# LECTURE NOTES ON EXISTENCE OF SOLUTIONS

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Abstract. In this note, we mainly study two methods for the existence of solutions to elliptic equations.

## Contents



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## 1. Introduction

In section 4 the solvability of the classical Dirichlet problem for quasilinear equations is reduced to the establishment of certain apriori estimates for solutions. This reduction is achieved through the application of topological fixed point theorems (section 3) in appropriate function spaces. In Section 5, we briefly introduce the Single- and Double-Layer Potentials Methods.

## 2. Preliminaries

**Theorem 2.1.** *(Global Schauder Estimate) Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of $Lu = f$ in $\Omega$, where $f \in C^{\alpha}(\overline{\Omega})$ and the coefficients of $L$ satisfy,for positive constants $\lambda$, $\Lambda$,*

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^n$$

*and*

$$|a^{ij}|_{0,\alpha;\Omega}, |b^i|_{0,\alpha;\Omega}, |c|_{0,\alpha;\Omega} \leq \Lambda.$$

*Let $\varphi(x) \in C^{2,\alpha}(\overline{\Omega})$, and suppose $u = \varphi$ on $\partial\Omega$. Then*

$$|u|_{2,\alpha;\Omega} \leq C(|u|_{0;\Omega} + |\varphi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega})$$

*where $C = C(n, \alpha, \lambda, \Lambda, \Omega)$.*

**Theorem 2.2.** *Let $L$ be strictly elliptic in a bounded domain $\Omega$, with $c \leq 0$, and let $f$ and the coefficients of $L$ belong to $C^{\alpha}(\overline{\Omega})$. Suppose that $\Omega$ is a $C^{2,\alpha}$ domain and that $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Then the Dirichlet problem,*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

*has a (unique) solution lying in $C^{2,\alpha}(\overline{\Omega})$.*

## 3. Fixed-Point Theorems

**Theorem 3.1.** *(Brouwer Fixed-Point Theorem) A continuous mapping of a closed ball in $\mathbb{R}^n$ into itself has at least one fixed point.*

*Proof.* M1. Algebraic topology.
M2. Milnor's proof.(cf.[2])

□

The Brouwer fixed point theorem can be extended to infinite dimensional spaces in various ways. We require first the following extension to Banach spaces.

**Theorem 3.2.** *(Schauder's Fixed-Point Theorem) Let $\mathcal{G}$ be a compact, convex set in a Banach space $X$, and let $T$ be a continuous mapping of $\mathcal{G}$ into itself. Then $T$ has a fixed point, that is, $Tx = x$ for some $x \in X$.*

*Proof.* Let $k$ be any positive integer. Since $\mathcal{G}$ is compact, there exists a finite number of points $x_1, ..., x_N \in \mathcal{G}$, where $N = N(k)$, such that the balls $B^i = B_{1/k}(x_i)$, $i = 1, ..., N$, cover $\mathcal{G}$. Let $\mathcal{G}_k \subset \mathcal{G}$ be the convex hull of $\{x_1, ..., x_N\}$, and define the mapping $J_k : \mathcal{G} \to \mathcal{G}_k$ by

$$J_k(x) = \frac{\sum dist(x, \mathcal{G} - B^i)x_i}{\sum dist(x, \mathcal{G} - B^i)}$$

Clearly $J_k$ is continuous and for any $x \in \mathcal{G}$

$$\|J_k(x) - x\| \leq \frac{\sum dist(x, \mathcal{G} - B^i)\|x_i - x\|}{\sum dist(x, \mathcal{G} - B^i)} < \frac{1}{k}.$$

The mapping $J_k \circ T$ when restricted to $\mathcal{G}_k$ is accordingly a continuous mapping of $\mathcal{G}_k$ into itself and hence, by virtue of the Brouwer fixed point theorem, possesses a fixed point $x^{(k)}$. (Note that $\mathcal{G}_k$ is homeomorphic to a closed ball in some Euclidean space.(cf.[3])) Since $\mathcal{G}$ is compact, a subsequence of the sequence $x^{(k)}$, $k = 1, 2, ...$, converges to some $x \in \mathcal{G}$. We claim that $x$ is a fixed point of $T$. For, applying the above inequality to $Tx^{(k)}$, we have

$$\|x^{(k)} - Tx^{(k)}\| = \|J_k \circ Tx^{(k)} - Tx^{(k)}\| < \frac{1}{k},$$

and, since $T$ is continuous, we have

$$\lim_{k\to\infty} x^{(k)} = x = Tx$$

for some $x \in \mathcal{G}$. □

**Corollary 3.3.** *Let $\mathcal{G}$ be a closed convex set in a Banach space $X$ and let $T$ be a continuous mapping of $\mathcal{G}$ into itself such that the image $T\mathcal{G}$ is precompact. Then $T$ has a fixed point.*

