THE HARNACK ESTIMATE FOR THE YAMABE FLOW ON CR MANIFOLDS OF DIMENSION 3

Our evolution equation, the so-called Yamabe flow, reads as follows:

$$\partial_t \theta_{(t)} = -2W_{(t)}\theta_{(t)}. \tag{0.1}$$

Write $\theta_{(t)} = e^{2\lambda_{(t)}}\hat{\theta}$ with respect to a fixed contact form $\hat{\theta}$. Then we can express (0.1) in terms of $\lambda_{(t)}$:

$$\lambda'_{(t)} = -W_{(t)}. \tag{0.2}$$

Since the linearization of $-W$ with respect to λ is a second-order subelliptic operator, the short time existence and uniqueness of a solution to (0.2) follows from a standard argument.

Define the following Harnack quantity:

$$Z(\theta, \eta) := 2\Delta_b W + W^2 + \frac{W}{t} + \langle \nabla_b W, \eta \rangle_{J,\theta} + \frac{1}{8} W |\eta|^2_{J,\theta}.$$

1. MAIN THEOREM

Theorem 1.1. *Let (M, ξ, J) be a closed spherical CR 3-manifold. Suppose there is a pseudo-Hermitian form $\hat{\theta}$ (together with J defining a positive pseudo-Hermitian structure) with vanishing torsion and positive Tanaka-Webester curvature. Then under the Yamabe flow (0.1),*

$$Z(\theta_{(t)}, \eta) \geq 0,$$

for any Legendrian vector field η.

Theorem 1.2. *Consider the Yamabe flow (0.1) under the same assumption as in Theorem 1.1. Then we have, for all points x_1, x_2 in M and $0 < t_1 < t_2$,*

$$\frac{W(x_2, t_2)}{W(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^{-2} \exp(-L/16),$$

where

$$L = \inf_{\gamma} \int_{t_1}^{t_2} |\dot{\gamma}|^2_{J,\theta_{(t)}} dt$$

and the infimum is taken over all Legendrian paths γ with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$.

2. Basics Derived from the Flow

Let $\hat{\theta}, \hat{\theta}^1, \hat{\theta}^{\bar{1}}$ satisfy $d\hat{\theta} = i\hat{\theta}^1 \wedge \hat{\theta}^{\bar{1}}$. Now consider the change of contact form: $\theta = e^{2\lambda}\hat{\theta}$. Choose $\theta^1 = e^{\lambda}(\hat{\theta}^1 + 2i\lambda_{\bar{1}}\hat{\theta})$ such that $h_{1\bar{1}} = \hat{h}_{\bar{1}1} = 1$. Then the associated connection form ω_1^1, torsion A_{11}, and Tanaka-Webster curvature W transform as follows:

$$\omega_1^1 = \hat{\omega}_1^1 + 3(\lambda_1\hat{\theta}^1 - \lambda_{\bar{1}}\hat{\theta}^{\bar{1}}) + i(\hat{\Delta}_b\lambda + 4|\hat{\nabla}_b\lambda|^2_{J,\hat{\theta}})\hat{\theta},$$

$$A_{11} = e^{-2\lambda}(\hat{A}_{11} + 2i\lambda_{11} - 4i(\lambda_1)^2),$$
$$W = e^{-2\lambda}(-4\hat{\Delta}_b\lambda - 4|\hat{\nabla}_b\lambda|^2_{J,\hat{\theta}} + \hat{W}).$$

Lemma 2.1. *Under the Yamabe flow* (0.1)*, we have*

$$\dot{W} = 4\Delta_b W + 2W^2, \tag{2.1}$$
$$\dot{A}_{11} = 2WA_{11} - 2iW_{11},$$

in which Δ_b, the torsion, and covariant derivatives are with respect to $\theta_{(t)}$ and induced coframes as shown previously.

Now applying the maximum principle to (2.1), we obtain

Corollary 2.2. *If $W_{(0)} \geq c > 0$, then the inequality $W \geq c > 0$ is preserved under the Yamabe flow* (0.1)*.*

Lemma 2.3. *Under the Yamabe flow* (0.1)*, we have*

$$\partial_t(\Delta_b f) = \Delta_b(\dot{f}) + 2W\Delta_b f - 2\langle \nabla_b W, \nabla_b f\rangle_{J,\theta_{(t)}},$$

for a smooth real-valued function $f = f(x,t)$ defined on $M \times \mathbb{R}$.