

# SUB-LAPLACIAN COMPARISON THEOREM

## 1. MAIN THEOREM

**Theorem 1.1.** *Suppose  $(M^{2m+1}, \theta)$  is a complete pseudo-Hermitian manifold. If for some  $k, k_1 \geq 0$ ,*

$$R_* \geq -k, \text{ and } |A|, |\operatorname{div} A| \leq k_1, \text{ in } B_R(x_0) \subset M \setminus \operatorname{Cut}(x_0),$$

*then there exists  $C_1 = C_1(m)$  such that*

$$\Delta_b r \leq C_1 \left( \frac{1}{r} + \sqrt{1 + k_1 + k_1^2 + k} \right) \text{ in } B_R(x_0),$$

*where  $r$  is the Riemannian distance from  $x_0$  and  $\operatorname{Cut}(x_0)$  is the cut locus of  $x_0$ .*

## 2. PROOF OF MAIN THEOREM

Now we formulate all Riemannian symbols with “hat” to distinguish with ones in pseudo-Hermitian geometry, such as Levi-Civita connection  $\hat{\nabla}$  and Riemannian curvature tensor  $\hat{R}$ . Recall the relation of Tanaka-Webster connection and Levi-Civita connection associated with Webster metric:

$$\hat{\nabla} = \nabla - \left( \frac{1}{2} d\theta + A \right) \otimes T + \tau \otimes \theta + \theta \odot J.$$

Hence the sub-Laplacian of  $r$  can also be calculated by Levi-Civita connection as follows:

$$\Delta_b r = \operatorname{trace}_{G_\theta} \widehat{\operatorname{Hess}} r|_{H(M) \times H(M)},$$

where  $\widehat{\operatorname{Hess}}$  is the Riemannian Hessian.

Let us recall the Index Lemma in Riemannian geometry.

**Lemma 2.1.** *Suppose  $\gamma : [0, a] \rightarrow M$  is a normal geodesic (with respect to  $\hat{\nabla}$ ) with no interior conjugate points,  $\gamma(0) = x_0$ ,  $X$  is a normal Jacobi field along  $\gamma$ , and  $V$  is any other piecewise smooth normal vector field along  $\gamma$  such that  $V(0) = X(0)$  and  $V(a) = X(a)$ . Then*

$$I_a(V, V) \geq I_a(X, X).$$

*In addition, assume that  $\gamma(a)$  is not conjugate to  $\gamma(0)$  along  $\gamma$ . We also have  $I_a(V, V) = I_a(X, X)$  if and only if  $V = X$ .*

Now let  $\gamma : [0, a] \rightarrow M$  is a normal geodesic with no conjugate points,  $\gamma(0) = x_0$  and  $\{e_B(a)\}_{B=1}^{2m}$  be an orthonormal basis of  $H(M)|_{\gamma(a)}$ . Set

$$e_B^\perp(a) = e_B(a) - \langle e_B(a), \dot{\gamma}(a) \rangle \dot{\gamma}(a) \in T_{\gamma(a)} M,$$

we know  $\dot{\gamma}(a) = \nabla r|_{\gamma(a)}$ . Since  $\widehat{\text{Hess}}r(\nabla r, \cdot) = 0$ , then

$$\Delta_b r|_{\gamma(a)} = \sum_{B=1}^{2m} \widehat{\text{Hess}}r(e_B(a), e_B(a)) = \sum_{B=1}^{2m} \widehat{\text{Hess}}r(e_B^\perp(a), e_B^\perp(a)).$$

By the two-point boundary problem for Jacobi Fields, there exists a normal Jacobi field  $U_B$  along  $\gamma$  with

$$U_B(0) = 0, U_B(a) = e_B^\perp(a).$$

Since every Jacobi field along  $\gamma$  is the variation field of a variation of  $\gamma$  through geodesics and  $\hat{\nabla}$  has zero torsion, we also have

$$\hat{\nabla}_\gamma U_B = \hat{\nabla}_{U_B} \dot{\gamma}.$$

Hence we obtain

$$\begin{aligned} \widehat{\text{Hess}}r(e_B^\perp(a), e_B^\perp(a)) &= \widehat{\text{Hess}}r(U_B(a), U_B(a)) \\ &= \langle U_B, \hat{\nabla}_{U_B} \nabla r \rangle|_{\gamma(a)} \\ &= \langle U_B, \hat{\nabla}_\gamma U_B \rangle|_{\gamma(a)} \\ &= \int_0^a \frac{d}{dt} \langle U_B, \hat{\nabla}_\gamma U_B \rangle dt \\ &= \int_0^a |\hat{\nabla}_\gamma U_B|^2 + \langle U_B, \hat{\nabla}_\gamma \hat{\nabla}_\gamma U_B \rangle dt \\ &= I_a(U_B, U_B). \end{aligned}$$

