Lecture Notes
on
Elliptic Differential Equations

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Preface

The purpose of these note is to present some basic methods for obtaining various a priori estimates for second order partial differential equations of elliptic type with particular emphasis on maximum principles, Harnack inequalities, and their applications. The equations we deal with are always linear, although most of the methods obviously apply to nonlinear problems. Students with some knowledge of real variables and Sobolev functions should be able to follow these notes without much difficulty.

It is not our intention to give a complete account of the related theory. Our goal is simply to provide these notes as a bridge between the books of John [**4**] and of Evans [**2**], which also study equations of other types, and the book of Gilbarg and Trudinger [**3**] which gives a relatively complete account of the theory of elliptic equations of second order. We also hope our notes can serve as a bridge between the book of Krylov [**5**] on the classical theory of elliptic equations developed before or around the 1960s and the book by Caffarelli and Cabré [**1**] which studies fully nonlinear elliptic equations, the theory obtained in the 1980s.

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CHAPTER 1

Harmonic Functions

In this chapter, we use several different methods to study harmonic functions. These include mean value properties, fundamental solutions, maximum principles and the energy method. Four sections in this chapter are relatively independent of each other.

1.1. Mean Value Properties

We begin this section with the definition of mean value properties. We assume that Ω is a connected domain in $\mathbb{R}^n$.

DEFINITION 1.1. For $u \in C(\Omega)$, (i) u satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int\limits_{\partial B_r(x)} u(y) d\sigma_y \quad \text{for any } B_r(x) \subset \Omega;$$

(ii) u satisfies the second mean value property if

$$u(x) = \frac{n}{\omega_n r^n} \int\limits_{B_r(x)} u(y) dy \quad \text{for any } B_r(x) \subset \Omega,$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$.

REMARK 1.2. These two definitions are equivalent. In fact, if we write (i) as

$$u(x) r^{n-1} = \frac{1}{\omega_n} \int\limits_{\partial B_r(x)} u(y) d\sigma_y,$$

we may integrate to get (ii). If we write (ii) as

$$u(x) r^n = \frac{n}{\omega_n} \int\limits_{B_r(x)} u(y) dy,$$

we may differentiate to get (i).

REMARK 1.3. We may write the mean value properties in the following equivalent ways:
(i) u satisfies the first mean value property if

$$u(x) = \frac{1}{\omega_n} \int\limits_{|y|=1} u(x + ry) d\sigma_y \quad \text{for any } B_r(x) \subset \Omega;$$

(ii) u satisfies the second mean value property if

$$u(x) = \frac{n}{\omega_n} \int_{|y| \le 1} u(x + ry)\ dy \quad \text{for any } B_r(x) \subset \Omega.$$

Now we prove the maximum principle for functions satisfying mean value properties.

THEOREM 1.4. *If $u \in C(\bar{\Omega})$ satisfies the mean value property in Ω, then u assumes its maximum and minimum only on $\partial\Omega$ unless u is constant.*

PROOF. We only prove for the maximum. Set

$$\Sigma = \left\{ x \in \Omega;\ u(x) = M \equiv \max_{\bar{\Omega}} u \right\} \subset \Omega.$$

It is obvious that Σ is relatively closed. Next we show that Σ is open. For any $x_0 \in \Sigma$, take $\bar{B}_r(x_0) \subset \Omega$ for some $r > 0$. By the mean value property, we have

$$M = u(x_0) = \frac{n}{\omega_n r^n} \int_{B_r(x_0)} u(y)dy \le M \frac{n}{\omega_n r^n} \int_{B_r(x_0)} dy = M.$$

This implies $u = M$ in $B_r(x_0)$. Hence Σ is both closed and open in Ω. Therefore either $\Sigma = \phi$ or $\Sigma = \Omega$. $\square$

Now we begin to discuss harmonic functions.

DEFINITION 1.5. A function $u \in C^2(\Omega)$ is harmonic if $\triangle u = 0$ in Ω.

THEOREM 1.6. *Let $u \in C^2(\Omega)$ be harmonic in Ω. Then u satisfies the mean value property in Ω.*

PROOF. Take any ball $B_r(x) \subset \Omega$. For any $\rho \in (0, r)$, we apply the divergence theorem in $B_\rho(x)$ and get

$$\begin{aligned} \int_{B_\rho(x)} \triangle u(y)dy &= \int_{\partial B_\rho} \frac{\partial u}{\partial \nu}\, d\sigma = \rho^{n-1} \int_{|w|=1} \frac{\partial u}{\partial \rho}(x + \rho w) d\sigma_w \\ &= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|w|=1} u(x + \rho w) d\sigma_w. \end{aligned} \tag{1}$$

Hence for the harmonic function u, we have for any $\rho \in (0, r)$

$$\frac{\partial}{\partial \rho} \int_{|w|=1} u(x + \rho w)\, d\sigma_w = 0.$$

Integrating from 0 to r, we obtain

$$\int_{|w|=1} u(x + rw)\, d\sigma_w = \int_{|w|=1} u(x)\, d\sigma_w = u(x)\omega_n,$$

or

$$u(x) = \frac{1}{\omega_n} \int_{|w|=1} u(x + rw)\, d\sigma_w = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) d\sigma_y.$$

This finishes the proof. $\square$

REMARK 1.7. For a function u satisfying the mean value property, u is not required to be smooth. However a harmonic function is required to be C^2. We now prove these two are equivalent.

THEOREM 1.8. *If $u \in C(\Omega)$ has the mean value property in Ω, then u is smooth and harmonic in Ω.*

PROOF. Choose $\varphi \in C_0^\infty(B_1)$ with $\int_{B_1} \varphi = 1$ and $\varphi(x) = \psi(|x|)$, i.e.

$$\omega_n \int_0^1 r^{n-1}\psi(r)\ dr = 1.$$

We define $\varphi_\varepsilon(z) = \dfrac{1}{\varepsilon^n}\varphi\big(\dfrac{z}{\varepsilon}\big)$ for $\varepsilon > 0$. Now for any $x \in \Omega$, consider $\varepsilon <$ dist $(x, \partial\Omega)$. Then we have

$$\begin{aligned}\int_\Omega u(y)\varphi_\varepsilon(y-x)dy &= \int u(x+y)\varphi_\varepsilon(y)dy = \frac{1}{\varepsilon^n}\int_{|y|<\varepsilon} u(x+y)\varphi\left(\frac{y}{\varepsilon}\right)dy \\ &= \int_{|y|<1} u(x+\varepsilon y)\varphi(y)dy \\ &= \int_0^1 r^{n-1}dr\int_{\partial B_1} u(x+\varepsilon rw)\varphi(rw)d\sigma_w \\ &= \int_0^1 \psi(r)r^{n-1}dr\int_{|w|=1} u(x+\varepsilon rw)d\sigma_w \\ &= u(x)\omega_n\int_0^1 \psi(r)r^{n-1}dr = u(x),\end{aligned}$$

where in the last equality we used the mean value property. Hence we get

$$u(x) = (\varphi_\varepsilon * u)(x) \quad \text{for any } x \in \Omega_\varepsilon = \{y \in \Omega;\ d(y, \partial\Omega) > \varepsilon\}.$$

Therefore, u is smooth. Moreover, by (1) in the proof of Theorem 1.6 and the mean value property, we have for any $B_r(x) \subset \Omega$

$$\int_{B_r(x)} \Delta u = r^{n-1}\frac{\partial}{\partial r}\int_{|w|=1} u(x+rw)d\sigma_w = r^{n-1}\frac{\partial}{\partial r}\big(\omega_n u(x)\big) = 0.$$

This implies $\Delta u = 0$ in Ω. □

REMARK 1.9. By combining Theorems 1.4-1.8, we conclude that harmonic functions are smooth and satisfy the mean value property. Hence harmonic functions satisfy the maximum principle, a consequence of which is the uniqueness of solutions of the following Dirichlet problem in a bounded domain

$$\begin{aligned}\Delta u &= f \text{ in } \Omega \\ u &= \varphi \text{ on } \partial\Omega,\end{aligned}$$

for $f \in C(\Omega)$ and $\varphi \in C(\partial\Omega)$. In general, the uniqueness does not hold for unbounded domains. Consider the following Dirichlet problem in an unbounded domain Ω

$$\begin{aligned}\Delta u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega.\end{aligned}$$

First, we consider the case $\Omega = \{x \in \mathbb{R}^n; |x| > 1\}$. Then we have a nontrivial solution u given by

$$u(x) = \begin{cases}\log|x| & \text{for } n = 2; \\ |x|^{2-n} - 1 & \text{for } n \geq 3.\end{cases}$$

Note $u \to \infty$ as $|x| \to \infty$ for $n = 2$ and u is bounded for $n \geq 3$. Next, we consider the upper half space $\Omega = \{x \in \mathbb{R}^n;\ x_n > 0\}$. Then $u(x) = x_n$ is a nontrivial solution, which is unbounded.

In the following, we discuss interior gradient estimates.

LEMMA 1.10. *Suppose $u \in C\big(\bar{B}_R(x_0)\big)$ is harmonic in $B_R(x_0)$. Then there holds*

$$|Du(x_0)| \leq \frac{n}{R} \max_{\bar{B}_R(x_0)} |u|.$$

PROOF. For simplicity, we assume $u \in C^1(\bar{B}_R)$. Since u is smooth, then $\Delta(D_{x_i}u) = 0$, i.e., $D_{x_i}u$ is also harmonic in B_R. Hence $D_{x_i}u$ satisfies the mean value property. By the divergence theorem, we have

$$D_{x_i}u(x_0) = \frac{n}{\omega_n R^n} \int_{B_R(x_0)} D_{x_i}u(y)dy = \frac{n}{\omega_n R^n} \int_{\partial B_R(x_0)} u(y)\ \nu_i d\sigma_y,$$

which implies

$$|D_{x_i}u(x_0)| \leq \frac{n}{\omega_n R^n} \max_{\partial B_R(x_0)} |u| \cdot \omega_n R^{n-1} \leq \frac{n}{R} \max_{\bar{B}_R(x_0)} |u|.$$

This finishes the proof. □

LEMMA 1.11. *Suppose $u \in C\big(\bar{B}_R(x_0)\big)$ is a nonnegative harmonic function in $B_R(x_0)$. Then there holds*

$$|Du(x_0)| \leq \frac{n}{R} u(x_0).$$

PROOF. As before, by the divergence theorem and the nonnegativeness of u, we have

$$|D_{x_i}u(x_0)| \leq \frac{n}{\omega_n R^n} \int_{\partial B_R(x_0)} u(y)\ d\sigma_y = \frac{n}{R} u(x_0),$$

where in the last equality we used the mean value property. □

COROLLARY 1.12. *A harmonic function in $\mathbb{R}^n$ bounded from above or below is constant.*

PROOF. Suppose u is a harmonic function in $\mathbb{R}^n$. We prove that u is a constant if $u \geq 0$. In fact for any $x \in \mathbb{R}^n$, we apply Lemma 1.11 to u in $B_R(x)$ and then let $R \to \infty$. We conclude $Du(x) = 0$ for any $x \in \mathbb{R}^n$. □

LEMMA 1.13. *Suppose $u \in C\big(\bar{B}_R(x_0)\big)$ is harmonic in $B_R(x_0)$. Then there holds for any multi-index α with $|\alpha| = m$*

$$|D^\alpha u(x_0)| \leq \frac{n^m e^{m-1} m!}{R^m} \max_{\bar{B}_R(x_0)} |u|.$$

PROOF. We will prove by an induction on $m \geq 1$. It holds for $m = 1$ by Lemma 1.10. We assume it holds for m and consider $m + 1$. For $0 < \theta < 1$, define $r = (1 - \theta)R \in (0, R)$. We apply Lemma 1.10 to u in $B_r(x_0)$ and get

$$|D^{m+1}u(x_0)| \leq \frac{n}{r} \max_{\bar{B}_r(x_0)} |D^m u|.$$

By the induction assumption, we have

$$\max_{\bar{B}_r(x_0)} |D^m u| \leq \frac{n^m \cdot e^{m-1} \cdot m!}{(R-r)^m} \max_{\bar{B}_R(x_0)} |u|.$$

Hence we obtain

$$|D^{m+1} u(x_0)| \leq \frac{n}{r} \cdot \frac{n^m e^{m-1} m!}{(R-r)^m} \max_{\bar{B}_R(x_0)} |u| = \frac{n^{m+1} e^{m-1} m!}{R^{m+1} \theta^m (1-\theta)} \max_{\bar{B}_R(x_0)} |u|.$$

By taking $\theta = \dfrac{m}{m+1}$, we have

$$\frac{1}{\theta^m (1-\theta)} = \left(1 + \frac{1}{m}\right)^m (m+1) < e(m+1).$$

Hence the result is established for any single derivative. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we note $\alpha_1! \cdots \alpha_n! \leq (|\alpha|)!$. $\square$

Theorem 1.14. *Harmonic functions are analytic.*

Proof. Suppose u is a harmonic function in Ω. For any fixed $x \in \Omega$, take $B_{2R}(x) \subset \Omega$ and $h \in \mathbb{R}^n$ with $|h| \leq R$. We have by the Taylor expansion

$$u(x+h) = u(x) + \sum_{i=1}^{m-1} \frac{1}{i!} \left[\left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n} \right)^i u \right] (x) + R_m(h),$$

where

$$R_m(h) = \frac{1}{m!} \left[\left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n} \right)^m u \right] (x_1 + \theta h_1, \ldots, x_n + \theta h_n),$$

for some $\theta \in (0,1)$. Note that $x + h \in B_R(x)$ for $|h| < R$. Hence by Lemma 1.13, we obtain

$$|R_m(h)| \leq \frac{1}{m!} |h|^m \cdot n^m \cdot \frac{n^m e^{m-1} m!}{R^m} \max_{\bar{B}_{2R}(x)} |u| \leq \left(\frac{|h| n^2 e}{R} \right)^m \max_{\bar{B}_{2R}(x)} |u|.$$

Then for any h with $|h| n^2 e < R/2$, $R_m(h) \to 0$ as $m \to \infty$. $\square$

Next we prove the Harnack inequality.

