chi}} \\ & - \left[C_2 + C_2\mu\chi + 16(\epsilon\mu\chi)^{-1}\right]\frac{|\mathrm{d}_b f|^2}{\Phi_{\mu\chi}}. \end{aligned}$$

The estimates (4.1) and Lemma 4.2 yield that

$$\begin{aligned} 0 \geq & -\frac{C_\nu}{\chi R} + \delta|\mathrm{d}_b f|^2 + \left(2m\epsilon - C_2\mu\chi - \frac{\mu C_\nu}{R}\right)\frac{|f_0|^2}{\Phi_{\mu\chi}} \\ & - \left[C_2 + C_2\mu\chi + 16(\epsilon\mu\chi)^{-1}\right]\frac{|\mathrm{d}_b f|^2}{\Phi_{\mu\chi}}, \end{aligned} \tag{4.16}$$

where $C_\nu = C_\nu(\nu, C_5)$ and δ is given by Lemma 4.1. By definition of $\Phi_{\mu\chi}$,

$$|f_0|^2 = \mu^{-1}\chi^{-1}(\Phi_{\mu\chi} - |\mathrm{d}_b f|^2)$$

which, together with (4.16), shows at x,

$$\begin{aligned} 0 \geq & \frac{1}{\chi}\left(2m\epsilon\mu^{-1} - C_2 - \frac{2C_\nu}{R}\right) \\ & + \left[\delta\chi\Phi_{\mu\chi} - 2m\epsilon\mu^{-1} - \left[C_2 + C_2\mu + 16(\epsilon\mu)^{-1}\right]\right]\frac{|\mathrm{d}_b f|^2}{\chi\Phi_{\mu\chi}}. \end{aligned} \tag{4.17}$$

To make the first bracket of the last line in (4.17) nonnegative, we can choose sufficiently small μ such that

$$\epsilon\mu^{-1} = C_2 + \frac{2C_\nu}{R},$$

which makes (4.10) right. Hence

$$(\chi\Phi_{\mu\chi})(x) \le C_6\delta^{-1}, \tag{4.18}$$

where

$$C_6 = (2m+1)C_2 + \frac{4mC_\nu}{R} + \frac{C_2}{2C_2 + 4C_\nu R^{-1}} + \frac{64\nu^2}{(2-\nu)^2}\left(C_2 + \frac{2C_\nu}{R}\right), \tag{4.19}$$

which implies

$$\max_{B_{2R}(x_0)} \chi F_{\mu\chi} \le \frac{\chi\Phi_{\mu\chi}}{(b-\psi\circ f)^\nu}(x) \le \frac{C_6}{\delta(b-\phi(D))^\nu}. \tag{4.20}$$

This shows that

$$\max_{B_R(x_0)} |\mathrm{d}_b f|^2 \le b^\nu \cdot \max_{B_R(x_0)} F_{\mu\chi} \le \frac{C_6 b^\nu}{\delta(b-\phi(D))^\nu}. \tag{4.21}$$

Note that the constants b, ν and δ depend on κ and D by Lemma 4.1. Hence the proof is finished by choosing a suitable constant C_3. □

5 Global Existence Theorem

Jost and Xu [15] studied the minimizing sequence of Dirichlet problem of subelliptic harmonic maps and obtained the existence theorem under some convexity conditions. Their results [15] seem to depend on the global fields which satisfy the Hörmander condition and the noncharacteristic assumption of the boundary. But the weak existence of Dirichlet problem and the interior continuity of weak solutions can be generalized to any sub-Riemannian manifolds with smooth boundaries, such as pseudo-Hermitian manifolds. Hence Theorem 1 in [15] can be generalized to pseudo-Hermitian manifolds with boundary as follows.

Theorem 5.1 *Suppose that (M,θ) is a pseudo-Hermitian manifold with smooth boundary and (N,h) is a Riemannian manifold with sectional curvature $K^N \le \kappa$ for some $\kappa \ge 0$. Let $B_D = B_D(p_0) \subset N$ be a regular ball. If $\varphi \in S_1^2(M,N)$ satisfies $\varphi(\overline{M}) \subset B_D(p_0)$, then there exists a weak pseudo-harmonic map $f \in C(M,N) \cap S_1^2(M,N)$ with*

$$f - \varphi \in S_{1,0}^2(M,N)$$

and

$$f(\overline{M}) \subset B_D(p_0).$$

For completeness, the proof will be given in Appendix.