**Remark 3.4.** *In the above theorems we note an essential difference from the contraction mapping principle in that the fixed points whose existence is asserted are not necessarily unique.*

A continuous mapping between two Banach spaces is called compact (or completely continuous) if the images of bounded sets are precompact (that is, their closures are compact). The following consequence of Corollary 3.3 is the fixed point result most often applied in our approach to the Dirichlet problem for quasilinear equations.

**Theorem 3.5.** *(Leray-Schauder Theorem) Let $T$ be a compact mapping of a Banach space $X$ into itself, and suppose there exists a constant $M$ such that*

$$\|u\|_X < M \tag{3.1}$$

*for all $x \in X$ and $\sigma \in [0, 1]$ satisfying $x = \sigma Tx$. Then $T$ has a fixed point.*

*Proof.* We can assume without loss of generality that $M = 1$. Let us define a mapping $T^*$ by

$$T^* = \begin{cases} Tx & if\ \|Tx\| \leq 1, \\ \frac{Tx}{\|Tx\|} & if\ \|Tx\| \geq 1. \end{cases}$$

Then $T^*$ is a continuous mapping of the closed unit ball $\overline{B}$ in $X$ into itself. Since $T\overline{B}$ is precompact the same is true of $T^*\overline{B}$. Hence by Corollary 3.3 the mapping $T^*$ has a fixed point $x$. We claim that $x$ is also a fixed point of $T$. For, suppose that $\|Tx\| \geq 1$. Then $x = T^*x = \sigma Tx$ if $\sigma = 1/\|Tx\|$, and $\|x\| = \|T^*\| = 1$ which contradicts (3.1) with $M = 1$. Hence $\|Tx\| \leq 1$ and consequently $x = T^*x = Tx$. □

**Remark 3.6.** *Theorem 3.5 implies that if $T$ is any compact mapping of a Banach space into itself (whether or not (3.1) holds), then for some $\sigma \in (0, 1]$ the mapping $\sigma T$ possesses a fixed point. Indeed, since $T\overline{B}$ is compact in $X$, there is an $A \geq 1$ such that $\|Tx\| \leq A$ for all $x \in \overline{B}$. Thus the mapping $\sigma T$ with $\sigma = \frac{1}{A}$ maps $\overline{B}$ into itself and our conclusion follows. Furthermore, if the estimate (3.1) holds then $\sigma T$ has a fixed point for all $\sigma \in [0, 1]$.*

## 4. Existence Results

In this section, we apply Leray-Schauder Theorem (Theorem 3.5) to the Dirichlet problem for quasilinear equations.

We fix a number $\beta \in (0, 1)$ and take the Banach space $X$ to be the Holder space $C^{1,\beta}(\overline{\Omega})$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$. Let $Q$ be the operator given by

$$Qu = a^{ij}(x, u, Du)D_{ij}u + b(x, u, Du) \tag{4.1}$$

and assume that $Q$ is elliptic in $\overline{\Omega}$, that is, the coefficient matrix $[a^{ij}(x, z, p)]$ is positive for all $(x, z, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. We also assume, for some $\alpha \in (0, 1)$, that the coefficients $a^{ij}, b \in C^{\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, that the boundary $\partial\Omega \in C^{2,\alpha}$ and that $\varphi$ is a given function in $C^{2,\alpha}(\overline{\Omega})$. For all $v \in C^{1,\beta}(\overline{\Omega})$, the operator $T$ is defined by letting $u = Tv$ be the unique solution in $C^{2,\alpha\beta}(\overline{\Omega})$ of the *linear* Dirichlet problem,

$$\begin{cases} a^{ij}(x, v, Dv)D_{ij}u + b(x, v, Dv) = 0 & \textit{in } \Omega, \\ u = \varphi & \textit{on } \partial\Omega. \end{cases} \tag{4.2}$$