Hence

$$\Delta_b r|_{\gamma(a)} = \sum_{B=1}^{2m} I_a(U_B, U_B).$$

**Lemma 2.2.** *Let  $e_B(t)$  be the parallel transport of  $e_B(a)$  along  $\gamma$  with respect to Tanaka-Webster connection. Suppose the curvature along  $\gamma$  satisfies*

$$(2.1) \quad \sum_{B=1}^{2m} \langle \hat{R}(e_B, \dot{\gamma})\dot{\gamma}, e_B \rangle \geq -\hat{k}$$

and the pseudo-Hermitian torsion is bounded, i.e.

$$|A(X, Y)| \leq k_1 g_\theta(X, Y), \text{ for any } X, Y \in TM,$$

for some  $\hat{k}, k_1 \geq 0$ . Then there is a constant  $C_4 = C_4(m)$  such that

$$\Delta_b r|_{\gamma(a)} \leq C_4 \left( 1/a + \sqrt{1 + k_1 + k_1^2 + \hat{k}} \right).$$

*Proof.*

$$\hat{\nabla}_\gamma e_B = - \left( \frac{1}{2} g_\theta(J\dot{\gamma}, e_B) + A(\dot{\gamma}, e_B) \right) T + \frac{1}{2} \theta(\dot{\gamma}) J e_B,$$

which implies that

$$\begin{aligned}
\sum_{B=1}^{2m} |\hat{\nabla}_{\dot{\gamma}} e_B|^2 &= m/2 |\theta(\dot{\gamma})|^2 + \sum_{B=1}^{2m} \left[ \frac{1}{4} g_{\theta}(J\dot{\gamma}, e_B)^2 + g_{\theta}(J\dot{\gamma}, e_B) A(\dot{\gamma}, e_B) + A(\dot{\gamma}, e_B)^2 \right] \\
&= \sum_{B=1}^{2m} \frac{1}{4} g_{\theta}(\dot{\gamma}, T)^2 + \frac{1}{4} g_{\theta}(J\dot{\gamma}, J\dot{\gamma}) + A(\dot{\gamma}, J\dot{\gamma}) + \sum_{B=1}^{2m} A(\dot{\gamma}, e_B)^2 \\
&\leq m/2 + A(\dot{\gamma}, J\dot{\gamma}) + \sum_{B=1}^{2m} A(\dot{\gamma}, e_B)^2 \leq m/2 + k_1 + 2mk_1^2.
\end{aligned}$$

Set

$$e_B^{\perp}(t) = e_B(t) - \langle e_B(t), \dot{\gamma}(t) \rangle \dot{\gamma}(t), V_B(t) = \frac{s_{\kappa}(t)}{s_{\kappa}(a)} e_B^{\perp}(t),$$

where

$$s_{\kappa}(t) = \frac{1}{\sqrt{\kappa}} \sinh(\sqrt{\kappa}t), \kappa = \frac{1}{4m} (4m + 8k_1 + 16mk_1^2 + \hat{k}).$$