Remark 5.2 Note that $B_D(p_0)$ can be covered by a geodesic normal coordinate $\{z^i\}$ and thus it can be viewed as an open set of $\mathbb{R}^n$ where $n = \dim N$. Hence the notion

$$S_1^2(M, N) = S_1^2(M, \mathbb{R}^n),$$

and $S_{1,0}^2(M, N)$ means the completion of all smooth $\mathbb{R}^n$-valued functions with compact support under S_1^2-norm. Moreover, the weak pseudo-harmonic map $f \in S_1^2(M, N)$ means that the following equations hold in the distribution sense

$$\Delta_b f^i + \sum_{j,k} \Gamma^i_{jk}(f)\langle \nabla_b f^j, \nabla_b f^k \rangle = 0, \quad \text{for all } i = 1, 2, \ldots n, \tag{5.1}$$

where $f^i = z^i \circ f$ and Γ^i_{jk}'s are Christoffel symbols of Levi-Civita connection in (N, h).

Since the Euler–Lagrange equations of pseudo-harmonic maps are quasilinear subelliptic systems, these weak solutions will be interior smooth by applying Theorem 1.1 in [27] to each coordinate neighborhood.

Theorem 5.3 *Suppose that (M, θ) is a pseudo-Hermitian manifold (with or without boundary) and (N, h) is a Riemannian manifold. Let $f : M \to N$ be a weak pseudo-harmonic map and $f \in S_1^2(M, N)$. If f is continuous inside M, then $f \in C^\infty(M, N)$.*

Now let us prove Theorem 1.4.

Proof of Theorem 1.4 Suppose that (M, θ) is a complete noncompact pseudo-Hermitian manifold and (N, h) is a Riemannian manifold with sectional curvature $K^N \leq \kappa$ for some $\kappa \geq 0$. Let $B_D(p_0) \subset N$ be a geodesic ball lying in the cut locus of p_0 and $D < \frac{\pi}{2\sqrt{\kappa}}$. Assume that $\varphi : M \to B_D(p_0)$ with $\varphi(x_0) = p_0$. We can choose a smooth exhaustion $\{\Omega_i\}$ of M such that $B_{2i}(x_0) \subset \Omega_i$. Theorem 5.1 and Theorem 5.3 guarantee that there is a smooth pseudo-harmonic map $f_i : \Omega_i \to B_D(p_0)$. One can find the constants $k(i)$ and $k_1(i)$ such that

$$R_*\big|_{B_{2i}(x_0)} \geq -k(i), \text{ and } \left|A|_{B_{2i}(x_0)}\right|, \left|\text{div } A|_{B_{2i}(x_0)}\right| \leq k_1(i). \tag{5.2}$$

Hence fixed i, for $j \geq i$, Theorem 1.2 controls the interior horizontal gradient of f_j on $B_i(x_0)$:

$$\max_{B_i(x_0)} |\mathrm{d}_b f_j|^2 \leq C_7(i), \tag{5.3}$$

where $C_7(i)$ only depends on $k(i), k_1(i), D, \kappa, i$. Arzelà–Ascoli theorem yields that by taking subsequence, f_j will uniformly converge to some continuous map in $B_i(x_0)$ as $j \to \infty$. By diagonalization, some subsequence of $\{f_i\}$ will internally closed uniformly converge to a continuous map $f : M \to B_D(p_0)$ as $i \to \infty$. Moreover, f is a weak solution of (5.1) and thus is smooth pseudo-harmonic by Theorem 5.3. □