Theorem 1.15. *Suppose u is a nonnegative harmonic function in Ω. Then for any compact subset K of Ω*

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y) \quad \textit{for any } x, y \in K,$$

where C is a positive constant depending only on Ω and K.

Proof. By the mean value property, we can prove easily that there holds for $B_{4R}(x_0) \subset \Omega$

$$\frac{1}{c} u(y) \leq u(x) \leq c u(y) \quad \text{for any } x, y \in B_R(x_0),$$

where c is a positive constant depending only on n.

Now for the given compact subset K, take $x_1, \ldots, x_N \in K$ such that $\{B_R(x_i)\}$ covers K with $4R < \text{dist}\,(K, \partial\Omega)$. Then we can choose $C = c^N$. $\square$

We finish this section by proving another characterization of harmonic functions. Suppose u is harmonic in Ω. Then we have by integrating by parts

$$\int_\Omega u\Delta\varphi = 0 \quad \text{for any } \varphi \in C_0^2(\Omega).$$

The converse is also true.

THEOREM 1.16. *Suppose $u \in C(\Omega)$ satisfies*

$$\int_\Omega u\Delta\varphi = 0 \quad \textit{for any } \varphi \in C_0^2(\Omega). \tag{1}$$

Then u is harmonic in Ω.

PROOF. We claim for any $B_r(x) \subset \Omega$

$$r\int_{\partial B_r(x)} u(y)d\sigma_y = n\int_{B_r(x)} u(y)dy. \tag{2}$$

Then we have

$$\frac{d}{dr}\left(\frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)d\sigma_y\right) = \frac{n}{\omega_n}\frac{d}{dr}\left(\frac{1}{r^n}\int_{B_r(x)} u(y)dy\right)$$

$$=\frac{n}{\omega_n}\left\{-\frac{n}{r^{n+1}}\int_{B_r(x)} u(y)dy + \frac{1}{r^n}\int_{\partial B_r(x)} u(y)d\sigma_y\right\} = 0.$$

This implies

$$\frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)dS_y = \text{const}.$$

This constant is $u(x)$ if we let $r \to 0$. Hence we have

$$u(x) = \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y)dS_y \quad \text{for any } B_r(x) \subset \Omega.$$

Next we prove (2) for $n \geq 3$. For simplicity, we assume $x = 0$. Set

$$\varphi(y,r) = \begin{cases} (|y|^2 - r^2)^n & |y| \leq r \\ 0 & |y| > r, \end{cases}$$

and then for $k = 2, 3, \dots, n$

$$\varphi_k(y,r) = (|y|^2 - r^2)^{n-k}\big(2(n-k+1)|y|^2 + n(|y|^2 - r^2)\big) \quad \text{for } |y| \leq r.$$

A direct calculation shows $\varphi(\cdot, r) \in C_0^2(\Omega)$ and

$$\Delta_y\varphi(y,r) = \begin{cases} 2n\varphi_2(y,r) & |y| \leq r \\ 0 & |y| > r. \end{cases}$$

By (1), we have

$$\int_{B_r(0)} u(y)\varphi_2(y,r)dy = 0.$$

Now we prove if for some $k = 2, \cdots, n-1$,

$$\int_{B_r} u(y)\varphi_k(y,r)dy = 0, \tag{3}$$

then

$$\int_{B_r} u(y)\varphi_{k+1}(y,r)dy = 0. \tag{4}$$

In fact, we differentiate (3) with respect to r and get

$$\int_{\partial B_r} u(y)\varphi_k(y,r)dy + \int_{B_r} u(y)\frac{\partial \varphi_k}{\partial r}(y,r)dy = 0.$$

For $2 \le k < n$, $\varphi_k(y,r) = 0$ for $|y| = r$. Then we have

$$\int_{B_r} u(y)\frac{\partial \varphi_k}{\partial r}(y,r)dy = 0.$$

We have (4) by noting

$$\frac{\partial \varphi_k}{\partial r}(y,r) = (-2r)(n-k+1)\varphi_{k+1}(y,r).$$

Therefore by taking $k = n-1$ in (4), we conclude

$$\int_{B_r} u(y)\big((n+2)|y|^2 - nr^2\big)dy = 0.$$

Differentiating with respect to r again, we get (2). □

1.2. Fundamental Solutions

We begin this section by seeking for a harmonic function u, i.e., $\Delta u = 0$, in $\mathbb{R}^n$ which depends only on $r = |x-a|$ for some fixed $a \in \mathbb{R}^n$. By setting $v(r) = u(x)$, we have

$$v'' + \frac{n-1}{r}v' = 0,$$

and hence

$$v(r) = \begin{cases} c_1 + c_2 \log r & n = 2, \\ c_3 + c_4 r^{2-n} & n \ge 3, \end{cases}$$

where c_i are constants for $i = 1,2,3,4$. We are interested in functions with a singularity such that

$$\int_{\partial B_r} \frac{\partial u}{\partial r} d\sigma = 1 \quad \text{for any } r > 0.$$

Hence we set for any fixed $a \in \mathbb{R}^n$

$$\Gamma(a,x) = \begin{cases} \frac{1}{2\pi}\log|a-x| & \text{for } n = 2, \\ \frac{1}{\omega_n(2-n)}|a-x|^{2-n} & \text{for } n \ge 3. \end{cases}$$

In summary, for any fixed $a \in \mathbb{R}^n$, $\Gamma(a,x)$ is harmonic at $x \ne a$, i.e.,

$$\Delta_x \Gamma(a,x) = 0 \quad \text{for any } x \ne a,$$

and has a singularity at $x = a$. Moreover, it satisfies

$$\int_{\partial B_r(a)} \frac{\partial \Gamma}{\partial n_x}(a,x)\, d\sigma_x = 1 \qquad \text{for any } r > 0.$$

The function Γ is called the fundamental solution of the Laplacian operator.

Now we prove the Green's identity.

THEOREM 1.17. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ and that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. Then for any $a \in \Omega$*

$$u(a) = \int_\Omega \Gamma(a,x)\Delta u(x)dx - \int_{\partial\Omega} \left(\Gamma(a,x)\frac{\partial u}{\partial n_x}(x) - u(x)\frac{\partial \Gamma}{\partial n_x}(a,x)\right) d\sigma_x.$$

REMARK 1.18. (i) For any $a \in \Omega$, $\Gamma(a,\cdot)$ is integrable in Ω although it has a singularity.

(ii) For $a \notin \bar{\Omega}$, the expression in the right hand side is zero.

(iii) By letting $u = 1$, we have

$$\int_{\partial\Omega} \frac{\partial \Gamma}{\partial n_x}(a,x)d\sigma_x = 1 \quad \text{for any } a \in \Omega.$$

PROOF. We apply the Green's formula to u and $\Gamma(a,\cdot)$ in $\Omega \setminus B_r(a)$ for small $r > 0$ and get

$$\int_{\Omega\setminus B_r(a)} (\Gamma\Delta u - u\Delta\Gamma)dx = \int_{\partial\Omega} \left(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial \Gamma}{\partial n}\right) d\sigma_x - \int_{\partial B_r(a)} \left(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial \Gamma}{\partial n}\right) d\sigma_x.$$

By Noting $\Delta\Gamma = 0$ in $\Omega \setminus B_r(a)$, we have

$$\int_\Omega \Gamma\Delta u dx = \int_{\partial\Omega} \left(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial \Gamma}{\partial n}\right) d\sigma_x - \lim_{r\to 0} \int_{\partial B_r(a)} \left(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial \Gamma}{\partial n}\right) d\sigma_x.$$

For $n \geq 3$, we get by the definition of Γ

$$\begin{aligned} \left| \int_{\partial B_r(a)} \Gamma\frac{\partial u}{\partial n}d\sigma \right| &= \left| \frac{1}{(2-n)\omega_n} r^{2-n} \int_{\partial B_r(a)} \frac{\partial u}{\partial n}d\sigma \right| \\ &\leq \frac{r}{n-2} \sup_{\partial B_r(a)} |Du| \to 0 \text{ as } r \to 0, \end{aligned}$$

and

$$\int_{\partial B_r(a)} u\frac{\partial \Gamma}{\partial n}\, dS = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(a)} u dS \to u(a) \text{ as } r \to 0.$$

For $n = 2$, we get the same conclusion similarly. □

REMARK 1.19. We may employ the local version of the Green's identity to get gradient estimates without using the mean value property. Suppose $u \in C(\bar{B}_1)$ is harmonic in B_1. For any fixed $0 < r < R < 1$, choose a cut-off function $\varphi \in C_0^\infty(B_R)$ such that $\varphi = 1$ in B_r and $0 \leq \varphi \leq 1$. Apply the Green's formula to u and $\varphi\Gamma(a,\cdot)$ in $B_1 \setminus B_\rho(a)$ for $a \in B_r$ and ρ small enough. We proceed as in the proof of Theorem 1.17 to obtain

$$u(a) = -\int_{r<|x|<R} u(x)\ \Delta_x\big(\varphi(x)\Gamma(a,x)\big)dx \quad \text{for any } a \in B_r.$$

Hence, we get

$$\sup_{B_{\frac{1}{2}}} |u| \leq c \left(\int_{B_1} |u|^p\right)^{\frac{1}{p}},$$

and

$$\sup_{B_{\frac{1}{2}}} |Du| \leq c \max_{B_1} |u|,$$

where c is a constant depending only on n.

Now we begin to discuss the Green's functions. Suppose Ω is a bounded domain in $\mathbb{R}^n$. For any $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$, we have by Theorem 1.17 for any $x \in \Omega$

$$u(x) = \int_\Omega \Gamma(x,y)\Delta u(y)dy - \int_{\partial\Omega} \left(\Gamma(x,y)\frac{\partial u}{\partial n_y}(y) - u(y)\frac{\partial \Gamma}{\partial n_y}(x,y) \right) d\sigma_y.$$

If u solves the following Dirichlet boundary value problem

$$(*) \qquad \begin{cases} \Delta u = f & \text{in } \ \Omega \\ u = \varphi & \text{on } \ \partial\Omega \end{cases}$$

for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$, then u can be expressed in terms of f and φ, with one *unknown term*. We intend to eliminate this term by adjusting Γ.

For any fixed $x \in \Omega$, consider

$$\gamma(x,y) = \Gamma(x,y) + \Phi(x,y),$$

for some $\Phi(x,\cdot) \in C^2(\bar{\Omega})$ with $\Delta_y \Phi(x,y) = 0$ in Ω. Then Theorem 1.17 can be expressed as follows for any $x \in \Omega$

$$u(x) = \int_\Omega \gamma(x,y)\Delta u(y)dy - \int_{\partial\Omega} \left(\gamma(x,y)\frac{\partial u}{\partial n_y}(y) - u(y)\frac{\partial \gamma}{\partial n_y}(x,y) \right) d\sigma_y,$$

since the extra $\Phi(x,\cdot)$ is harmonic. Now by choosing Φ appropriately, we are led to the important concept of Green's functions.

For each fixed $x \in \Omega$, choose $\Phi(x,\cdot) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} \Delta_y \Phi(x,y) = 0 & \text{for } \ y \in \Omega \\ \Phi(x,y) = -\Gamma(x,y) & \text{for } \ y \in \partial\Omega. \end{cases}$$

Denote by $G(x,y)$ the resulting $\gamma(x,y)$, which is called the Green's function. If such a G exists, then the solution u of the Dirichlet problem $(*)$ can be expressed by

$$(**) \qquad u(x) = \int_\Omega G(x,y)f(y)dy + \int_{\partial\Omega} \varphi(y)\frac{\partial G}{\partial n_y}(x,y)d\sigma_y.$$

Note that the Green's function $G(x,y)$ is defined as a function of $y \in \bar{\Omega}$ for each fixed $x \in \Omega$. Now we discuss some properties of G as a function of x and y. We first note by the maximum principle that the Green's function is unique, since the difference of two Green's functions is harmonic in Ω with a zero boundary value.