Hence  $V_B(0) = 0$ ,  $V_B(a) = e'_B(a)$ ,  $V_B \perp \dot{\gamma}$  and

$$\begin{aligned}
\sum_{B=1}^{2m} |\hat{\nabla}_{\dot{\gamma}} V_B|^2 &= \sum_{B=1}^{2m} \left| \frac{\dot{s}_{\kappa}(t)}{s_{\kappa}(a)} e_B^{\perp}(t) + \frac{s_{\kappa}(t)}{s_{\kappa}(a)} \hat{\nabla}_{\dot{\gamma}} e_B^{\perp}(t) \right|^2 \\
&\leq 2 \sum_{B=1}^{2m} \left| \frac{\dot{s}_{\kappa}(t)}{s_{\kappa}(a)} e_B^{\perp}(t) \right|^2 + 2 \sum_{B=1}^{2m} \left| \frac{s_{\kappa}(t)}{s_{\kappa}(a)} \hat{\nabla}_{\dot{\gamma}} e_B^{\perp}(t) \right|^2 \\
&\leq 4m \left| \frac{\dot{s}_{\kappa}(t)}{s_{\kappa}(a)} \right|^2 + 8(m/2 + k_1 + 2mk_1^2) \left| \frac{s_{\kappa}(t)}{s_{\kappa}(a)} \right|^2
\end{aligned}$$

By the curvature assumption and the Index Lemma, we have

$$\begin{aligned}
\Delta_b r|_{\gamma(a)} &\leq \sum_{B=1}^{2m} I_a(V_B, V_B) = \sum_{B=1}^{2m} \int_0^a |\hat{\nabla}_{\dot{\gamma}} V_B|^2 - \langle \hat{R}(V_B, \dot{\gamma}) \dot{\gamma}, V_B \rangle \\
&\leq 4m \int_0^a \left( \left| \frac{\dot{s}_{\kappa}(t)}{s_{\kappa}(a)} \right|^2 + \kappa \left| \frac{s_{\kappa}(t)}{s_{\kappa}(a)} \right|^2 \right) \\
&= 4m \sqrt{\kappa} \coth \sqrt{\kappa} a \leq 4m \left( \frac{1}{a} + \sqrt{\kappa} \right),
\end{aligned}$$

which finishes the proof.  $\square$

To prove Theorem 1.1, it suffices to demonstrate (2.1). It can be expressed by pseudo-Hermitian data due to the relationship between the Riemannian curvature tensor  $\hat{R}$  and the curvature tensor  $R$  associated with Tanaka-Webster connection  $\nabla$ :

$$\begin{aligned}
(2.2) \quad \hat{R}(X, Y)Z &= R(X, Y)Z + (LX \wedge LY)Z + \frac{1}{2} d\theta(X, Y)JZ \\
&\quad - g_{\theta}(S(X, Y), Z)T + \theta(Z)S(X, Y) \\
&\quad - g_{\theta}(\theta \wedge \mathcal{O}(X, Y), Z)T + \theta(Z)(\theta \wedge \mathcal{O})(X, Y),
\end{aligned}$$

where

$$\begin{aligned} S(X, Y) &= (\nabla_X \tau)Y - (\nabla_Y \tau)X \\ \mathcal{O} &= \tau^2 + J\tau - \frac{1}{4}I \\ L &= \tau + \frac{1}{2}J. \end{aligned}$$

**Lemma 2.3.** *For  $X, Y \in TM$ , we have*

$$\begin{aligned} \sum_{B=1}^{2m} \langle \hat{R}(e_B, X)Y, e_B \rangle &= \sum_{B=1}^{2m} \langle R(e_B, X)Y, e_B \rangle - \frac{3}{4} \langle \pi_H X, \pi_H Y \rangle \\ &\quad + \langle \tau X, \tau Y \rangle + (m/2 - |\tau|^2) \theta(X) \theta(Y) + (\operatorname{div} \tau)(X) \theta(Y), \end{aligned}$$

where

$$(\operatorname{div} \tau)(\cdot) = \sum_{B=1}^{2m} \langle (\nabla_{e_B} \tau)(\cdot), e_B \rangle.$$

*Proof.* By (2.2), we have

$$\begin{aligned} \sum_{B=1}^{2m} \langle \hat{R}(e_B, X)Y, e_B \rangle &= \sum_{B=1}^{2m} \langle R(e_B, X)Y, e_B \rangle + \sum_{B=1}^{2m} \langle (Le_B \wedge LX)Y, e_B \rangle \\ &\quad + \sum_{B=1}^{2m} \frac{1}{2} d\theta(e_B, X) \langle JY, e_B \rangle + \sum_{B=1}^{2m} \theta(Y) \langle S(e_B, X), e_B \rangle \\ &\quad + \sum_{B=1}^{2m} \theta(Y) \langle (\theta \wedge \mathcal{O}(e_B, X)), e_B \rangle. \end{aligned}$$