It is notable that the pseudo-harmonic map given by Theorem 1.4 will depend on the initial map. By Theorem 1.3, it is always trivial if the domain has nonnegative pseudo-Hermitian Ricci curvature. At the end of this paper, we will give a nontrivial example when the domain has negative pseudo-Hermitian Ricci curvature. One model of Sasakian space form with constant negative pseudo-Hermitian sectional curvature is the Riemannian submersion

$$\pi : B^n_{\mathbb{C}} \times \mathbb{R} \to B^n_{\mathbb{C}},$$

where $B^n_{\mathbb{C}} \subset \mathbb{C}^n$ is the complex ball with Bergman metric ω (cf. Example 7.3.22 in [4]). Let ω_0 be the canonical Kähler form on $\mathbb{C}^n$. Since the identity I of $B^n_{\mathbb{C}}$ is a holomorphic map from $B^n_{\mathbb{C}}$ to $\mathbb{C}^n$, then it is also a harmonic map from $(B^n_{\mathbb{C}}, \omega)$ to $(\mathbb{C}^n, \omega_0)$. The lift of I is denoted by $\tilde{I}$ such that

$$\tilde{I} = I \circ \pi : B^n_{\mathbb{C}} \times \mathbb{R} \to \mathbb{C}^n.$$

Then by the composition rule,

$$\hat{\nabla} \mathrm{d}\tilde{I} = \hat{\nabla} dI(d\pi, d\pi) + dI(\hat{\nabla} d\pi), \tag{5.4}$$

where the Levi-Civita connections of $(B^n_{\mathbb{C}}, \omega)$ and $(\mathbb{C}^n, \omega_0)$ are both denoted by $\hat{\nabla}$. Suppose that ∇ is the Tanaka–Webster connection of $B^n_{\mathbb{C}} \times \mathbb{R}$. Their relation is given by (cf. [12, Lemma 1.3])

$$\hat{\nabla} = \nabla - \mathrm{d}\theta \otimes \xi + 2\theta \odot J, \tag{5.5}$$

where $2\theta \odot J = \theta \otimes J + J \otimes \theta$. Assume that $\{e_B\}_{B=1}^{2n}$ is a orthonormal frame in $(B^n_{\mathbb{C}}, \omega)$ with $e_{\alpha+n} = J e_\alpha$ for $1 \le \alpha \le n$ and $\tilde{e}_B$ is the horizontal lift of e_B. On one hand, the relation (5.5) guarantees that

$$\begin{aligned} \tau_H(\tilde{I}) &= \sum_{B=1}^{2n} (\nabla_{\tilde{e}_B} \mathrm{d}\tilde{I})(\tilde{e}_B) \\ &= \sum_{B=1}^{2n} \hat{\nabla}_{\tilde{e}_B} \left(\mathrm{d}\tilde{I}(\tilde{e}_B) \right) - \sum_{B=1}^{2n} \mathrm{d}\tilde{I} \left(\nabla_{\tilde{e}_B} \tilde{e_B} \right) \end{aligned}$$

$$= \sum_{B=1}^{2n} \hat{\nabla}_{\tilde{e}_B} \left(\mathrm{d}\tilde{I}(\tilde{e}_B) \right) - \sum_{B=1}^{2n} \mathrm{d}\tilde{I} \left(\hat{\nabla}_{\tilde{e}_B} \tilde{e_B} \right)$$

$$= \sum_{B=1}^{2n} (\hat{\nabla}_{\tilde{e}_B} \mathrm{d}\tilde{I})(\tilde{e}_B). \tag{5.6}$$

On the other hand, by the relation of Levi-Civita connection and metric, we have

$$\sum_{B=1}^{2n} \mathrm{d}\pi \left(\hat{\nabla}_{\tilde{e}_B} \tilde{e}_B \right) = \sum_{B=1}^{2n} \hat{\nabla}_{e_B} e_B$$

which implies that

$$\sum_{i=1}^{2n} \left(\hat{\nabla}_{\tilde{e}_B} \mathrm{d}\pi \right) (\tilde{e}_B) = 0. \tag{5.7}$$

Taking the horizontal trace of (5.4) and using (5.6), (5.7), we obtain that

$$\tau_H(\tilde{I}) = \sum_{i=1}^{2n} \left(\hat{\nabla}_{e_B} \mathrm{d}I \right) (e_B) = 0,$$

since I is harmonic. Hence $\tilde{I}$ is nontrivial pseudo-harmonic. But the image of $\tilde{I}$ is exactly the unit ball in $\mathbb{C}^n$ which is a regular ball. So this is a nontrivial pseudo-harmonic example when the domain has negative pseudo-Hermitian Ricci curvature.