The unique solvability of the problem (4.2) is guaranteed by the linear existence result, Theorem 2.2. The solvability of the Dirichlet problem,

$$\begin{cases} Qu = 0 & \textit{in } \Omega, \\ u = \varphi & \textit{on } \partial\Omega, \end{cases}$$

in the space $C^{2,\alpha}(\overline{\Omega})$ is thus equivalent to the solvability of the equation $u = Tu$ in the Banach space $X = C^{1,\beta}(\overline{\Omega})$. The equation $u = \sigma Tu$ in $X$ is equivalent to the Dirichlet problem

$$\begin{cases} Q_\sigma u = a^{ij}(x, u, Du)D_{ij}u + b(x, u, Du) = 0 & \textit{in } \Omega, \\ u = \sigma\varphi & \textit{on } \partial\Omega. \end{cases} \tag{4.3}$$

By applying Theorem 3.5, we can then prove the following criterion for existence.

**Theorem 4.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and suppose that $Q$ is elliptic in $\overline{\Omega}$ with coefficients $a^{ij}, b \in C^{\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, $0 < \alpha < 1$. Let $\partial\Omega \in C^{2,\alpha}$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$.*

*Then, if for some $\beta > 0$ there exists a constant $M$, independent of $u$ and $\sigma$, such that every $C^{2,\alpha}(\overline{\Omega})$ solution of the Dirichlet problems,*

$$\begin{cases} Q_\sigma u = 0 & \text{in } \Omega, \\ u = \sigma\varphi & \text{on } \partial\Omega, \end{cases}$$

*$0 \leq \sigma \leq 1$, satisfies*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} < M,$$

*it follows that the Dirichlet problem,*

$$\begin{cases} Qu = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

*is solvable in $C^{2,\alpha}(\overline{\Omega})$.*

*Proof.* In view of the remarks preceding the statement of the theorem, it only remains to show that the operator $T$ is continuous and compact. By virtue of the global Schauder estimate, Theorem 2.1, $T$ maps bounded sets in $C^{1,\beta}(\overline{\Omega})$ into bounded sets in $C^{2,\alpha\beta}(\overline{\Omega})$ which (by Arzela's theorem) are precompact in $C^2(\overline{\Omega})$ and $C^{1,\beta}(\overline{\Omega})$. In order to show the continuity of $T$, we let $v_m$, $m = 1, \ldots$ converge to $v$ in $C^{1,\beta}(\overline{\Omega})$. Then, since the sequence $\{Tv_m\}$ is precompact in $C^2(\overline{\Omega})$, every subsequence in turn has a convergent subsequence. Let $\{T\overline{v}_m\}$ be such a convergent subsequence with limit $u \in C^2(\overline{\Omega})$. Then since

$$\begin{aligned} & a^{ij}(x, v, Dv)D_{ij}u + b(x, v, Dv) \\ = & \lim_{m\to\infty} \{a^{ij}(x, \overline{v}_m, D\overline{v}_m)D_{ij}u + b(x, \overline{v}_m, D\overline{v}_m)\} \\ = & 0 \end{aligned} \tag{4.4}$$

we must have $u = Tv$, and hence the sequence $\{Tv_m\}$ itself converges to $u$. □

**Remark 4.2.** *Theorem 4.1 reduces the solvability of the Dirichlet problem*

$$\begin{cases} Qu = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

*to the apriori estimation in the space $C^{1,\beta}(\overline{\Omega})$, for some $\beta > 0$, of the solutions of a related family of problems.*

In practice it is desirable to break the derivation of the apriori estimates into four stages(cf.[1]):

I. Estimation of $\sup_\Omega |u|$;
II. Estimation of $\sup_{\partial\Omega} |Du|$;
III. Estimation of $\sup_\Omega |Du|$;
IV. Estimation of $[Du]_{\beta;\Omega}$, for some $\beta > 0$.

We shall briefly mention how assumptions in Theorem 4.1 can be verified for the minimal surface equation. For simplicity, we consider the case where $\Omega$ is a uniformly convex, $C^{2,\alpha}-$bounded domain in $\mathbb{R}^n$ and the following Dirichlet problem:

$$\begin{cases} div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & in\ \Omega. \\ u = \phi & on\ \partial\Omega, \end{cases} \tag{4.5}$$

We assume that $\phi \in C^{2,\alpha}(\partial\Omega)$.