LEMMA 1.20. *The Green's function $G(x,y)$ is symmetric in $\Omega \times \Omega$, i.e., $G(x,y) = G(y,x)$ for $x \neq y \in \Omega$.*

PROOF. For any $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$, take $r > 0$ small such that $B_r(x_1) \cap B_r(x_2) = \phi$. Set $G_1(y) = G(x_1,y)$ and $G_2(y) = G(x_2,y)$. We apply the Green's formula in $\Omega \setminus B_r(x_1) \cup B_r(x_2)$ and get

$$\int_{\Omega\setminus B_r(x_1)\cup B_r(x_2)} (G_1\Delta G_2 - G_2\Delta G_1) = \int_{\partial\Omega}\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right)d\sigma$$
$$-\int_{\partial B_r(x_1)}\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right)d\sigma - \int_{\partial B_r(x_2)}\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right)d\sigma.$$

Since G_i is harmonic for $y \neq x_i$, $i = 1,2$, and vanishes on $\partial\Omega$, we have

$$\int_{\partial B_r(x_1)}\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right)d\sigma + \int_{\partial B_r(x_2)}\left(G_1\frac{\partial G_2}{\partial n} - G_2\frac{\partial G_1}{\partial n}\right)d\sigma = 0.$$

Note that the left hand side has the same limit as the left hand side in the following as $r \to 0$

$$\int_{\partial B_r(x_1)}\left(\Gamma\frac{\partial G_2}{\partial n} - G_2\frac{\partial \Gamma}{\partial n}\right)d\sigma + \int_{\partial B_r(x_2)}\left(G_1\frac{\partial \Gamma}{\partial n} - \Gamma\frac{\partial G_1}{\partial n}\right)d\sigma = 0.$$

On the other hand, we have

$$\int_{\partial B_r(x_1)}\Gamma\frac{\partial G_2}{\partial n}d\sigma \to 0, \quad \int_{\partial B_r(x_2)}\Gamma\frac{\partial G_1}{\partial n}d\sigma \to 0 \quad \text{as } r \to 0,$$

and

$$\int_{\partial B_r(x_1)}G_2\frac{\partial \Gamma}{\partial n}d\sigma \to G_2(x_1), \quad \int_{\partial B_r(x_2)}G_1\frac{\partial \Gamma}{\partial n}d\sigma \to G_1(x_2) \quad \text{as } r \to 0.$$

This implies $G_2(x_1) - G_1(x_2) = 0$, or $G(x_2, x_1) = G(x_1, x_2)$. □

PROPOSITION 1.21. *Let G be the Green's function on Ω. Then for any $x, y \in \Omega$ with $x \neq y$*

$$0 > G(x,y) > \Gamma(x,y) \quad \textit{for } n \geq 3,$$
$$0 > G(x,y) > \Gamma(x,y) - \frac{1}{2\pi}\log \mathit{diam}(\Omega) \quad \textit{for } n = 2.$$

PROOF. Fix an $x \in \Omega$ and write $G(y) = G(x,y)$. Since $\lim_{y\to x} G(y) = -\infty$, there exists an $r > 0$ such that $G(y) < 0$ in $B_r(x)$. Note that G is harmonic in $\Omega \setminus B_r(x)$ with $G = 0$ on $\partial\Omega$ and $G < 0$ on $\partial B_r(x)$. The maximum principle implies $G(y) < 0$ in $\Omega \setminus B_r(x)$. Next, we discuss the other part of the inequality. Recall the definition of the Green's function

$$G(x,y) = \Gamma(x,y) + \Phi(x,y),$$

where

$$\Delta\Phi = 0 \quad \text{in } \Omega$$
$$\Phi = -\Gamma \quad \text{on } \partial\Omega.$$

For $n \geq 3$, we have

$$\Gamma(x,y) = \frac{1}{(2-n)\omega_n}|x-y|^{2-n} < 0 \quad \text{for } y \in \partial\Omega,$$

which implies $\Phi(x,\cdot) > 0$ on $\partial\Omega$. By the maximum principle, we have $\Phi > 0$ in Ω. For $n = 2$, we have

$$\Gamma(x,y) = \frac{1}{2\pi}\log|x-y| \leq \frac{1}{2\pi}\log \mathrm{diam}(\Omega) \quad \text{for } y \in \partial\Omega.$$

Hence the maximum principle implies $\Phi > -\frac{1}{2\pi}\log\operatorname{diam}(\Omega)$ in Ω. $\square$

Now, we calculate Green's functions explicitly for some special domains.

THEOREM 1.22. *The Green's function for the ball $B_R \subset \mathbb{R}^n$ is given by*
(i) for $n \geq 3$

$$G(x,y) = \frac{1}{(2-n)\omega_n}\left(|x-y|^{2-n} - \left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|^{2-n}\right);$$

(ii) for $n = 2$

$$G(x,y) = \frac{1}{2\pi}\left(\log|x-y| - \log\left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|\right).$$

PROOF. Fix an $x \neq 0$ with $|x| < R$, and consider $X \in \mathbb{R}^n \setminus \bar{B}_R$ given by $X = \dfrac{R^2}{|x|^2}x$. In other words, X and x are reflexive of each other with respect to the sphere ∂B_R. Note that the map $x \mapsto X$ is conformal, i.e., preserves angles. For $|y| = R$, we have by the similarity of triangles

$$\frac{|x|}{R} = \frac{R}{|X|} = \frac{|y-x|}{|y-X|},$$

which implies

$$|y-x| = \frac{|x|}{R}|y-X| = \left|\frac{|x|}{R}y - \frac{R}{|x|}x\right| \quad \text{for any } y \in \partial B_R. \tag{1}$$

Therefore, in order to have a zero boundary value, we take for $n \geq 3$

$$G(x,y) = \frac{1}{(2-n)\omega_n}\left(\frac{1}{|x-y|^{n-2}} - \left(\frac{R}{|x|}\right)^{n-2}\frac{1}{|y-X|^{n-2}}\right).$$

The case $n = 2$ is similar. $\square$

Next, we calculate the normal derivative of the Green's function on the sphere.

COROLLARY 1.23. *Suppose G is the Green's function in B_R. Then*

$$\frac{\partial G}{\partial n_y}(x,y) = \frac{R^2 - |x|^2}{\omega_n R|x-y|^n} \quad \textit{for any } x \in B_R \textit{ and } y \in \partial B_R.$$

PROOF. We only consider the case $n \geq 3$. Recall with $X = R^2x/|x|^2$

$$G(x,y) = \frac{1}{(2-n)\omega_n}\left(|x-y|^{2-n} - \left(\frac{R}{|x|}\right)^{n-2}|y-X|^{2-n}\right),$$

for any $x \in B_R$ and $y \in \partial B_R$. Hence we have for such x and y

$$D_{y_i}G(x,y) = -\frac{1}{\omega_n}\left(\frac{x_i - y_i}{|x-y|^n} - \left(\frac{R}{|x|}\right)^{n-2}\cdot\frac{X_i - y_i}{|X-y|^n}\right) = \frac{y_i}{\omega_n R^2}\frac{R^2 - |x|^2}{|x-y|^n},$$

by (1) in the proof of Theorem 1.22. With $n_i = \dfrac{y_i}{R}$ for $|y| = R$, we obtain

$$\frac{\partial G}{\partial n_y}(x,y) = \sum_{i=1}^n n_i D_{y_i}G(x,y) = \frac{1}{w_n R}\cdot\frac{R^2 - |x|^2}{|x-y|^n}.$$

This finishes the proof. $\square$

Denote by $K(x, y)$ the function in Corollary 1.23 for $x \in \Omega, y \in \partial\Omega$. It is called the Poisson kernel and has the following properties:

(K1) $K(x, y)$ is smooth for $x \neq y$.

(K2) $K(x, y) > 0$ for $|x| < R$ and $|y| = R$.

(K3) For any fixed $|x_0| = R$, $\lim_{x \to x_0, |x| < R} K(x, y) = 0$ uniformly in y for $|y - x_0| > \delta > 0$.

(K4) $\Delta_x K(x, y) = 0$ for $|x| < R$ and $|y| = R$.

(K5) $\int_{|y|=R} K(x, y) dS_y = 1$ for any $|x| < R$.

Here (K1), (K2) and (K3) follow easily from the explicit expression for K in Corollary 1.23 and (K4) follows easily from the definition $K(x, y) = \partial_{n_y} G(x, y)$ and the fact that $G(x, y)$ is harmonic in x. An easy derivation of (K5) is based on $(**)$. By taking a $C^2(\bar{B}_R)$ harmonic function u in $(**)$, we conclude

$$u(x) = \int_{\partial B_R} K(x, y) u(y) d\sigma_y \quad \text{for any } |x| < R.$$

This is called the Poisson integral formula. Then we have (K5) easily by taking $u \equiv 1$.

The following result yields the existence of harmonic functions in balls with the prescribed Dirichlet boundary value.

THEOREM 1.24. *For $\varphi \in C(\partial B_R)$, the function u defined by*

$$u(x) = \int_{\partial B_R} K(x, y) \varphi(y) d\sigma_y \quad \textit{for any } |x| < R \tag{1}$$

is smooth in B_R and continuous up to ∂B_R and satisfies

$$\begin{cases} \Delta u = 0 & \textit{in } \Omega \\ u = \varphi & \textit{on } \partial\Omega. \end{cases}$$

PROOF. By (K1) and (K4), we conclude easily that u defined by (1) is smooth and harmonic in B_R. We only need to prove the continuity of u up to the boundary ∂B_R. Let $x_0 \in \partial B_R$ and $x \in B_R$. By (K5), we have

$$u(x) - \varphi(x_0) = \int_{|y|=R} K(x, y) \big(\varphi(y) - \varphi(x_0)\big) d\sigma_y = I_1 + I_2,$$

where

$$I_1 = \int_{|y - x_0| < \delta, |y| = R} \cdots, \quad I_2 = \int_{|y - x_0| > \delta, |y| = R} \cdots,$$

for a positive constant δ to be determined. For any given $\varepsilon > 0$, we choose $\delta = \delta(\varepsilon) > 0$ so small that

$$|\varphi(y) - \varphi(x_0)| < \varepsilon \quad \text{for any } |y - x_0| < \delta, \ |y| = R,$$

by the continuity of φ. Then $|I_1| \leq \varepsilon$ by (K2) and (K5). Let $M = \sup_{\partial B_R} |\varphi|$. By (K3), we find a δ' such that

$$K(x, y) \leq \frac{\varepsilon}{2M\omega_n R^{n-1}} \quad \text{for any } |x - x_0| < \delta', \ |y - x_0| > \delta,$$

where δ' depends on ε and $\delta = \delta(\varepsilon)$, and hence only on ε. Then $|I_2| < \varepsilon$. Hence

$$|u(x) - \varphi(x_0)| < 2\varepsilon \quad \text{for any } |x - x_0| < \delta', |x| < R.$$

This shows that u is continuous at the boundary point x_0 and hence completes the proof. $\square$

REMARK 1.25. In the Poisson integral formula, by letting $x = 0$, we have

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R} u(y) d\sigma_y,$$

which is the mean value property.

As an application, we prove the Harnack inequality.

LEMMA 1.26. *Suppose u is harmonic in $B_R(x_0)$ and $u \geq 0$. Then*

$$\left(\frac{R}{R+r}\right)^{n-2} \frac{R-r}{R+r} u(x_0) \leq u(x) \leq \left(\frac{R}{R-r}\right)^{n-2} \frac{R+r}{R-r} u(x_0),$$

where $r = |x - x_0| < R$.

PROOF. We assume $x_0 = 0$ and $u \in C(\bar{B}_R)$. Note that u is given by the Poisson integral formula

$$u(x) = \frac{1}{\omega_n R} \int_{\partial B_R} \frac{R^2 - |x|^2}{|y-x|^n} u(y) d\sigma_y.$$

Since $R - |x| \leq |y - x| \leq R + |x|$ for $|y| = R$, we have

$$\begin{aligned} &\frac{1}{\omega_n R} \cdot \frac{R-|x|}{R+|x|} \left(\frac{1}{R+|x|}\right)^{n-2} \int_{\partial B_R} u(y) d\sigma_y \leq u(x) \\ \leq &\frac{1}{\omega_n R} \cdot \frac{R+|x|}{R-|x|} \left(\frac{1}{R-|x|}\right)^{n-2} \int_{\partial B_R} u(y) d\sigma_y. \end{aligned}$$

The mean value property implies

$$u(0) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R} u(y) d\sigma_y.$$

This finishes the proof. $\square$

COROLLARY 1.27. *If u is a harmonic function in $\mathbb{R}^n$ and bounded above or below, then $u \equiv const.$*

PROOF. We assume $u \geq 0$ in $\mathbb{R}^n$. Take any point $x \in \mathbb{R}^n$ and apply Lemma 1.26 to any ball $B_R(0)$ with $R > |x|$. We obtain

$$\left(\frac{R}{R+|x|}\right)^{n-2} \frac{R-|x|}{R+|x|} u(0) \leq u(x) \leq \left(\frac{R}{R-|x|}\right)^{n-2} \frac{R+|x|}{R-|x|} u(0).$$

This implies $u(x) = u(0)$ by letting $R \to +\infty$. $\square$

Next, we prove a result concerning the removable singularity.

THEOREM 1.28. *Suppose u is harmonic in $B_R \setminus \{0\}$ and satisfies*

$$u(x) = \begin{cases} o(\log |x|), & n = 2 \\ o(|x|^{2-n}), & n \geq 3 \end{cases} \quad \text{as } |x| \to 0.$$

Then u can be defined at 0 so that it is C^2 and harmonic in B_R.

PROOF. Assume u is continuous in $0 < |x| \leq R$. Let v solve

$$\begin{cases} \Delta v = 0 \text{ in } B_R \\ v = u \text{ on } \partial B_R. \end{cases}$$

We prove $u = v$ in $B_R \setminus \{0\}$. Set $w = v - u$ in $B_R \setminus \{0\}$ and $M_r = \max_{\partial B_r} |w|$. We only consider the case $n \geq 3$. It is obvious that

$$|w(x)| \leq M_r \cdot \frac{r^{n-2}}{|x|^{n-2}} \text{ on } \partial B_r.$$

Note that w and $\dfrac{1}{|x|^{n-2}}$ are harmonic in $B_R \setminus B_r$. Hence the maximum principle implies

$$|w(x)| \leq M_r \cdot \frac{r^{n-2}}{|x|^{n-2}} \text{ for any } x \in B_R \setminus B_r,$$

where

$$M_r = \max_{\partial B_r} |v - u| \leq \max_{\partial B_r} |v| + \max_{\partial B_r} |u| \leq M + \max_{\partial B_r} |u|,$$

with $M = \max_{\partial B_R} |u|$. Then we have for each fixed $x \neq 0$

$$|w(x)| \leq \frac{r^{n-2}}{|x|^{n-2}} M + \frac{1}{|x|^{n-2}} r^{n-2} \max_{\partial B_r} |u| \to 0 \text{ as } r \to 0.$$

This implies $w = 0$ in $B_R \setminus \{0\}$. □

1.3. Maximum Principles

In this section, we use the maximum principle to derive the interior gradient estimate and the Harnack inequality. We first give another proof of the maximum principle without using mean value properties.