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Appendix

This section will deduce Theorems 2.4 and 5.1 by the theory of subelliptic analysis. Suppose that (M, θ) is a pseudo-Hermitian manifold of real dimension $2m + 1$. Let Ω be a coordinate neighborhood in M and $\{e_B\}_{B=1}^{2m}$ be an orthonormal basis of $HM\big|_{\Omega}$ with $Je_i = e_{i+m}$ for $i = 1, 2, \ldots, m$. Since

$$-\theta([e_i, Je_i]) = \mathrm{d}\theta(e_i, Je_i) = G_\theta(e_i, e_i) = 1, \quad \text{for } i = 1, 2, \ldots, m,$$

then each $[e_i, Je_i]$ is transversal with horizontal distribution which implies that HM satisfies the strong bracket generating hypothesis. Moreover, by identifying Ω with a domain in $\mathbb{R}^{2m+1}$, the vector fields $\{e_1, \ldots, e_{2m}\}$ satisfy the Hörmander's condition. Let e_B^* be the formal adjoint of e_B. For any $u \in C^\infty(\Omega)$, we have

$$\Delta_b u = -\sum_{B=1}^{2m} e_B^* e_B u,$$

which shows that the sub-Laplacian operator is subelliptic. One can refer to [12, Sect. 2.2] for more discussions. Since Tanaka–Webster connection preserves the horizontal distribution, then the higher-order horizontal covariant derivative on Ω can be expressed as follows:

$$\begin{aligned}\nabla_b^l u(e_{B_1}, \cdots, e_{B_l}) &= \nabla_{e_{B_l}}\left[\nabla^{l-1}u(e_{B_1}, \cdots, e_{B_{l-1}})\right]\\ &\quad -\sum_{i=1}^{l}\nabla^{l-1}u(e_{B_1}, \cdots, \nabla_{e_{B_l}}e_{B_i}, \cdots, e_{B_l})\\ &= \cdots\\ &= e_{B_l}e_{B_{l-1}}\cdots e_{B_1}u + \text{ lower order terms},\end{aligned}$$

for any $B_1, \cdots, B_l \in \{1, 2, \cdots, 2m\}$, which implies that the S_k^p-norm on Ω is equivalent with the local Folland–Stein Sobolev norm (cf. [12, p. 193]). Hence local results of subelliptic analysis always hold for the sub-Laplacian operator on a coordinate neighborhood of pseudo-Hermitian manifolds. By partition of unity, the domain can be generalized to a relatively compact domain in a pseudo-Hermitian manifold. Let us use this idea to prove Theorem 2.4 by the following local version.

Theorem 6.1 ([12, Theorem 3.17] and [23, Theorem 16]) *Suppose that (M, θ) is a pseudo-Hermitian manifold and $\Omega \Subset M$ is a coordinate neighborhood. Assume that $u, v \in L^1_{loc}(\Omega)$ and $\Delta_b u = v$ in the distribution sense. For any $\chi \in C_0^\infty(\Omega)$, if $v \in S_k^p(\Omega)$ with $p > 1$ and $k \in \mathbb{N}$, then $\chi u \in S_{k+2}^p(\Omega)$ and*

$$||\chi u||_{S_{k+2}^p(\Omega)} \leq C_\chi\left(||u||_{L^p(\Omega)} + ||v||_{S_k^p(\Omega)}\right), \tag{6.1}$$

where C_χ only depends on χ.

Proof of Theorem 2.4 Let $\{\Omega_\alpha\}$ be a finite open cover of supp χ and $\{\chi_\alpha\}$ be a partition of unity subordinating to $\{\Omega_\alpha\}$. Since $\Delta_b u = v$ holds in each Ω_α, Theorem 1 guarantees that

$$||\chi_\alpha \chi u||_{S_{k+2}^p(\Omega_\alpha)} \leq C_{\chi_\alpha\chi}\left(||u||_{L^p(\Omega_\alpha)} + ||v||_{S_k^p(\Omega_\alpha)}\right),$$

which implies that

$$||\chi u||_{S_{k+2}^p(\Omega)} \leq \sum_\alpha ||\chi u||_{S_{k+2}^p(\Omega_\alpha)} \leq \left(\sum_\alpha C_{\chi_\alpha\chi}\right)\left(||u||_{L^p(\Omega)} + ||v||_{S_k^p(\Omega)}\right).$$

The proof is finished by setting $C_\chi = \sum_\alpha C_{\chi_\alpha\chi}$. □

Next let us prove Theorem 5.1.