Suppose $u$ is a $C^{2,\alpha}-$solution of (4.5); then the maximum principle implies that

$$\|u\|_{L^\infty(\overline{\Omega})} \leq \|\phi\|_{L^\infty(\partial\Omega)} \equiv C_0 < \infty. \tag{4.6}$$

Next, by the uniform convexity of $\partial\Omega$ and the $C^{2,\alpha}-$regularity of $\phi$, we can check that(cf.[1]), for every $x_0 \in \partial\Omega$, there exist linear functions $l^{\pm}_{x_0}(x)$ such that

$$l^{\pm}_{x_0}(x) = \phi(x_0) \quad and \quad l^{-}_{x_0}(x) \leq \phi(x) \leq l^{+}_{x_0}(x)$$

for all $x \in \partial\Omega$. Since linear functions are solutions of $div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$ in $\Omega$, from the maximum principle we conclude that

$$l^{-}_{x_0}(x) \leq u(x) \leq l^{+}_{x_0}(x), \quad x \in \overline{\Omega};$$

in particular,

$$|\nabla u(x_0)| \leq \max |l^{\pm}_{x_0}(x_0)| \equiv C_1 < \infty.$$

On the other hand, if $u$ is a $C^{2,\alpha}-$solution of (4.5), then $u_i = \frac{\partial}{\partial x_i}u$, $i = 1, ..., n$, satisfies

$$\frac{\partial}{\partial x_i}(F_{P_iP_j}(Du)u_j) = 0.$$

Here $F(Du) = \sqrt{1 + |\nabla u|^2}$, hence $(F_{P_iP_j}(Du)) > 0$. Thus $u_i$ satisfies the maximum principle. Therefore we have

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \|\nabla u\|_{L^\infty(\partial\Omega)} \equiv C_1 < \infty.$$

From the above inequalities we further deduce that

$$\|\nabla u\|_{C^\beta(\overline{\Omega})} \leq C(C_0, C_1, C_2) < \infty. \tag{4.7}$$

where $C_2 = \|\phi\|_{C^{2,\alpha}(\partial\Omega)}$. This follows from De Giorgi - Moser theory.

We rewrite $div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$ in $\Omega$ as

$$\Delta u - \frac{u_i u_j}{1 + |\nabla u|^2} u_{ij} = 0 \tag{4.8}$$

and combine (4.7) with (4.8) and the Schauder estimates to obtain

$$\|\nabla u\|_{C^{2,\beta}(\overline{\Omega})} \leq C(C_0, C_1, C_2, \Omega) < \infty.$$

where we may assume that $0 < \beta \leq \alpha$.

## 5. Single- and Double-Layer Potentials Methods

Now we assume that $\Omega$ is a bounded, connected domain in $\mathbb{R}^n$, $n \geq 3$, with a $C^2-$boundary.(Here we assume $n \neq 2$ to simplify matters and avoid technicalities.) Consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $f \in C(\partial\Omega)$.

Let $\gamma(x) = \frac{C_n}{|x|^{n-2}}$ be the fundamental solution of the Laplace operator in $\mathbb{R}^n$; here

$$C_n = \frac{-1}{(n-2)\omega_n} = \frac{-1}{(n-2)} \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}}.$$

**Definition 5.1.** *Set $R(x,y) = -\gamma(x-y)$, and for $f \in C(\partial\Omega)$, we define the double-layer potential*

$$\mathcal{D}f(x) = \int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}_y} R(x,y) f(y) d\mathcal{H}^{n-1}(y), \quad x \notin \partial\Omega, \tag{5.2}$$

*and the single-layer potential*

$$\mathcal{S}f(x) = \int_{\partial\Omega} R(x,y) f(y) d\mathcal{H}^{n-1}(y), \quad x \notin \partial\Omega. \tag{5.3}$$

*Here $\mathbf{n}_y$ is the outward unit normal for $\partial\Omega$ at $y$.*

**Remark 5.1.** *It is easy to check that*

$$\Delta \mathcal{D}f(x) = 0, \quad for\ x \in \mathbf{R}^n \backslash \partial\Omega.$$

We need to understand the boundary behavior of $\mathcal{D}f(x)$ on $\partial\Omega$.

**Lemma 5.2.** *If $f \in C(\partial\Omega)$, then*
*(i) $\mathcal{D}f(x) \in C(\overline{\Omega})$,*
*(ii) $\mathcal{D}f(x) \in C(\overline{\Omega^c})$.*

In other words, $\mathcal{D}f(x)$ can be extended continuously from inside $\Omega$ to $\overline{\Omega}$ and from outside $\Omega$ to $\overline{\Omega^c}$.