THEOREM 1.29. *Suppose $u \in C^2(B_1) \cap C(\bar{B}_1)$ is a subharmonic function in B_1, i.e., $\triangle u \geq 0$. Then*

$$\sup_{B_1} u \leq \sup_{\partial B_1} u.$$

PROOF. For any $\varepsilon > 0$, we consider $u_\varepsilon(x) = u(x) + \varepsilon |x|^2$ in B_1. Then a simple calculation yields

$$\triangle u_\varepsilon = \triangle u + 2n\varepsilon \geq 2n\varepsilon > 0.$$

It is easy to see, by a contradiction argument, that u_ε can not have an interior maximum. This implies in particular

$$\sup_{B_1} u_\varepsilon \leq \sup_{\partial B_1} u_\varepsilon.$$

Therefore, we have

$$\sup_{B_1} u \leq \sup_{B_1} u_\varepsilon \leq \sup_{\partial B_1} u + \varepsilon.$$

We finish the proof by letting $\varepsilon \to 0$. □

The following result is called the comparison principle.

Corollary 1.30. *Suppose $u, v \in C^2(B_1) \cap C(\bar{B}_1)$ satisfy*

$$\begin{aligned} \triangle u \geq& \triangle v \quad \text{in } B_1 \\ u \leq& v \quad \text{on } \partial B_1. \end{aligned}$$

Then $u \leq v$ in B_1.

Remark 1.31. Theorem 1.29 and Corollary 1.30 still hold if B_1 is replaced by any bounded domain.

The next result is the interior gradient estimate for harmonic functions.

Theorem 1.32. *Suppose u is harmonic in B_1. Then*

$$\sup_{B_{\frac{1}{2}}} |Du| \leq c \sup_{\partial B_1} |u|,$$

where c is a positive constant depending only on n.

By Theorem 1.32, we have for any $\alpha \in [0, 1]$

$$|u(x) - u(y)| \leq c|x - y|^{\alpha} \sup_{\partial B_1} |u| \quad \text{for any } x, y \in B_{\frac{1}{2}},$$

where c is a positive constant depending only on n.

Proof. A direct calculation shows

$$\triangle(|Du|^2) = 2 \sum_{i,j=1}^{n} (D_{ij}u)^2 + 2 \sum_{i=1}^{n} D_i u D_i(\triangle u) = 2 \sum_{i,j=1}^{n} (D_{ij}u)^2,$$

where we used $\triangle u = 0$ in B_1. Hence $|Du|^2$ is a subharmonic function. To get interior estimates we need a cut-off function. For any $\varphi \in C_0^1(B_1)$, we have

$$\triangle(\varphi |Du|^2) = (\triangle \varphi)|Du|^2 + 4 \sum_{i,j=1}^{n} D_i \varphi D_j u D_{ij} u + 2\varphi \sum_{i,j=1}^{n} (D_{ij}u)^2.$$

By taking $\varphi = \eta^2$ for some $\eta \in C_0^1(B_1)$ with $\eta \equiv 1$ in $B_{1/2}$, we obtain by the Cauchy inequality

$$\begin{aligned} &\triangle(\eta^2 |Du|^2) \\ =& 2\eta \triangle \eta |Du|^2 + 2|D\eta|^2 |Du|^2 + 8\eta \sum_{i,j=1}^{n} D_i \eta D_j u D_{ij} u + 2\eta^2 \sum_{i,j=1}^{n} (D_{ij}u)^2 \\ \geq& \Big(2\eta \triangle \eta - 6|D\eta|^2 \Big) |Du|^2 \geq -C|Du|^2, \end{aligned}$$

where C is a positive constant depending only on η. Note

$$\triangle(u^2) = 2|Du|^2 + 2u\triangle u = 2|Du|^2,$$

since u is harmonic. By taking α large enough, we get

$$\triangle(\eta^2 |Du|^2 + \alpha u^2) \geq 0.$$

We apply Theorem 1.29, the maximum principle, to get the result. □

Next we derive the Harnack inequality.

LEMMA 1.33. *Suppose u is a positive harmonic function in B_1. Then*

$$\sup_{B_{\frac{1}{2}}} |D \log u| \leq c,$$

where c is a positive constant depending only on n.

PROOF. Set $v = \log u$. Then a direct calculation shows

$$\triangle v = -|Dv|^2.$$

We need an interior gradient estimate on v. Set $w = |Dv|^2$. Then we get

$$\triangle w + 2\sum_{i=1}^{n} D_i v D_i w = 2\sum_{i,j=1}^{n} (D_{ij} v)^2.$$

As before, we need to introduce a cut-off function. First note

$$\sum_{i,j=1}^{n} (D_{ij}v)^2 \geq \sum_{i}^{n} (D_{ii}v)^2 \geq \frac{1}{n}(\triangle v)^2 = \frac{|Dv|^4}{n} = \frac{w^2}{n}. \tag{1}$$

Take a nonnegative function $\varphi \in C_0^1(B_1)$. We obtain by the Cauchy inequality

$$\begin{aligned}
&\triangle(\varphi w) + 2\sum_{i=1}^{n} D_i v D_i(\varphi w) \\
=&2\varphi \sum_{i,j=1}^{n} (D_{ij}v)^2 + 4\sum_{i,j=1}^{n} D_i\varphi D_j v D_{ij} v + 2w\sum_{i=1}^{n} D_i\varphi D_i v + (\triangle\varphi) w \\
\geq&\varphi \sum_{i,j=1}^{n} (D_{ij}v)^2 - 2|D\varphi||Dv|^3 - \left(|\triangle\varphi| + C\frac{|D\varphi|^2}{\varphi}\right)|Dv|^2,
\end{aligned}$$

if we choose φ such that $|D\varphi|^2/\varphi$ is bounded in B_1. Choose $\varphi = \eta^4$ for some $\eta \in C_0^1(B_1)$. For such a fixed η, we obtain by (1)

$$\begin{aligned}
&\triangle(\eta^4 w) + 2\sum_{i=1}^{n} D_i v D_i(\eta^4 w) \\
\geq&\frac{1}{n}\eta^4|Dv|^4 - C\eta^3|D\eta||Dv|^3 - 4\eta^2(\eta\triangle\eta + C|D\eta|^2)|Dv|^2 \\
\geq&\frac{1}{n}\eta^4|Dv|^4 - C\eta^3|Dv|^3 - C\eta^2|Dv|^2,
\end{aligned}$$

where C is a positive constant depending only on n and η. Hence we get by the Cauchy inequality

$$\triangle(\eta^4 w) + 2\sum_{i=1}^{n} D_i v D_i(\eta^4 w) \geq \frac{1}{n}\eta^4 w^2 - C,$$

where C is a positive constant depending only on n and η.

Suppose $\eta^4 w$ attains its maximum at $x_0 \in B_1$. Then $D(\eta^4 w) = 0$ and $\triangle(\eta^4 w) \leq 0$ at x_0. Hence

$$\eta^4 w^2(x_0) \leq C(n, \eta).$$

If $w(x_0) \geq 1$, then $\eta^4 w(x_0) \leq C(n)$. Otherwise $\eta^4 w(x_0) \leq w(x_0) \leq \eta^4(x_0)$. In both cases we conclude

$$\eta^4 w \leq C(n, \eta) \quad \text{in } B_1.$$

This finishes the proof. $\square$

COROLLARY 1.34. *Suppose u is a nonnegative harmonic function in B_1. Then*

$$u(x_1) \le cu(x_2) \text{ for any } x_1, x_2 \in B_{\frac{1}{2}},$$

where c is a positive constant depending only on n.

PROOF. We may assume $u > 0$ in B_1. For any $x_1, x_2 \in B_{\frac{1}{2}}$, by a simple integration we obtain with Lemma 1.33

$$\log \frac{u(x_1)}{u(x_2)} \le |x_1 - x_2| \int_0^1 |D \log u(tx_2 + (1-t)x_1)| dt \le C|x_1 - x_2|.$$

This finishes the proof. $\square$

Next we prove a quantitative Hopf Lemma.

THEOREM 1.35. *Suppose $u \in C(\bar{B}_1)$ is a harmonic function in B_1. If $u(x) < u(x_0)$ for any $x \in \bar{B}_1$ and some $x_0 \in \partial B_1$, then*

$$\frac{\partial u}{\partial n}(x_0) \ge c\big(u(x_0) - u(0)\big),$$

where c is a positive constant depending only on n.

PROOF. Consider a positive function v in B_1 defined by

$$v(x) = e^{-\alpha|x|^2} - e^{-\alpha}.$$

It is easy to see

$$\triangle v(x) = e^{-\alpha|x|^2}(-2\alpha n + 4\alpha^2|x|^2) > 0 \text{ for any } |x| \ge \frac{1}{2},$$

if $\alpha \ge 2n+1$. Hence for such a fixed α, the function v is subharmonic in the region $A = B_1 \setminus B_{1/2}$. Now define for $\varepsilon > 0$

$$h_\varepsilon(x) = u(x) - u(x_0) + \varepsilon v(x).$$

This is also a subharmonic function, i.e., $\triangle h_\varepsilon \ge 0$ in A. Obviously $h_\varepsilon \le 0$ on ∂B_1 and $h_\varepsilon(x_0) = 0$. Since $u(x) < u(x_0)$ for $|x| = 1/2$, we take $\varepsilon > 0$ small such that $h_\varepsilon(x) < 0$ for $|x| = 1/2$. Therefore by Theorem 1.29, h_ε assumes at the point x_0 its maximum in A. This implies

$$\frac{\partial h_\varepsilon}{\partial n}(x_0) \ge 0,$$

or

$$\frac{\partial u}{\partial n}(x_0) \ge -\varepsilon \frac{\partial v}{\partial n}(x_0) = 2\alpha\varepsilon e^{-\alpha} > 0.$$

Note that we only used the subharmonicity of u so far. We estimate ε as follows. Set $w(x) = u(x_0) - u(x) > 0$ in B_1. Obviously w is a harmonic function in B_1. By Corollary 1.34, the Harnack inequality, we have

$$\inf_{B_{\frac{1}{2}}} w \ge cw(0),$$

or

$$\max_{B_{\frac{1}{2}}} u \le u(x_0) - c\big(u(x_0) - u(0)\big),$$

where c is a positive constant depending only on n. Hence we take

$$\varepsilon = \delta c\big(u(x_0) - u(0)\big),$$

for δ small, depending on n. This finishes the proof. $\square$

To finish this section, we prove a result on the global Hölder continuity for harmonic functions.

LEMMA 1.36. *Suppose $u \in C(\bar{B}_1)$ is a harmonic function in B_1 with $u = \varphi$ on ∂B_1. If $\varphi \in C^\alpha(\partial B_1)$ for some $\alpha \in (0,1)$, then $u \in C^{\alpha/2}(\bar{B}_1)$ and*

$$\|u\|_{C^{\frac{\alpha}{2}}(\bar{B}_1)} \leq c\|\varphi\|_{C^\alpha(\partial B_1)},$$

where c is a positive constant depending only on n and α.

PROOF. First, the maximum principle implies that

$$\inf_{\partial B_1} \varphi \leq u \leq \sup_{\partial B_1} \varphi \quad \text{in } B_1.$$

Next, we claim for any $x_0 \in \partial B_1$

$$\sup_{x \in B_1} \frac{|u(x) - u(x_0)|}{|x - x_0|^{\frac{\alpha}{2}}} \leq 2^{\frac{\alpha}{2}} \sup_{x \in \partial B_1} \frac{|\varphi(x) - \varphi(x_0)|}{|x - x_0|^\alpha}. \tag{1}$$

Lemma 1.36 follows easily from (1).

For any $x, y \in B_1$, set $d_x = \operatorname{dist}(x, \partial B_1)$ and $d_y = \operatorname{dist}(y, \partial B_1)$, and assume $d_y \leq d_x$. Take $x_0, y_0 \in \partial B_1$ such that $|x - x_0| = d_x$ and $|y - y_0| = d_y$. Assume first that $|x - y| \leq d_x/2$. Then $y \in \bar{B}_{d_x/2}(x) \subset B_{d_x}(x) \subset B_1$. We apply Theorem 1.32 (scaled version) to $u - u(x_0)$ in $B_{d_x}(x)$ and get by (1)

$$d_x^{\frac{\alpha}{2}} \frac{|u(x) - u(y)|}{|x - y|^{\frac{\alpha}{2}}} \leq C|u - u(x_0)|_{L^\infty(B_{d_x}(x))} \leq C d_x^{\frac{\alpha}{2}} \|\varphi\|_{C^\alpha(\partial B_1)}.$$

Hence, we obtain

$$|u(x) - u(y)| \leq C|x - y|^{\frac{\alpha}{2}} \|\varphi\|_{C^\alpha(\partial B_1)}.$$

Assume now that $d_y \leq d_x \leq 2|x - y|$. Then by (1) again we have

$$\begin{aligned}
|u(x) - u(y)| &\leq |u(x) - u(x_0)| + |u(x_0) - u(y_0)| + |u(y_0) - u(y)| \\
&\leq C(d_x^{\frac{\alpha}{2}} + |x_0 - y_0|^{\frac{\alpha}{2}} + d_y^{\frac{\alpha}{2}})\|\varphi\|_{C^\alpha(\partial B_1)} \\
&\leq C|x - y|^{\frac{\alpha}{2}} \|\varphi\|_{C^\alpha(\partial B_1)},
\end{aligned}$$

since $|x_0 - y_0| \leq d_x + |x - y| + d_y \leq 5|x - y|$.