Proof of Theorem 5.1 Under the exponential map at $p_0 \in N$, the regular ball $B_D = B_D(p_0)$ is diffeomorphic to the ball B_D with radius D and centered at the origin in $\mathbb{R}^n$ where $n = \dim N$. Let $\{z^i\}_{i=1}^n$ be the geodesic normal coordinates at p_0 and $f^i = z^i \circ f$ be the components of a function $f : M \to B_D$. Denote

$$\mathcal{S} = \left\{ f \in S_1^2(M, \mathbb{R}^n) \middle| \, f - \varphi \in S_{1,0}^2(M, \mathbb{R}^n), \ \sup_M |f| \le D \right\},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Consider the minimizing problem

$$\lambda = \inf_{f \in \mathcal{S}} E_H(f) = \inf_{f \in \mathcal{S}} \int_M h_{ij}(f) \langle \nabla_b f^i, \nabla_b f^j, \rangle \tag{6.2}$$

where $h_{ij} = h(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j})$. Since $\varphi \in \mathcal{S}$, then λ is finite. Let $\{f_s\}_{s=1}^\infty$ be a minimizing sequence of (6.2) which have uniform S_1^2-norm bound. By CR compact embedding theorem of Folland–Stein space (cf. Theorem 3.15 in [12]), there are a $f \in S_1^2(M, \mathbb{R}^n)$ and a subsequence of $\{f_s\}$ (also denoted by $\{f_s\}$) such that

(i) $f_s \to f$ strongly in $L^2(M, \mathbb{R}^n)$;
(ii) $f_s \rightharpoonup f$ weakly in $S_1^2(M, \mathbb{R}^n)$.

By (1), f_s converges to f almost everywhere on M which implies that $|f| \le D$; by (1), $f - \varphi \in S_{1,0}^2(M, \mathbb{R}^n)$ which is closed in $S_1^2(M, \mathbb{R}^n)$. Hence $f \in \mathcal{S}$.

We claim that

$$E_H(f) \le \liminf_{s \to \infty} E_H(f_s). \tag{6.3}$$

It suffices to show that for any domain $\Omega \subset M$ with an orthonormal basis $\{e_A\}_{A=1}^{2m}$ of $HM\big|_\Omega$,

$$\sum_{i,j,A} \int_\Omega h_{ij}(f) e_A f^i \, e_A f^j \le \liminf_{s \to \infty} \sum_{i,j,A} \int_\Omega h_{ij}(f_s) e_A f_s^i \, e_A f_s^j. \tag{6.4}$$

For any $\varepsilon > 0$, since $f^i \in S_1^2(\Omega)$ and $f_s^i \to f^i$ strongly in $L^2(\Omega)$, there is a compact set $K \subset \Omega$ such that

$$\sum_{i,j,A} \int_{\Omega \setminus K} h_{ij}(f) e_A f^i \, e_A f^j < \varepsilon \quad \text{and} \quad f_s^i \rightrightarrows f^i \quad \text{on } K,$$

where "$\rightrightarrows$" means "uniform convergence". The positivity of (h_{ij}) implies that

$$\begin{aligned} 0 &\le \sum_{i,j,A} h_{ij}(f_s) e_A(f_s^i - f^i) \, e_A(f_s^j - f^j) \\ &= \sum_{i,j,A} h_{ij}(f_s) e_A f_s^i \, e_A f_s^j - \sum_{i,j,A} h_{ij}(f_s) e_A f^i \, e_A f^j \end{aligned}$$