Now we see each term in the right side except the first one. Note that

$$\begin{aligned} \sum_{B=1}^{2m} \langle (Le_B \wedge LX)Y, e_B \rangle &= \sum_{B=1}^{2m} \langle Le_B, Y \rangle \langle LX, Y \rangle \langle Le_B, e_B \rangle \\ &= \langle \tau X, \tau Y \rangle - \frac{1}{4} \langle \pi_H X, \pi_H Y \rangle, \end{aligned}$$

$$\sum_{B=1}^{2m} \frac{1}{2} d\theta(e_B, X) \langle JY, e_B \rangle = -\frac{1}{2} \langle \pi_H X, \pi_H Y \rangle,$$

$$\sum_{B=1}^{2m} \theta(Y) \langle S(e_B, X), e_B \rangle = (\operatorname{div} \tau)(X) \theta(Y),$$

and

$$\sum_{B=1}^{2m} \theta(Y) \langle (\theta \wedge \mathcal{O}(e_B, X)), e_B \rangle = -\theta(X) \theta(Y) (|\tau|^2 - m/2).$$

□

Tanaka obtained the following version of first Bianchi identity of  $R$ :

$$\mathcal{S}(R(X, Y)Z) = \frac{1}{2}\mathcal{S}(d\theta(X, Y)\tau(Z)),$$

where  $\mathcal{S}$  stands for the cyclic sum with respect to  $X, Y, Z \in H(M)$ .

**Lemma 2.4.** *For  $X, Y \in TM$ , we have*

$$\langle R_*X, Y \rangle = \sum_{B=1}^{2m} \langle R(e_B, \pi_H X)\pi_H Y, e_B \rangle + \frac{1}{2}(m-1)A(X, JY).$$

*Proof.* First, recall the definition of  $R_*$ ,

$$R_*X = -i \sum_{\alpha=1}^m R(T_\alpha, T_{\bar{\alpha}})JX.$$

By the first Bianchi identity of  $R$ , we have

$$\begin{aligned} & -i \sum_{\alpha=1}^m R(T_\alpha, T_{\bar{\alpha}})JX - i \sum_{\alpha=1}^m R(T_{\bar{\alpha}}, JX)T_\alpha - i \sum_{\alpha=1}^m R(JX, T_\alpha)T_{\bar{\alpha}} \\ & = -i \sum_{\alpha=1}^m \frac{1}{2}d\theta(T_\alpha, T_{\bar{\alpha}})\tau JX - i \sum_{\alpha=1}^m \frac{1}{2}d\theta(T_{\bar{\alpha}}, JX)\tau T_\alpha \\ & \quad - i \sum_{\alpha=1}^m \frac{1}{2}d\theta(JX, T_\alpha)\tau T_{\bar{\alpha}} = \frac{m-1}{2}\tau JX. \end{aligned}$$

$\Rightarrow$

$$\langle R_*X, Y \rangle = \langle i \sum_{\alpha=1}^m R(T_{\bar{\alpha}}, JX)T_\alpha + i \sum_{\alpha=1}^m R(JX, T_\alpha)T_{\bar{\alpha}}, Y \rangle + \frac{m-1}{2}A(JX, Y).$$

□

**Lemma 2.5.** *For  $X, Y \in TM$ , we have*

$$\begin{aligned} \sum_{B=1}^{2m} \langle \hat{R}(e_B, X)Y, e_B \rangle & = \langle R_*X, Y \rangle - \frac{1}{2}(m-1)A(X, JY) \\ & \quad - \frac{3}{4}\langle \pi_H X, \pi_H Y \rangle + \langle \tau X, \tau Y \rangle + (m/2 - |\tau|^2)\theta(X)\theta(Y) \\ & \quad + (\operatorname{div}\tau)(X)\theta(Y) + (\operatorname{div}\tau)(Y)\theta(X). \end{aligned}$$