Let $\mathcal{D}_+f(x)$ and $\mathcal{D}_-f(x)$ be the restrictions of these two functions to $\partial\Omega$. Set

$$K(x,y) = \frac{\partial}{\partial \mathbf{n}_y} R(x,y) = \frac{1}{\omega_n} \frac{\langle x-y, \mathbf{n}_y \rangle}{|x-y|^n}.$$

Thus

$$K(x,y) \in C(\partial\Omega \times \partial\Omega \backslash \{(x,x) : x \in \partial\Omega\})$$

and

$$|K(x,y)| \leq \frac{C}{|x-y|^{n-2}}$$

for $x, y \in \partial\Omega$ and some $C < \infty$. The latter estimate follows from the $C^2$ property of $\partial\Omega$.

We shall define, for $f \in C(\partial\Omega)$, the operator

$$\mathcal{T}f(x) = \int_{\partial\Omega} K(x,y)f(y)d\mathcal{H}^{n-1}(y), \quad x \in \partial\Omega. \tag{5.4}$$

We have the following:

**Lemma 5.3.** *(Jump Relations for $\mathcal{D}$)*
*(i)* $\mathcal{D}_+ = \frac{1}{2}I + \mathcal{T}$
*(ii)* $\mathcal{D}_- = -\frac{1}{2}I + \mathcal{T}$
*Moreover,* $\mathcal{T} : C(\partial\Omega) \to C(\partial\Omega)$ *is compact.*

*Proof.* We first verify that $\mathcal{T}$ defined by (5.4) is a compact operator from $C(\partial\Omega) \to C(\partial\Omega)$. Let

$$K_N(x,y) = signK(x,y) \cdot \min\{N, |K(x,y)|\}, \quad N \in \mathbb{Z}_+.$$

Thus $K_N$ is continuous on $\partial\Omega \times \partial\Omega$, and the Arzela-Ascoli theorem implies that $\mathcal{T}_N f(x) = \int_{\partial\Omega} K_N(x,y)f(y)d\mathcal{H}^{n-1}(y)$ is compact on $C(\partial\Omega)$. Furthermore, since $\|\mathcal{T}_N\| \leq \sup_{x\in\partial\Omega} \|K_N(x,y)\|_{L^1(\partial\Omega)} \leq C < \infty$ where $C$ is independent of $N$, it is rather easy to see that

$$\|\mathcal{T}_N - \mathcal{T}_{N+1}\| \leq C\big[(\frac{1}{N})^{\frac{1}{n-2}} - (\frac{1}{N+1})^{\frac{1}{n-2}}\big] \leq CN^{-1-\frac{1}{n-2}}.$$

We therefore conclude that $\mathcal{T} = \lim_{N\to\infty} \mathcal{T}_N$ is a compact operator on $C(\partial\Omega)$.

Next we apply the divergence theorem on $\Omega|B_\delta(x)$ for small positive $\delta$ with $\delta \to 0^+$ to obtain

$$\mathcal{D}f(x) = \int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}_y} R(x,y)d\mathcal{H}^{n-1}(y) = 1, \quad if\ x \in \Omega, \tag{5.5}$$

$$\mathcal{T}f(x) = \int_{\partial\Omega} K(x,y)d\mathcal{H}^{n-1}(y) = \frac{1}{2}, \quad if\ x \in \partial\Omega. \tag{5.6}$$

Let $x_0 \in \partial\Omega$ and $x \in \Omega$ such that $x \to x_0$. We want to verify that

$$\mathcal{D}f(x) \Rightarrow \frac{1}{2}f(x_0) + \mathcal{T}f(x_0). \tag{5.7}$$

Here we observe that $\int_{\partial\Omega} |\frac{\partial}{\partial \mathbf{n}_y} R(x,y)|d\mathcal{H}^{n-1}(y) \leq C < \infty$ for all $x \notin \partial\Omega$. Thus, in particular, $\|\mathcal{D}f\|_{L^\infty(R^n\backslash\partial\Omega)} \leq C\|f\|_{L^\infty(\partial\Omega)}$.