$$-2\sum_{i,j,A} h_{ij}(f_s)e_A f^i\, e_A(f_s^j - f^j),$$

which yields that

$$\begin{aligned}\sum_{i,j,A}\int_K h_{ij}(f_s)e_A f_s^i\, e_A f_s^j &\geq \sum_{i,j,A}\int_K h_{ij}(f_s)e_A f^i\, e_A f^j \\ &\quad +2\sum_{i,j,A}\int_K h_{ij}(f_s)e_A f^i\, e_A(f_s^j - f^j) \\ &= \sum_{i,j,A}\int_K h_{ij}(f_s)e_A f^i\, e_A f^j \\ &\quad +2\sum_{i,j,A}\int_K (h_{ij}(f_s)-h_{ij}(f))e_A f^i\, e_A(f_s^j - f^j) \\ &\quad +2\sum_{i,j,A}\int_K h_{ij}(f)e_A f^i\, e_A(f_s^j - f^j). \qquad (6.5)\end{aligned}$$

For the first term of (6.5), since $f_s^i \rightrightarrows f^i$ on K, then by mean value theorem, we have

$$\begin{aligned}&\left|\sum_{i,j,A}\int_K (h_{ij}(f_s)-h_{ij}(f))e_A f^i\, e_A f^j\right| \\ &\leq \sum_{i,j,k,A}\max_{B_D}\left|\frac{\partial h_{ij}}{\partial z^k}\right|\int_K |f_s^k - f^k|\,|e_A f^i|\,|e_A f^j| \to 0,\end{aligned}$$

as $s \to \infty$, which implies that

$$\lim_{s\to\infty}\sum_{i,j,A}\int_K h_{ij}(f_s)e_A f^i\, e_A f^j = \sum_{i,j,A}\int_K h_{ij}(f)e_A f^i\, e_A f^j. \qquad (6.6)$$

Similarly, since $e_A f_s^j$ and $e_A f^j$ are uniformly bounded in $L^2(K)$, then

$$\lim_{s\to\infty}\sum_{i,j,A}\int_K (h_{ij}(f_s)-h_{ij}(f))e_A f^i\, e_A(f_s^j - f^j) = 0. \qquad (6.7)$$

For the third term of (6.5), define an operator $T_A : S_1^2(M) \to L^2(K)$ by

$$T_A(u) = e_A u\big|_K.$$

T_A is continuous due to the following calculation:

$$||T_A(u)||^2_{L^2(K)} = \int_K |e_A u|^2 \leq \int_M |\nabla_b u|^2 \leq ||u||^2_{S_1^2(M)}.$$

Since any continuous operator between two Banach spaces preserves weak convergence, then $e_A f_s^i \rightharpoonup e_A f^i$ weakly in $L^2(K)$ for any A and i. Hence

$$\lim_{s\to\infty} \sum_{i,j,A} \int_K h_{ij}(f) e_A f^i \, e_A (f_s^j - f^j) = 0. \tag{6.8}$$

Using (6.6), (6.7) and (6.8), we find that

$$\sum_{i,j,A} \int_K h_{ij}(f) e_A f^i \, e_A f^j \le \liminf_{s\to\infty} \sum_{i,j,A} \int_K h_{ij}(f_s) e_A f_s^i \, e_A f_s^j,$$

which implies that

$$\begin{aligned}\sum_{i,j,A} \int_\Omega h_{ij}(f) e_A f^i \, e_A f^j &\le \sum_{i,j,A} \int_K h_{ij}(f) e_A f^i \, e_A f^j + \varepsilon \\ &\le \liminf_{s\to\infty} \sum_{i,j,A} \int_K h_{ij}(f_s) e_A f_s^i \, e_A f_s^j + \varepsilon \\ &\le \liminf_{s\to\infty} \sum_{i,j,A} \int_\Omega h_{ij}(f_s) e_A f_s^i \, e_A f_s^j + \varepsilon.\end{aligned}$$

By taking $\varepsilon \to 0$, we obtain (6.4) and thus $E_H(f) \le \lambda$.

Obviously, $E_H(f) \ge \lambda$ and then $E_H(f) = \lambda$ which shows that f has the minimal horizontal energy in $\mathcal{S}$ and satisfies

$$\Delta_b f^i + \Gamma^i_{jk}(f) \langle \nabla_b f^j, \nabla_b f^k \rangle = 0,$$

in the distribution sense. By applying Theorem 2 in [15] to f on each coordinate neighborhood $\Omega \Subset M$, we obtain the interior continuity of f. □

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