*Proof.* It suffices to prove that

$$\sum_{B=1}^{2m} \langle R(e_B, T)Y, e_B \rangle = (\operatorname{div}\tau)(Y).$$

for  $Y \in H(M)$ . Indeed,

$$\begin{aligned}
\sum_{B=1}^{2m} \langle R(e_B, T)Y, e_B \rangle &= \sum_{B=1}^{2m} \langle \hat{R}(e_B, T)Y, e_B \rangle \\
&= \sum_{B=1}^{2m} \langle \hat{R}(e_B, Y)T, e_B \rangle \\
&= \sum_{B=1}^{2m} \langle S(e_B, Y), e_B \rangle = (\operatorname{div} \tau)(Y).
\end{aligned}$$

□

$$\begin{aligned}
\sum_{B=1}^{2m} \langle \hat{R}(e_B, \dot{\gamma})\dot{\gamma}, e_B \rangle &= \langle R_* \dot{\gamma}, \dot{\gamma} \rangle - \frac{1}{2}(m-1)A(\dot{\gamma}, J\dot{\gamma}) \\
&\quad - \frac{3}{4} \langle \pi_H \dot{\gamma}, \pi_H \dot{\gamma} \rangle + \langle \tau \dot{\gamma}, \tau \dot{\gamma} \rangle + (m/2 - |\tau|^2) \theta(\dot{\gamma}) \theta(\dot{\gamma}) \\
&\quad + (\operatorname{div} \tau)(\dot{\gamma}) \theta(\dot{\gamma}) + (\operatorname{div} \tau)(\dot{\gamma}) \theta(\dot{\gamma}).
\end{aligned}$$

## 3. THE ESTIMATES OF PSEUDOHARMONIC MAPS

Suppose that  $(M, \theta)$  is a pseudo-Hermitian manifold of real dimension  $2m + 1$ . Let  $\{T_\alpha\}$  be a local orthonormal frame of  $T_{1,0}(M)$  and let  $\nabla$  be the Tanaka-Webster connection.

Suppose that  $(N, h)$  is a Riemannian manifold. Let  $\{\xi_i\}$  be a local orthonormal frame of  $TN$  and  $\sigma^i$  is its dual coframe. Let  $\hat{\nabla}$  and  $\hat{R}$  be the Levi-Civita connection and the Riemannian curvature of  $(N, h)$ , respectively.

Assume that  $f : M \rightarrow N$  is a smooth map,  $f^*TN$  is the pullback bundle, define

$$d_b f := \pi_H df = f_\alpha^i \theta^\alpha \otimes \xi_i + f_{\bar{\alpha}}^i \theta^{\bar{\alpha}} \otimes \xi_i \in \Gamma(T^*M \otimes f^*TN),$$

$$f_0 := f_0^i \xi_i \in f^*TN.$$

Now, we can determine a connection  $\nabla^f$  in  $T^*M \otimes f^*TN$  by

$$\nabla_X^f(\omega \otimes \xi := \nabla_X \omega \otimes \xi + \omega \otimes \hat{\nabla}_{df(X)} \xi, X \in TM, \omega \in \Gamma(T^*M), \xi \in \Gamma(f^*TN).$$

We write

$$\nabla^f df = f_{AB}^i \theta^A \otimes \theta^B \otimes \xi_i,$$

set  $\nabla_b^f d_b f := \pi_H \nabla^f df$ .

**Definition 3.1.** A smooth map  $f : M \rightarrow N$  is called pseudo-harmonic if the tensor field

$$\tau(f; \theta, \hat{\nabla}) = \text{trace}_{G_\theta}(\nabla_b^f d_b f) \in \Gamma(f^*TN),$$

we say that  $f$  is pseudoharmonic if  $\tau(f; \theta, \hat{\nabla}) = 0$ . We have  $\tau(f; \theta, \hat{\nabla}) = (f_{\alpha\bar{\alpha}}^i + f_{\bar{\alpha}\alpha}^i) \xi_i$ .

A.Greenleaf obtained the commutation relations of various covariant derivatives of smooth functions and established Bochner-type formulas of smooth functions. We shall need the commutation relations of various covariant derivatives of smooth maps and Bochner-type formulas of smooth maps.