To prove (1), we assume $B_1 = B_1\big((1, 0, \cdots, 0)\big)$, $x_0 = 0$ and $\varphi(0) = 0$. Define $K = \sup_{x \in \partial B_1} |\varphi(x)|/|x|^\alpha$. Note $|x|^2 = 2x_1$ for $x \in \partial B_1$. Then

$$\varphi(x) \leq K|x|^\alpha \leq 2^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}} \quad \text{for any } x \in \partial B_1.$$

Define $v(x) = 2^{\alpha/2} K x_1^{\alpha/2}$ in B_1. Then we have

$$\triangle v(x) = 2^{\frac{\alpha}{2}} K \cdot \frac{\alpha}{2}(\frac{\alpha}{2} - 1) x_1^{\frac{\alpha}{2} - 2} < 0 \text{ in } B_1.$$

Corollary 1.30 implies

$$u(x) \leq v(x) = 2^{\frac{\alpha}{2}} K x_1^{\frac{\alpha}{2}} \leq 2^{\frac{\alpha}{2}} K|x|^{\frac{\alpha}{2}} \quad \text{for any } x \in B_1.$$

Considering $-u$ similarly, we get

$$|u(x)| \le 2^{\frac{\alpha}{2}} K|x|^{\frac{\alpha}{2}} \quad \text{for any } x \in B_1.$$

This proves (1). $\square$

1.4. Energy Methods

In this section, we discuss harmonic functions by using energy methods. In general, we assume throughout this section that $a_{ij} \in C(B_1)$ satisfies

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } x \in B_1 \text{ and } \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ. We consider the function $u \in C^1(B_1)$ satisfying

$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \quad \text{for any } \varphi \in C_0^1(B_1).$$

It is easy to check by integration by parts that harmonic functions satisfy the above equation for $a_{ij} = \delta_{ij}$.

The following result is referred to as the Cacciopolli inequality.

Lemma 1.37. *Suppose $u \in C^1(B_1)$ satisfies*

$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \text{ for any } \varphi \in C_0^1(B_1).$$

Then for any function $\eta \in C_0^1(B_1)$

$$\int_{B_1} \eta^2 |Du|^2 \le c \int_{B_1} |D\eta|^2 u^2,$$

where c is a positive constant depending only on λ and Λ.

Proof. For any $\eta \in C_0^1(B_1)$, set $\varphi = \eta^2 u$. Then we have

$$\lambda \int_{B_1} \eta^2 |Du|^2 \le \Lambda \int_{B_1} \eta |u||D\eta||Du|.$$

A simple application of the Cauchy inequality yields the result. $\square$

Corollary 1.38. *Let u be as in Lemma 1.37. Then for any $0 \le r < R \le 1$*

$$\int_{B_r} |Du|^2 \le \frac{C}{(R-r)^2} \int_{B_R} u^2,$$

where C is a positive constant depending only on λ and Λ.

Proof. Take η such that $\eta = 1$ on B_r, $\eta = 0$ outside B_R and $|D\eta| \le 2(R-r)^{-1}$. $\square$

Corollary 1.39. *Let u be as in Lemma 1.37. Then for any $0 < R \le 1$*

$$\int_{B_{\frac{R}{2}}} u^2 \le \theta \int_{B_R} u^2,$$

and

$$\int_{B_{\frac{R}{2}}} |Du|^2 \le \theta \int_{B_R} |Du|^2,$$

where $\theta \in (0,1)$ is a positive constant depending only on n, λ and Λ.

Proof. Take an $\eta \in C_0^1(B_R)$ with $\eta = 1$ on $B_{R/2}$ and $|D\eta| \le 2R^{-1}$. Then Lemma 1.37 yields

$$\int_{B_R} |D(\eta u)|^2 \le C \int_{B_R} |D\eta|^2 u^2 \le \frac{C}{R^2} \int_{B_R \setminus B_{\frac{R}{2}}} u^2,$$

since $D\eta = 0$ in $B_{R/2}$. Hence, by the Poincaré inequality, we get

$$\int_{B_R} (\eta u)^2 \le CR^2 \int_{B_R} |D(\eta u)|^2.$$

Therefore we obtain

$$\int_{B_{\frac{R}{2}}} u^2 \le C \int_{B_R \setminus B_{\frac{R}{2}}} u^2,$$

and hence

$$(C+1) \int_{B_{\frac{R}{2}}} u^2 \le C \int_{B_R} u^2.$$

For the second inequality, observe that Lemma 1.37 holds for $u - a$ for an arbitrary constant a. Then as before, we have

$$\int_{B_R} \eta^2 |Du|^2 \le C \int_{B_R} |D\eta|^2 (u-a)^2 \le \frac{C}{R^2} \int_{B_R \setminus B_{\frac{R}{2}}} (u-a)^2.$$

The Poincaré inequality implies with $a = |B_R \setminus B_{\frac{R}{2}}|^{-1} \int_{B_R \setminus B_{\frac{R}{2}}} u$

$$\int_{B_R \setminus B_{\frac{R}{2}}} (u-a)^2 \le c(n) R^2 \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2.$$

Hence we obtain

$$\int_{B_{\frac{R}{2}}} |Du|^2 \le C \int_{B_R \setminus B_{\frac{R}{2}}} |Du|^2,$$

and hence

$$(C+1) \int_{B_{\frac{R}{2}}} |Du|^2 \le C \int_{B_R} |Du|^2.$$

This finishes the proof. □

Remark 1.40. Corollary 1.39 implies, in particular, that a harmonic function in $\mathbb{R}^n$ with a finite L^2-norm is identically zero and that a harmonic function in $\mathbb{R}^n$ with a finite Dirichlet integral is constant.

Remark 1.41. By iterating the result in Corollary 1.39, we have the following estimates. Let u be as in Lemma 1.37. Then for any $0 < \rho < r \le 1$

$$\int_{B_\rho} u^2 \le C \Big(\frac{\rho}{r}\Big)^\mu \int_{B_r} u^2,$$

and

$$\int_{B_\rho} |Du|^2 \le C \Big(\frac{\rho}{r}\Big)^\mu \int_{B_r} |Du|^2,$$

for some positive constant μ depending only on n, λ and Λ. Later on, we will prove that we can take $\mu \in (n-2, n)$ in the second inequality. For harmonic functions, we have better results.

LEMMA 1.42. *Suppose $\{a_{ij}\}$ is a constant positive definite matrix with*

$$\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n,$$

for some constants $0 < \lambda \le \Lambda$. Suppose $u \in C^1(B_1)$ satisfies

$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \text{ for any } \varphi \in C_0^1(B_1). \tag{1}$$

Then for any $0 < \rho \le r$

$$\int_{B_\rho} |u|^2 \le c\left(\frac{\rho}{r}\right)^n \int_{B_r} |u|^2, \tag{2}$$

and

$$\int_{B_\rho} |u - u_\rho|^2 \le c\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |u - \bar{u}_r|^2, \tag{3}$$

where c is a positive constant depending only on λ and Λ and $\bar{u}_r$ denotes the average of u in B_r.

PROOF. By a simple dilation, we need only consider $r = 1$. We only prove (2) and (3) for $\rho \in \left(0, \frac{1}{2}\right]$, since they are trivial for $\rho \in (\frac{1}{2}, 1]$.

Now we claim

$$|u|^2_{L^\infty(B_{\frac{1}{2}})} + |Du|^2_{L^\infty(B_{\frac{1}{2}})} \le c\int_{B_1} |u|^2,$$

where c is a positive constant depending only on λ and Λ. This implies for $\rho \in \left(0, \frac{1}{2}\right]$

$$\int_{B_\rho} |u|^2 \le \rho^n |u|^2_{L^\infty(B_{\frac{1}{2}})} \le c\rho^n \int_{B_1} |u|^2,$$

and

$$\int_{B_\rho} |u - \bar{u}_\rho|^2 \le \int_{B_\rho} |u - u(0)|^2 \le \rho^{n+2} |Du|^2_{L^\infty(B_{\frac{1}{2}})} \le c\rho^{n+2} \int_{B_1} |u|^2.$$

If u is a solution of (1), so is $u - u_1$. With u replaced by $u - u_1$ in the above inequality, we have

$$\int_{B_\rho} |u - u_\rho|^2 \le c\rho^{n+2} \int_{B_1} |u - u_1|^2.$$

We now present two methods to prove the claim.

Method 1. By a rotation, we may assume (a_{ij}) is a diagonal matrix. Hence (1) becomes

$$\sum_{i=1}^n \lambda_i D_{ii} u = 0,$$

with $0 < \lambda \le \lambda_i \le \Lambda$ for $i = 1, \cdots n$. It is easy to see that there exists a constant $r_0 \in (0, 1/2)$, depending only on λ and Λ, such that for any $x_0 \in B_{\frac{1}{2}}$ the rectangle

$$\left\{x; \frac{|x_i - x_{0i}|}{\sqrt{\lambda_i}} < r_0\right\}$$

is contained in B_1. Consider the change the coordinate

$$x_i \mapsto y_i = \frac{x_i}{\sqrt{\lambda_i}},$$

and set

$$v(y) = u(x).$$

Then v is harmonic in $\{y; \sum_{i=1}^n \lambda_i y_i^2 < 1\}$. In the ball $\{y; |y-y_0| < r_0\}$, we use the interior estimates to yield

$$|v(y_0)|^2 + |Dv(y_0)|^2 \leq c \int\limits_{B_{r_0}(y_0)} v^2 \leq c \int\limits_{\{\sum_{i=1}^n \lambda_i y_i^2 < 1\}} v^2,$$

where c is a positive constant depending only on λ and Λ. We transform back to u to get

$$|u(x_0)|^2 + |Du(x_0)|^2 \leq c \int\limits_{|x|<1} u^2.$$

Method 2. If u is a solution of (1), so are any derivatives of u. By applying Corollary 1.38 to derivatives of u, we conclude for any positive integer k

$$\|u\|_{H^k(B_{\frac{1}{2}})} \leq c\|u\|_{L^2(B_1)},$$

where c is a positive constant depending only on n, λ and Λ. By taking $k = [\frac{n}{2}] + 1$, we have $H^k(B_{\frac{1}{2}})$ continuously embedded in $C^1(\bar{B}_{\frac{1}{2}})$. We then get

$$|u|_{L^\infty(B_{\frac{1}{2}})} + |Du|_{L^\infty(B_{\frac{1}{2}})} \leq c\|u\|_{L^2(B_1)}.$$

This finishes the proof. $\square$

CHAPTER 2

Maximum Principles

In this chapter, we discuss maximum principles and their applications. Two classes of maximum principles will be discussed, one due to Hopf and the other to Alexandroff. The former gives estimates of solutions in terms of the L^∞-norm of the nonhomogenous terms while the latter gives the estimates in terms of the L^n-norm. Applications include various a priori estimates and the moving plane method.

2.1. Strong Maximum Principle

Suppose Ω is a bounded and connected domain in $\mathbb{R}^n$. Consider the operator L in Ω

$$Lu \equiv a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u,$$

for $u \in C^2(\Omega) \cap C(\bar{\Omega})$. We always assume that a_{ij}, b_i and c are continuous and hence bounded in $\bar{\Omega}$ and that L is uniformly elliptic in Ω in the following sense

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n,$$

for some positive constant λ.

Lemma 2.1. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu > 0$ in Ω with $c(x) \leq 0$ in Ω. If u has a nonnegative maximum in $\bar{\Omega}$, then u cannot attain this maximum in Ω.*

Proof. Suppose u attains its nonnegative maximum of $\bar{\Omega}$ at $x_0 \in \Omega$. Then $D_iu(x_0) = 0$ and the matrix $\big(D_{ij}(x_0)\big)$ is semi-negative definite. By the ellipticity condition, the matrix $\big(a_{ij}(x_0)\big)$ is positive definite. This implies $Lu(x_0) = a_{ij}(x_0)D_{ij}u(x_0) \leq 0$, which is a contradiction. □

Remark 2.2. If $c(x) \equiv 0$, then the requirement for the nonnegativeness on u can be removed. This remark holds for many results in the rest of this section.

The next result is referred to as the weak maximum principle.