If $x_0 \notin$ support of $f$, then it is obvious that

$$\int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}_y} R(x,y)d\mathcal{H}^{n-1}(y) \stackrel{x\to x_0}{\longrightarrow} \int_{\partial\Omega} K(x_0,y)d\mathcal{H}^{n-1}(y) = \mathcal{T}f(x_0).$$

If $x_0 \in$ support of $f$ and $f(x_0) = 0$, then we let $\{f_k\} \subset C(\partial\Omega)$ such that

$$\|f - f_k\|_{L^\infty(\partial\Omega)} \stackrel{k\to\infty}{\longrightarrow} 0,$$

and $x_0 \notin$ support of $f_k$ for each $k$, $k = 1, 2...$ Then

$$\begin{aligned}|\mathcal{D}f(x) - \mathcal{T}f(x)| \leq &|\mathcal{D}(f - f_k)(x)| + |\mathcal{T}(f - f_k)(x)| + |\mathcal{D}f_k(x) - \mathcal{T}f_k(x)| \\ \leq &C\|f - f_k\|_{L^\infty(\partial\Omega)} + \|\mathcal{T}\|\|f - f_k\|_{L^\infty(\partial\Omega)} + \|\mathcal{D}f_k(x) - \mathcal{T}f_k(x)\|\end{aligned}$$

We initially choose $k$ large so that the first two terms on the right-hand side of the above inequality will be small. We then observe that for fixed $k$ (large) as $x \to x_0$, the last term in the inequality also goes to 0.

To complete the proof it suffices to verify the case when $f = 1$, for which the result is trivial. If we replace $\Omega$ by $\Omega^c$ then all the other statements in Lemmas 5.2 and 5.3 follow.

□

To conclude our consideration of double-layer potentials we need to show how to use them to solve the Dirichlet problem (5.1).

We begin with a $g \in C(\partial\Omega)$ and let $u(x) = \mathcal{D}g(x)$ for $x \in \Omega$. It is clear from our previous discussion that $\Delta u = 0$ in $\Omega$ and $u \in C(\overline{\Omega})$; moreover, $u|_{\partial\Omega} = (\frac{1}{2}I + \mathcal{T})g$. Therefore, we need to solve for $g$ a given $f \in C(\partial\Omega)$, $f = (\frac{1}{2}I + \mathcal{T})g$. Since $\mathcal{T}$ is compact, $(\frac{1}{2}I + \mathcal{T})$ is obviously a $1:1$ map on $C(\partial\Omega)$; hence it is also an onto map from $C(\partial\Omega)$ to $C(\partial\Omega)$. This last statement follows from Theorem 3.5.

Finally, we shall state without proof the results corresponding to those for single-layer potentials (5.3). All of the proofs are similar to those for Lemmas 5.2 and 5.3 above.

Once again we assume $\Omega$ to be class $C^2$ and $f \in C(\partial\Omega)$.

**Lemma 5.4.** *If $f \in C(\partial\Omega)$, then*
*(i) $\mathcal{D}_+\mathcal{S}(x) = grad\mathcal{S}(f) \in C(\overline{\Omega_{\delta_0}})$,*
*(ii) $\mathcal{D}_-\mathcal{S}(x) = grad\mathcal{S}(f) \in C(\overline{\Omega^c_{\delta_0}})$.*
*Here $\overline{\Omega_{\delta_0}} = \{x \in \overline{\Omega} : dist(x, \partial\Omega) \leq \delta_0\}$ for some small $\delta_0 > 0$.*

Let $K^*(x,y) = K(y,x)$ and define

$$\mathcal{T}^* f(x) = \int_{\partial\Omega} K^*(x,y) f(y) d\mathcal{H}^{n-1}(y), \quad x \in \partial\Omega.$$

**Lemma 5.5.** *(Jump Relations for $\mathcal{DS}(f)$)*
*(i) $\mathcal{D}_+\mathcal{S}(f) = -\frac{1}{2}I + \mathcal{T}^*$,*
*(ii) $\mathcal{D}_-\mathcal{S}(f) = \frac{1}{2}I + \mathcal{T}^*$.*

Single-layer potentials can be used to solve the Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = f & \text{on } \partial\Omega. \end{cases}$$

**Remark 5.6.** *Layer potentials can be used to solve more general elliptic equations (and systems) with constant coefficients on smooth domains. This method can be further generalized to $C^{1,\alpha}-$domains for general elliptic equations of second order with $C^\alpha-$coefficients (or general first-order elliptic systems with suitably smooth coefficients). The latter is often referred to as ADN theory due to Agmon, Douglis, and Nirenberg.*

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