**Lemma 3.1.**

$$\begin{aligned} f_{\alpha\beta}^i &= f_{\beta\alpha}^i, \\ f_{\alpha\bar{\beta}}^i - f_{\bar{\beta}\alpha}^i &= i f_0^i \delta_{\alpha\bar{\beta}}, \\ f_{0\alpha}^i - f_{\alpha 0}^i &= f_\beta^i A_\alpha^{\bar{\beta}}. \end{aligned}$$

and

$$\begin{aligned} f_{\alpha\beta\gamma}^i - f_{\alpha\gamma\beta}^i &= i f_\beta^i A_{\alpha\gamma} - i f_\gamma^i A_{\alpha\beta} - f_\alpha^j f_\beta^k f_\gamma^l \hat{R}_{jkl}^i, \\ f_{\alpha\bar{\beta}\bar{\gamma}}^i - f_{\alpha\bar{\gamma}\bar{\beta}}^i &= i f_\mu^i A_{\bar{\gamma}}^\mu \delta_{\alpha\bar{\beta}} - i f_\mu^i A_{\bar{\beta}}^\mu \delta_{\alpha\bar{\gamma}} - f_\alpha^j f_\beta^k f_\gamma^l \hat{R}_{jkl}^i, \\ f_{\alpha\beta\bar{\gamma}}^i - f_{\alpha\bar{\gamma}\beta}^i &= f_\mu^i R_{\alpha\beta\bar{\gamma}}^\mu + i f_{\alpha 0}^i \delta_{\beta\bar{\gamma}} - f_\alpha^j f_\beta^k f_\gamma^l \hat{R}_{jkl}^i, \\ f_{\alpha\beta 0}^i - f_{\alpha 0\beta}^i &= f_\gamma^i A_{\alpha\beta, \bar{\gamma}} - f_{\alpha\bar{\gamma}}^i A_\beta^\gamma - f_\alpha^j f_\beta^k f_0^l \hat{R}_{jkl}^i, \\ f_{\alpha\bar{\beta} 0}^i - f_{\alpha 0\bar{\beta}}^i &= -f_\gamma^i A_{\bar{\gamma}\bar{\beta}, \alpha} - f_{\alpha\gamma}^i A_{\bar{\beta}}^\gamma - f_\alpha^j f_\beta^k f_0^l \hat{R}_{jkl}^i, \end{aligned}$$

**Lemma 3.2.** (*CR Bochner-type formulas*) For any smooth map  $f : M \rightarrow N$ , we have

$$\begin{aligned} \frac{1}{2}\Delta_b|\nabla_b f|^2 &= |\nabla_b^f d_b f|^2 + \langle \nabla_b^f \tau(f; \theta, \hat{\nabla}), d_b f \rangle \\ &\quad + 2f_{\bar{\alpha}}^i f_{\mu}^i R_{\alpha\bar{\mu}} - i(m-2)(f_{\alpha}^i f_{\mu}^i A_{\bar{\mu}\bar{\alpha}} - f_{\bar{\alpha}}^i f_{\bar{\mu}}^i A_{\mu\alpha}) \\ &\quad + 2(f_{\bar{\alpha}}^i f_{\beta}^j f_{\bar{\beta}}^k f_{\alpha}^l \hat{R}_{ijkl} + f_{\alpha}^i f_{\beta}^j f_{\bar{\beta}}^k f_{\bar{\alpha}}^l \hat{R}_{ijkl}) \\ &\quad + 2i(f_{\bar{\alpha}}^i f_{0\alpha}^i - f_{\alpha}^i f_{0\bar{\alpha}}^i), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\Delta_b|df(T)|^2 &= 2f_{0\bar{\beta}}^i f_{0\beta}^i + \langle df(T), \nabla_T^f \tau(f; \theta, \hat{\nabla}) \rangle + 2f_0^i f_{\alpha}^j f_{\bar{\alpha}}^k f_0^l \hat{R}_{ijkl} \\ &\quad + 2(f_0^i f_{\beta}^i A_{\bar{\beta}\bar{\alpha},\alpha} + f_0^i f_{\bar{\beta}}^i A_{\beta\alpha,\bar{\alpha}} + f_0^i f_{\bar{\beta}\bar{\alpha}}^i A_{\beta\alpha} + f_0^i f_{\beta\alpha}^i A_{\bar{\beta}\bar{\alpha}}). \end{aligned}$$

For the proof of the above two lemma, you can read the paper [Liouville theorem for pseudoharmonic maps from Sasakian manifolds].