Lemma 2.3. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω with $c(x) \leq 0$ in Ω. Then u attains on $\partial\Omega$ its nonnegative maximum in $\bar{\Omega}$.*

Proof. For any $\varepsilon > 0$, consider $w(x) = u(x) + \varepsilon e^{\alpha x_1}$ with α to be determined. Then we have

$$Lw = Lu + \varepsilon e^{\alpha x_1}(a_{11}\alpha^2 + b_1\alpha + c).$$

Since b_1 and c are bounded and $a_{11}(x) \geq \lambda > 0$ for any $x \in \Omega$, by choosing $\alpha > 0$ large enough we get

$$a_{11}(x)\alpha^2 + b_1(x)\alpha + c(x) > 0 \quad \text{for any } x \in \Omega.$$

This implies $Lw > 0$ in Ω. By Lemma 2.1, w attains its nonnegative maximum only on $\partial\Omega$, i.e.,

$$\sup_\Omega w \le \sup_{\partial\Omega} w^+.$$

Then we obtain

$$\sup_\Omega u \le \sup_\Omega w \le \sup_{\partial\Omega} w^+ \le \sup_{\partial\Omega} u^+ + \varepsilon \sup_{x\in\partial\Omega} e^{\alpha x_1}.$$

We finish the proof by letting $\varepsilon \to 0$. □

As an application, we have the uniqueness of the solution $u \in C^2(\Omega)\cap C(\bar{\Omega})$ of the following Dirichlet boundary value problem for $f \in C(\Omega)$ and $\varphi \in C(\partial\Omega)$

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

if $c(x) \le 0$ in Ω.

Remark 2.4. The boundedness of the domain Ω is essential, since it guarantees the existence of maximum and minimum of u in $\bar{\Omega}$. The uniqueness does not hold if the domain is unbounded. Some examples are given in Section 1.1. Equally important is the nonpositiveness of the coefficient c.

Example 2.5. Set $\Omega = \{(x,y) \in \mathbb{R}^2; 0 < x < \pi, 0 < y < \pi\}$. Then $u = \sin x \sin y$ is a nontrivial solution of the problem

$$\begin{aligned} \triangle u + 2u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The next result is called the Hopf lemma.

Theorem 2.6. *Let B be an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$. Suppose $u \in C^2(B)\cap C(B\cup\{x_0\})$ satisfies $Lu \ge 0$ in B with $c(x) \le 0$ in B. Assume in addition that*

$$u(x) < u(x_0) \quad \textit{for any } x \in B \textit{ and } u(x_0) \ge 0.$$

Then for each outward direction ν at x_0 with $\nu \cdot \mathbf{n}(x_0) > 0$ there holds

$$\liminf_{t\to 0^+} \frac{1}{t}[u(x_0) - u(x_0 - t\nu)] > 0.$$

Remark 2.7. If, in addition, $u \in C^1(B\cup\{x_0\})$, we have

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Proof. We assume that B is centered at the origin with radius r. We assume further that $u \in C(\bar{B})$ and $u(x) < u(x_0)$ for any $x \in \bar{B}\backslash\{x_0\}$, since we can construct a ball $B_* \subset B$ tangent to B at x_0.

Consider $v(x) = u(x) + \varepsilon h(x)$ for some nonnegative function h. We will choose $\varepsilon > 0$ appropriately such that v attains its nonnegative maximum only at x_0. Set $\Sigma = B \cap B_{r/2}(x_0)$ and $h(x) = e^{-\alpha|x|^2} - e^{-\alpha r^2}$ with α to be determined. We check in the following that

$$Lh > 0 \text{ in } \Sigma.$$

A direct calculation yields

$$\begin{aligned}Lh =& e^{-\alpha|x|^2}\big\{4\alpha^2\sum_{i,j=1}^n a_{ij}(x)x_ix_j - 2\alpha\sum_{i=1}^n a_{ii}(x) - 2\alpha\sum_{n=1}^n b_i(x)x_i + c\big\} \\ &- ce^{-\alpha r^2} \\ \geq & e^{-\alpha|x|^2}\big\{4\alpha^2\sum_{i,j=1}^n a_{ij}(x)x_ix_j - 2\alpha\sum_{i=1}^n [a_{ii}(x) + b_i(x)x_i] + c\big\}.\end{aligned}$$

By the ellipticity, we have

$$\sum_{i,j=1}^n a_{ij}(x)x_ix_j \geq \lambda|x|^2 \geq \lambda\left(\frac{r}{2}\right)^2 > 0 \text{ in } \Sigma.$$

We conclude $Lh > 0$ in Σ for α large enough. With such an h, we have $Lv = Lu + \varepsilon Lh > 0$ in Σ for any $\varepsilon > 0$. By Lemma 2.1, v cannot attain its nonnegative maximum inside Σ.

Next, we prove, for some small $\varepsilon > 0$, v attains at x_0 its nonnegative maximum. Consider v on the boundary $\partial\Sigma$.

(i) For $x \in \partial\Sigma \cap B$, since $u(x) < u(x_0)$, so $u(x) < u(x_0) - \delta$ for some $\delta > 0$. Take ε small such that $\varepsilon h < \delta$ on $\partial\Sigma \cap B$. Hence, for such an ε, we have $v(x) < u(x_0)$ for any $x \in \partial\Sigma \cap B$.

(ii) On $\Sigma \cap \partial B$, $h(x) = 0$ and $u(x) < u(x_0)$ for $x \neq x_0$. Hence $v(x) < u(x_0)$ on $\Sigma \cap \partial B \setminus \{x_0\}$ and $v(x_0) = u(x_0)$.

Therefore, we conclude for any small $t > 0$

$$\frac{1}{t}\big(v(x_0) - v(x_0 - t\nu)\big) \geq 0.$$

By letting $t \to 0$, we obtain

$$\liminf_{t\to 0}\frac{1}{t}\big(u(x_0) - u(x_0 - t\nu)\big) \geq -\varepsilon\frac{\partial h}{\partial\nu}(x_0).$$

By the definition of h, we have

$$\frac{\partial h}{\partial\nu}(x_0) = \frac{\partial h}{\partial n}(x_0)\mathbf{n}\cdot\nu = -2\alpha r e^{-\alpha r^2}\mathbf{n}\cdot\nu < 0.$$

This finishes the proof. $\square$

Next, we are ready to prove the strong maximum principle.

Theorem 2.8. *Let $u \in C^2(\Omega)\cap C(\bar\Omega)$ satisfy $Lu \geq 0$ with $c(x) \leq 0$ in Ω. Then the nonnegative maximum of u in $\bar\Omega$ can be assumed only on $\partial\Omega$ unless u is a constant.*

Proof. Let M be the nonnegative maximum of u in $\bar\Omega$. Set $\Sigma = \{x \in \Omega; u(x) = M\}$. It is relatively closed in Ω. We need to show $\Sigma = \Omega$.

We prove by contradiction. If Σ is a proper subset of Ω, then we may find an open ball $B \subset \Omega\setminus\Sigma$ with a point on its boundary belonging to Σ. (In fact, we may choose a point $p \in \Omega\setminus\Sigma$ such that $d(p,\Sigma) < d(p,\partial\Omega)$ first and then extend the ball centered at p. It hits Σ before hitting $\partial\Omega$.) Suppose $x_0 \in \partial B\cap\Sigma$. Obviously we have $Lu \geq 0$ in B and

$$u(x) < u(x_0) \quad \text{for any } x \in B \text{ and } u(x_0) = M \geq 0.$$

Theorem 2.6 implies $\dfrac{\partial u}{\partial n}(x_0) > 0$, where $\mathbf{n}$ is the outward normal direction at x_0 to the ball B. While x_0 is the interior maximal point of Ω, hence $Du(x_0) = 0$. This leads to a contradiction. □

The following result is often referred to as the comparison principle.

Corollary 2.9. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω with $c(x) \leq 0$ in Ω. If $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω. In fact, either $u < 0$ in Ω or $u \equiv 0$ in Ω.*

In order to discuss the boundary value problem with a general boundary condition, we need the following result, which is just a corollary of Theorem 2.6 and Theorem 2.8.

Corollary 2.10. *Suppose Ω has the interior sphere property and that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω with $c(x) \leq 0$. Assume u attains its nonnegative maximum at $x_0 \in \bar{\Omega}$. Then $x_0 \in \partial\Omega$ and for any outward direction ν at x_0 to $\partial\Omega$*

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

unless u is a constant in $\bar{\Omega}$.

Corollary 2.11. *Suppose Ω is bounded in $\mathbb{R}^n$ and satisfies the interior sphere property. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a solution of the following boundary value problem*

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$. Assume in addition that $c(x) \leq 0$ in Ω and $\alpha(x) \geq 0$ on $\partial\Omega$. Then u is the unique solution if $c \not\equiv 0$ or $\alpha \not\equiv 0$. If $c \equiv 0$ and $\alpha \equiv 0$, u is unique up to additive constants.

Proof. Suppose u is a solution of the following homogeneous equation

$$\begin{aligned} Lu &= 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Case 1. $c \not\equiv 0$ or $\alpha \not\equiv 0$. Suppose that u has a positive maximum at $x_0 \in \bar{\Omega}$. If u is a positive constant, there leads to a contradiction to the condition $c \not\equiv 0$ in Ω or $\alpha \not\equiv 0$ on $\partial\Omega$. Otherwise $x_0 \in \partial\Omega$ and $\frac{\partial u}{\partial n}(x_0) > 0$ by Corollary 2.10, which contradicts the boundary value. Therefore $u \equiv 0$.

Case 2. $c \equiv 0$ and $\alpha \equiv 0$. Suppose u is a nonconstant solution. Then its maximum in $\bar{\Omega}$ is assumed only on $\partial\Omega$ by Theorem 2.8, say at $x_0 \in \partial\Omega$. Again Corollary 2.10 implies $\frac{\partial u}{\partial n}(x_0) > 0$. This is a contradiction. Therefore, u is a constant. □

The following theorem generalizes the comparison principle under no restrictions on $c(x)$.

Theorem 2.12. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$. If $u \leq 0$ in Ω, then either $u < 0$ in Ω or $u \equiv 0$ in Ω.*

PROOF. *Method 1.* Suppose $u(x_0) = 0$ for some $x_0 \in \Omega$. We will prove that $u \equiv 0$ in Ω. Write $c(x) = c^+(x) - c^-(x)$, where $c^+(x)$ and $c^-(x)$ are the positive and negative part of $c(x)$ respectively. Then u satisfies

$$a_{ij}D_{ij}u + b_iD_iu - c^-u \geq -c^+u \geq 0.$$

So we have $u \equiv 0$ by Theorem 2.8.

Method 2. Set $v = ue^{-\alpha x_1}$ for some $\alpha > 0$ to be determined. By $Lu \geq 0$, we have

$$a_{ij}D_{ij}v + \big(\alpha(a_{1i} + a_{i1}) + b_i\big)D_iv + (a_{11}\alpha^2 + b_1\alpha + c)v \geq 0.$$

Choose α large enough such that $a_{11}\alpha^2 + b_1\alpha + c > 0$. Therefore v satisfies

$$a_{ij}D_{ij}v + \big(\alpha(a_{1i} + a_{i1}) + b_i\big)D_iv \geq 0.$$

Hence we apply Theorem 2.8 to v to conclude that either $v < 0$ in Ω or $v \equiv 0$ in Ω. $\square$

The next result is the general maximum principle for the operator L with no restriction on $c(x)$.

THEOREM 2.13. *Suppose there exists a $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying $w > 0$ in $\bar{\Omega}$ and $Lw \leq 0$ in Ω. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω, then u/w cannot attain in Ω its nonnegative maximum unless u/w is constant. If, in addition, u/w assumes its nonnegative maximum at $x_0 \in \partial\Omega$ and $u/w \not\equiv$ const., then for any outward direction ν at x_0 to $\partial\Omega$*

$$\frac{\partial}{\partial \nu}\left(\frac{u}{w}\right)(x_0) > 0,$$

provided $\partial\Omega$ has the interior sphere property at x_0.

PROOF. Set $v = \dfrac{u}{w}$. Then v satisfies

$$a_{ij}D_{ij}v + B_iD_iv + \big(\frac{Lw}{w}\big)v \geq 0,$$

where $B_i = b_i + \dfrac{2}{w}a_{ij}D_{ij}w$. We simply apply Theorem 2.6 and Corollary 2.9 to v. $\square$

REMARK 2.14. If the operator L in Ω satisfies the condition of Theorem 2.13, then L has the comparison principle. In particular, the Dirichlet boundary value problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned}$$

has at most one solution.

The next result is the so-called maximum principle for *narrow domains.*

PROPOSITION 2.15. *Let d be a positive number and $\mathbf{e}$ be a unit vector such that $|(y - x) \cdot \mathbf{e}| < d$ for any $x, y \in \Omega$. Then there exists a $d_0 > 0$, depending only on λ and the sup-norm of b_i and c^+, such that the assumptions of Theorem 2.13 are satisfied if $d \leq d_0$.*

PROOF. By choosing $\mathbf{e} = (1, 0, \cdots, 0)$, we suppose $\bar{\Omega}$ lies in $\{0 < x_1 < d\}$. Assume, in addition, $|b_i|, c^+ \leq N$ for some positive constant N. We construct w as follows. Set $w = e^{\alpha d} - e^{\alpha x_1} > 0$ in $\bar{\Omega}$. A direct calculation yields

$$Lw = -(a_{11}\alpha^2 + b_1\alpha)e^{\alpha x_1} + c(e^{\alpha d} - e^{\alpha x_1}) \leq -(a_{11}\alpha^2 + b_1\alpha) + Ne^{\alpha d}.$$

Choose α large to have

$$a_{11}\alpha^2 + b_1\alpha \geq \lambda\alpha^2 - N\alpha \geq 2N.$$

Hence $Lw \leq -2N + Ne^{\alpha d} = N(e^{\alpha d} - 2) \leq 0$ if d is small with $e^{\alpha d} \leq 2$. $\square$

REMARK 2.16. Some results in this section, including Proposition 2.15, hold for unbounded domains.

2.2. Poisson Equations

In this section, we use the maximum principle to discuss solutions of Poisson equations $\Delta u = f$.

As the first application, we derive interior gradient estimates for solutions of Poisson equations.

LEMMA 2.17. *Suppose $u \in C^2(B_R) \cap C(\bar{B}_R)$ satisfies*

$$\triangle u = f \text{ in } B_R,$$

for some $f \in C(\bar{B}_R)$. Then

$$|Du(0)| \leq \frac{n}{R} \max_{\partial B_R} |u| + \frac{R}{2} \max_{B_R} |f|.$$

PROOF. We write $B = B_R$ and consider $D_n u(0)$. Set $M = \max_{\partial B} |u|$ and $F = \max_B |f|$. Consider

$$v(x', x_n) = \frac{1}{2}\big(u(x', x_n) - u(x', -x_n)\big) \quad \text{in } B^+.$$

Then v satisfies

$$\begin{aligned} &|\triangle v| \leq F \quad \text{in } B^+, \\ &|v| \leq M \quad \text{on } \partial B^+, \\ &v(x', 0) = 0 \quad \text{for } |x'| < R. \end{aligned}$$

Consider an auxiliary function

$$w(x', x_n) = A|x'|^2 + Bx_n + Cx_n^2.$$

By choosing A, B, C appropriately, we assume

(i) $w(x', 0) = A|x'|^2 \geq 0$ for $|x'| \leq R$;

(ii) $w(x', x_n) = AR^2 + (C - A)x_n^2 + Bx_n \geq M$ for $|x'|^2 + x_n^2 = R^2$;

(iii) $\triangle w = 2(n-1)A + 2C \leq -F$ in B^+.

To achieve this, we need to require

$$\begin{aligned} A &\geq 0, \\ AR^2 &\geq M, \\ (C - A)x_n^2 + Bx_n &\geq 0 \quad \text{for any } 0 \leq x_n \leq R, \\ 2(n-1)A + 2C &\leq -F. \end{aligned}$$

So we take

$$A = \frac{M}{R^2},$$

and

$$C = -\frac{F}{2} - (n-1)A,$$

which implies

$$A - C = \frac{F}{2} + nA.$$

For B, we need $B + (C - A)x_n \geq 0$, or

$$B \geq \left(\frac{F}{2} + \frac{n}{R^2}M\right)x_n \quad \text{for any } 0 \leq x_n \leq R.$$

Hence we may take

$$B = \left(\frac{F}{2} + \frac{n}{R^2}M\right)R = \frac{R}{2}F + \frac{n}{R}M.$$

With such a choice of w, we have $\triangle w \leq \triangle v$ in B^+ and $w \geq v$ on ∂B^+. By Corollary 2.9, the comparison principle, we obtain $w \geq v$ in B^+. Taking $x' = 0$, we obtain

$$\frac{1}{2}\left(\frac{u(0,x_n) - u(0,-x_n)}{x_n}\right) \leq B + Cx_n \quad \text{for any } 0 < x_n < R.$$

Therefore, we have by letting $x_n \to 0$

$$D_n u(0) \leq \frac{n}{R}M + \frac{R}{2}F.$$

By considering $-u$ similarly, we obtain

$$|D_n u(0)| \leq \frac{n}{R}M + \frac{R}{2}F.$$

This finishes the proof. □

Next, we derive an estimate of the modules of continuity for gradients of solutions.

LEMMA 2.18. *Suppose $u \in C^2(B_{2R}) \cap C(\bar{B}_{2R})$ satisfies*

$$\triangle u = f \ \text{ in } B_{2R},$$

for some $f \in C(\bar{B}_{2R})$. Then for any $x, y \in B_R$ with $x \neq y$,

$$R^2 \frac{|Du(x) - Du(y)|}{|x-y|} \leq c\left(\sup_{B_{2R}} |u| + R^2 \sup_{B_{2R}} |f|\right)\left(\log \frac{2R}{|x-y|} + 1\right),$$

where c is a positive constant depending only on n.

PROOF. In the following, we use cubes Q_R, Q_{2R} instead of balls B_R, B_{2R}. Set $M = \sup_{Q_{2R}} |u|$ and $F = \sup_{Q_{2R}} |f|$. We will prove for any $x_n \in \left(0, \frac{R}{4}\right)$,

$$\frac{1}{2}|Du(0,x_n) - Du(0,-x_n)| \leq cx_n\left(\frac{M}{R^2} + F\right)\left(\log\frac{2R}{x_n} + 1\right), \tag{1}$$

where c is a positive constant depending only on n. By Lemma 2.17, we have

$$\sup_{Q_R} |Du| \leq c\left(\frac{M}{R} + RF\right). \tag{2}$$

Let Q' be the domain in $\mathbb{R}^{n+1}$ given by

$$Q' = \left\{ (x_1, \ldots, x_{n-1}, y, z);\ |x_i| < \frac{R}{2}, i = 1, \ldots, n-1, \quad 0 < y, z < \frac{R}{4} \right\},$$

and define in Q' the function

$$v(x', y, z) = \frac{1}{4} \left\{ u(x', y+z) - u(x', y-z) - u(x', -y+z) + u(x', -y-z) \right\}.$$

Define an operator L in $\mathbb{R}^{n+1}$ by

$$L \equiv \sum_{i=1}^{n-1} D_{x_i x_i} + \frac{1}{2} D_{yy} + \frac{1}{2} D_{zz}.$$

It is easy to see in Q'

(i) $|Lv| \leq F$ in Q';

(ii) $v(x', 0, z) = v(x', y, 0) = 0$;

(iii) $|v| \leq M$ on $|x_i| = \frac{R}{2}, i = 1, \ldots, n-1$;

(iv) $|v(x', \frac{R}{4}, z)| \leq \mu z$ and $|v(x', y, \frac{R}{4})| \leq \mu y$,

where $|Du| \leq \mu$ in Q_R with μ given in terms of M and F by (2). Choose a comparison function w in Q' of the form

$$w(x', y, z) = \frac{4M|x'|^2}{R^2} + \frac{4\mu}{R} yz + kyz \log \frac{2R}{y+z},$$

where k is a positive constant to be determined. Note first that $|v| \leq w$ on $\partial Q'$. Since

$$Lw(x', y, z) = \frac{8(n-1)}{R^2} M + k \left(-1 + \frac{yz}{(y+z)^2} \right) \leq \frac{8(n-1)}{R^2} M - \frac{3}{4} k,$$

we see that $Lw \leq -F$ provided

$$k \geq \frac{4}{3} \left(F + \frac{8(n-1)M}{R^2} \right).$$

With such a choice of k, the function

$$w(x', y, z) = \frac{4M|x'|^2}{R^2} + yz \left(\frac{4\mu}{R} + k \log \frac{2R}{y+z} \right)$$

satisfies the conditions $L(w \pm v) \leq 0$ in Q' and $w \pm v \geq 0$ on $\partial Q'$. By Corollary 2.9, the comparison principle, $|v| \leq w$ in Q'. Letting $x' = 0$ in this inequality, then dividing by z and letting z tend to zero, we obtain

$$\frac{1}{2} |D_y u(0, y) - D_y u(0, -y)| \leq \frac{4\mu}{R} y + ky \log \frac{2R}{y}.$$

We proved (1) for D_n.

With a slight modification we can derive (1) for $D_i, i = 1, \ldots, n-1$. We work in $\mathbb{R}^n$ in this case. In the domain

$$Q' = \left\{ (x_1, \ldots, x_{n-2}, y, z);\ |x_i| < \frac{R}{2}, i = 1, \ldots, n-2, 0 < y, z < \frac{R}{2} \right\},$$

we define

$$v(\hat{x}, y, z) = \frac{1}{4} \{ u(\hat{x}, y, z) - u(\hat{x}, -y, -z) - u(\hat{x}, y, -z) + u(\hat{x}, -y, -z) \}.$$

We choose the comparison function of the form

$$w(\hat{x}, y, z) = \frac{4M|x|^2}{R^2} + yz\left(\frac{4\mu}{R} + \bar{k}\log\frac{2R}{y+z}\right),$$

with μ as before and

$$\bar{k} \geq \frac{2}{3}\left(F + \frac{8(n-2)M}{R^2}\right).$$

We verify easily that $\triangle(w \pm v) \leq 0$ in Q' and $w \pm v \geq 0$ on $\partial Q'$. Hence $|v| \leq w$ in Q'. As above, if we set $\hat{x} = 0$, then divide by y and let y tend to zero, we obtain

$$\frac{1}{2}|D_{n-1}u(0,z) - D_{n-1}u(0,-z)| \leq \frac{4\mu}{R}z + \bar{k}z\log\frac{2R}{z}.$$

Obviously the same result holds if D_{n-1} is replaced by D_i for $i = 1, \ldots, n-2$. $\square$

Despite the elementary character of its proof, Lemma 2.18 is essentially sharp and the estimate cannot be improved without further continuity assumptions on f.

As the second application, we discuss the Schauder theory. We will show that, if $u \in C^2(B_1)$ satisfies

$$\Delta u = f \text{ in } B_1,$$

for some Hölder continuous function f in B_1, then D^2u is Hölder continuous with the same exponent. We will use the maximum principle approach, avoiding the potential integrals.

Let us recall the definition of the Hölder continuity. A function u is C^α at 0 if

$$|u(x) - u(0)| \leq c|x|^\alpha.$$

Here $|x|^\alpha$ is of course smaller than any constants. It gives a quantitative speed of how functions approach constant. We define the Hölder semi-norm of u at 0 by

$$[u]_{C^\alpha}(0) \equiv \sup_{|x|\leq 1}\frac{|u(x) - u(0)|}{|x|^\alpha}.$$

Similarly, we can define $C^{1,\alpha}$ and $C^{2,\alpha}$. For example, u is $C^{2,\alpha}$ at 0 if there exists a second order polynomial $P(x)$ such that

$$|u(x) - P(x)| \leq c|x|^{2+\alpha}.$$

LEMMA 2.19. *Suppose $u \in C^2(B_1)$ satisfies*

$$\Delta u = f \textit{ in } B_1,$$

for some $f \in C(B_1)$. Then for any $\alpha \in (0,1)$, there exist constants $c_0 > 0, \mu \in (0,1)$ and $\varepsilon_0 > 0$, depending only on n and α, such that, if $|u| \leq 1$ and $|f| \leq \varepsilon_0$ in B_1, there exists a second order harmonic polynomial

$$p(x) = \frac{1}{2}x^TAx + B \cdot x + C,$$

satisfying

$$|u(x) - p(x)| \leq \mu^{2+\alpha} \textit{ for } |x| \leq \mu,$$

and

$$|A| + |B| + |C| \leq c_0.$$

PROOF. Suppose v is the harmonic function satisfying

$$\begin{aligned} \Delta v &= 0 \text{ in } B_1, \\ v &= u \text{ on } \partial B_1. \end{aligned}$$

Then the function $u - v$ satisfies

$$\begin{aligned} \Delta(u-v) &= f \quad \text{in} \quad B_1, \\ u - v &= 0 \quad \text{on } \partial B_1. \end{aligned}$$

By Corollary 2.9, the comparison principle, we have

$$|u(x) - v(x)| \leq \frac{1-|x|^2}{2n} \sup_{B_1} |f| \quad \text{for any } x \in B_1.$$

Clearly $|v| \leq 1$ in B_1. Hence its second order Taylor polynomial at 0

$$p(x) = \frac{1}{2} x^T A x + B \cdot x + C$$

has universal bounded coefficients and is harmonic. By the mean value theorem, we have

$$\begin{aligned} |u(x) - p(x)| &\leq |v(x) - p(x)| + \frac{1}{2n} \sup_{B_1} |f| \\ &\leq C|x|^3 \sup_{B_{\frac{1}{2}}} |D^3 v| + \frac{1}{2n} \sup_{B_1} |f| \quad \text{for any } x \in B_{\frac{1}{2}}. \end{aligned}$$

Now take μ small enough such that the first term is less than or equal to

$$\frac{1}{2}\mu^{2+\alpha} \text{ for } |x| \leq \mu,$$

and then take ε_0 such that

$$\frac{1}{2} \sup_{B_1} |f| \leq \frac{1}{2}\varepsilon_0 \leq \frac{1}{2}\mu^{2+\alpha}.$$

This finishes the proof. □

THEOREM 2.20. *Suppose $u \in C^2(B_1)$ satisfies*

$$\Delta u = f \textit{ in } B_1,$$

where f is Hölder continuous at 0. Then $u(x)$ is $C^{2,\alpha}$ at 0, i.e., there exists a second order polynomial

$$p(x) = \frac{1}{2} x^T A x + Bx + C$$

satisfying

$$|u(x) - p(x)| \leq c_0 |x|^{2+\alpha} \big(|u|_{L^\infty} + |f(0)| + [f]_{C^\alpha}(0)\big) \textit{ for any } x \in B_1,$$

and

$$|A| + |B| + |C| \leq c_0 \big(|u|_{L^\infty} + |f(0)| + [f]_{C^\alpha}(0)\big),$$

where c_0 is a positive constant depending only on n and α.

PROOF. Without loss of generality, we assume $f(0) = 0$. In general, we set $v = u - \frac{1}{2n} f(0)|x|^2$. Then $\Delta v = f - f(0)$ in B_1. Furthermore, we assume $|u| \le 1$ and $[f]_{C^\alpha}(0) \le \varepsilon_0$ for small $\varepsilon_0 > 0$. The general case can be recovered by considering

$$\frac{u}{|u|_{L^\infty} + \frac{1}{\varepsilon_0}[f]_{C^\alpha}(0)}.$$

First we claim that there are harmonic polynomials for any $k = 1, 2, \cdots$,

$$P_k(x) = \frac{1}{2} x^T A_k x + B_k x + C_k,$$

satisfying

$$|u(x) - P_k(x)| \le \mu^{(2+\alpha)k} \quad \text{for any } |x| \le \mu^k,$$

and

$$\begin{aligned} |A_k - A_{k+1}| &\le c\mu^{\alpha k}, \\ |B_k - B_{k+1}| &\le c\mu^{(\alpha+1)k}, \\ |C_k - C_{k+1}| &\le c\mu^{(\alpha+2)k}, \end{aligned}$$

where $\mu \in (0,1)$ and c are positive constants depending only on n and α.

Note that the case $k = 1$ corresponds to Lemma 2.19. Let us assume it is true for k. Set

$$w(y) = \frac{(u - P_k)(\mu^k y)}{\mu^{(2+\alpha)k}} \quad \text{for } |y| \le 1.$$

Then we have

$$\Delta w(y) = \frac{f(\mu^k y)}{\mu^{\alpha k}} \quad \text{for } y \in B_1.$$

By Lemma 2.19, there is a harmonic polynomial p_0 with bounded coefficients such that

$$|w(y) - p_0(y)| \le \mu^{2+\alpha} \text{ for } |y| \le \mu,$$

provided

$$\sup_{|y| \le 1} \frac{|f(\mu^k y)|}{\mu^{\alpha k}} \le [f]_{C^0}(0) \le \varepsilon_0.$$

Now we scale back to get

$$|u(x) - P_k(x) - \mu^{(2+\alpha)k} p_0\left(\frac{x}{\mu^k}\right)| \le \mu^{(k+1)(2+\alpha)} \text{ for } |x| \le \mu^{k+1}.$$

Clearly we proved the $(k+1)$-th step by letting

$$P_{k+1}(x) = P_k(x) + \mu^{(2+\alpha)k} p_0\left(\frac{x}{\mu^k}\right).$$

It is easy to see that A_k, B_k and C_k converge and the limiting polynomial

$$p(x) = \frac{1}{2} x^T A_\infty x + B_\infty x + C_\infty$$

satisfies

$$|P_k(x) - p(x)| \le c\big\{|x|^2 \mu^{\alpha k} + |x|\mu^{(\alpha+1)k} + \mu^{(\alpha+2)k}\big\} \le c\mu^{(2+\alpha)k},$$

for any $|x| \le \mu^k$. Hence we have for $|x| \le \mu^k$

$$|u(x) - p(x)| \le |u(x) - P_k(x)| + |P_k(x) - p(x)| \le c\mu^{(\alpha+2)k},$$

or

$$|u(x) - p(x)| \le c|x|^{2+\alpha} \quad \text{for any } x \in B_1.$$

This finishes the proof. $\square$

As the third application, we use the moving plane method to discuss the symmetry of solutions. The following result was first proved by Gidas, Ni and Nirenberg.

THEOREM 2.21. *Suppose $u \in C^2(\bar{B}_1)$ is a positive solution of*

$$\Delta u = f(u) \text{ in } B_1,$$
$$u = 0 \text{ on } \partial B_1,$$

where f is locally Lipschitz in $\mathbb{R}$. Then u is radially symmetric in B_1 and $\frac{\partial u}{\partial r}(x) < 0$ for $x \neq 0$.

We need a lemma first.

LEMMA 2.22. *Let u be as in Theorem 2.21. Then for any $x_0 \in \partial B_1$ and a unit vector ν with $\nu \cdot x_0 > 0$, there exists an $r > 0$ such that*

$$\frac{\partial u}{\partial \nu}(x) < 0 \quad \text{for any } x \in B_1 \cap B_r(x_0).$$

PROOF. We consider two cases.

(i) $f(0) \le 0$. Then we have

$$\Delta u = f(u) \le f(u) - f(0) \le \left(\frac{f(u) - f(0)}{u}\right)^+ u(x) = c(x)u(x),$$

where $c(x)$ is a nonnegative bounded function. Theorem 2.6, the Hopf Lemma, implies $\frac{\partial u}{\partial \nu}(x_0) < 0$.

(ii) $f(0) > 0$. Clearly, we have $\frac{\partial u}{\partial r}(x) \le 0$ for any $x \in \partial B_1$. Suppose there exists a sequence $x_i \to x_0 \in \partial B_1$ such that

$$\frac{\partial u}{\partial r}(x_i) \ge 0.$$

Then $\frac{\partial u}{\partial r}(x_0) = 0$. Hence $\frac{\partial u}{\partial r}$ is maximized at x_0 along ∂B_1, and then

$$\frac{\partial^2 u}{\partial T \partial r}(x_0) = 0,$$

for any tangential direction T. Clearly $\frac{\partial^2 u}{\partial T^2}(x_0) = 0$. Therefore we have

$$\frac{\partial^2 u}{\partial n^2}(x_0) = \Delta u(x_0) = f(0) > 0.$$

We may assume $x_0 = (1, 0, \cdots, 0)$ and then apply Taylor expansion to $\frac{\partial u}{\partial n}(x)$ at x_0. Then we get

$$\begin{aligned}\frac{\partial u}{\partial n}(x) &= \frac{\partial u}{\partial n}(x_0) + D\frac{\partial u}{\partial n}(x_0) \cdot (x - x_0) + o(|x - x_0|) \\ &= \frac{\partial^2 u}{\partial n^2}(x_0)(x_1 - 1) + o(|x - x_0|).\end{aligned}$$

Hence for $x_1 < 1$ small, we have $\dfrac{\partial u}{\partial n}(x) < 0$. This contradiction finishes the proof. $\square$

PROOF OF THEOREM 2.21. We only need to show that u is symmetric in x_1 direction. Define for $\lambda > 0$

$$\begin{aligned}
\Sigma_\lambda &= \{x \in B_1; x_1 > \lambda\},\\
T_\lambda &= \{x_1 = \lambda\},\\
\Sigma'_\lambda &= \text{the reflection of } \Sigma_\lambda \text{ with respect to } T_\lambda,\\
x_\lambda &= (2\lambda - x_1, x_2, \cdots x_n).
\end{aligned}$$

First we write the following statement

$$(*) \qquad u(x) < u(x_\lambda) \text{ for any } x \in \Sigma_\lambda \text{ and } u_{x_1} < 0 \text{ on } B_1 \cap T_\lambda.$$

We claim that $\Lambda = \{\lambda \in (0,1); (*) \text{ holds for } \lambda\}$ is (0,1).

To this end, we need to show that Λ is non-empty, open and closed in $(0,1)$. It is clear that Λ is non-empty by Lemma 2.22. For the openness, we take a $\lambda_0 \in \Lambda$ and prove that $(*)$ holds for $\lambda < \lambda_0$ if λ is closed to λ_0. First we have

$$u(x) < u(x_{\lambda_0}) \quad \text{for any } x \in \Sigma_{\lambda_0}.$$

Take any $\lambda_1 > \lambda_0$ with λ_1 closed to λ_0. Then we get

$$u(x) < u(x_{\lambda_0}) \quad \text{for any } x \in \overline{\Sigma}_{\lambda_1}.$$

By the continuity, this implies for λ close to λ_0

$$u(x) < u(x_\lambda) \quad \text{for any } x \in \overline{\Sigma}_{\lambda_1}.$$

Next, Lemma 2.22 and the fact

$$D_{x_1}u(x) < 0 \text{ on } B_1 \cap T_{\lambda_0}$$

imply that

$$D_{x_1}u(x) < 0 \text{ on } B_1 \cap T_\lambda,$$

for any λ close to λ_0. Therefore, we have $\lambda \in \Lambda$ for any λ close to λ_0. For the closedness, we prove that positive $\lambda \in \Lambda$ if $u(x) \le u(x_\lambda)$ for $x \in \Sigma_\lambda$ and $D_{x_1}u(x) \le 0$ for $x \in T_\lambda$. For such a λ, define $v(x) = u(x_\lambda)$ in Σ'_λ. Then, we have

$$\Delta v = f(v) \text{ in } \Sigma'_\lambda.$$

Consider $w = u - v$ in Σ'_λ. We have $w \ge 0$ and $w \ne 0$ in Σ'_λ. In fact, $w > 0$ on $\partial\Sigma'_\lambda \setminus T_\lambda$. Moreover, we get

$$\Delta w = f(u) - f(v) = c(x)w,$$

by the mean value theorem. Theorem 2.13 implies $w > 0$ in Σ_λ. We note $w \equiv 0$ on T_λ. By Theorem 2.6, the Hopf lemma, we get

$$2D_{x_1}u = D_{x_1}w < 0 \text{ on } T_\lambda \cap B_1.$$

Hence $\lambda \in \Lambda$. This finishes the proof of Theorem 2.21. $\square$

REMARK 2.23. The method above depends on the smoothness of domains and the smoothness of solutions up to the boundary. In fact, such conditions can be removed. See Section 2.6 for details.

2.3. A Priori Estimates

In this section, we derive a priori estimates for solutions of the Dirichlet problem and the Neumann problem.

Suppose Ω is a bounded and connected domain in $\mathbb{R}^n$. Consider the operator L in Ω

$$Lu \equiv a_{ij}(x)D_{ij}u + b_i(x)D_iu + c(x)u,$$

for $u \in C^2(\Omega) \cap C(\bar{\Omega})$. We assume that a_{ij}, b_i and c are continuous and hence bounded in $\bar{\Omega}$ and that L is uniformly elliptic in Ω, i.e.,

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n,$$

where λ is a positive constant. We denote by Λ the sup-norms of a_{ij} and b_i, i.e.,

$$\max_{\Omega} |a_{ij}| + \max_{\Omega} |b_i| \leq \Lambda.$$

THEOREM 2.24. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

for some $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$. If $c(x) \leq 0$, then

$$\max_{\Omega} |u| \leq \max_{\partial\Omega} |\varphi| + C \max_{\Omega} |f|,$$

where C is a positive constant depending only on λ, Λ and diam(Ω).

PROOF. We will construct a function w in Ω such that
(i) $L(w \pm u) = Lw \pm f \leq 0$, or $Lw \leq \mp f$ in Ω;
(ii) $w \pm u = w \pm \varphi \geq 0$, or $w \geq \mp\varphi$ on $\partial\Omega$.
Set $F = \max_{\Omega} |f|$ and $\Phi = \max_{\partial\Omega} |\varphi|$. We need

$$\begin{aligned} Lw &\leq -F \quad \text{in } \Omega \\ w &\geq \Phi \quad \text{on } \partial\Omega. \end{aligned}$$

Suppose the domain Ω lies in the set $\{0 < x_1 < d\}$ for some $d > 0$. Set for some $\alpha > 0$ to be chosen later

$$w = \Phi + (e^{\alpha d} - e^{\alpha x_1})F.$$

Then we have by a direct calculation

$$\begin{aligned} -Lw &= (a_{11}\alpha^2 + b_1\alpha)Fe^{\alpha x_1} - c\Phi - c(e^{\alpha d} - e^{\alpha x_1})F \\ &\geq (a_{11}\alpha^2 + b_1\alpha)F \geq (\alpha^2\lambda + b_1\alpha)F \geq F, \end{aligned}$$

by choosing α large such that $\alpha^2\lambda + b_1(x)\alpha \geq 1$ for any $x \in \Omega$. Hence w satisfies (i) and (ii). By Corollary 2.9, the comparison principle, we conclude $-w \leq u \leq w$ in Ω, and in particular,

$$\sup_{\Omega} |u| \leq \Phi + (e^{\alpha d} - 1)F,$$

where α is a positive constant depending only on λ and Λ. $\square$

THEOREM 2.25. *Suppose $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$Lu = f \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial n} + \alpha(x)u = \varphi \quad \text{on } \partial\Omega,$$

where $\mathbf{n}$ is the outward normal direction to $\partial\Omega$. If $c(x) \le 0$ in Ω and $\alpha(x) \ge \alpha_0 > 0$ on $\partial\Omega$, then

$$\max_{\Omega} |u| \le C\big(\max_{\partial\Omega} |\varphi| + \max_{\Omega} |f|\big),$$

where C is a positive constant depending only on λ, Λ, α_0 and $diam(\Omega)$.

PROOF. We first consider a special case $c(x) \le -c_0 < 0$. We will show

$$|u(x)| \le \frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi \quad \text{for any } x \in \Omega.$$

Set

$$v = \frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi \pm u.$$

Then we have

$$Lv = c(x)\left(\frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi\right) \pm f \le -F \pm f \le 0 \quad \text{in } \Omega,$$

and

$$\frac{\partial v}{\partial n} + \alpha v = \alpha\left(\frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi\right) \pm \varphi \ge \Phi \pm \varphi \ge 0 \quad \text{on } \partial\Omega.$$

If v has a negative minimum in $\bar{\Omega}$, then v attains it on $\partial\Omega$, say at $x_0 \in \partial\Omega$, by Theorem 2.3. This implies $\frac{\partial v}{\partial n}(x_0) \le 0$ for $\mathbf{n} = \mathbf{n}(x_0)$, the outward normal direction at x_0. Therefore, we get

$$\left(\frac{\partial v}{\partial n} + \alpha v\right)(x_0) \le \alpha v(x_0) < 0,$$

which is a contradiction. Hence we have $v \ge 0$ in $\bar{\Omega}$, and in particular,

$$|u(x)| \le \frac{1}{c_0}F + \frac{1}{\alpha_0}\Phi \quad \text{for any } x \in \Omega.$$

Note that c_0 and α_0 are independent of λ and Λ for this special case.

Next, we consider the general case $c(x) \le 0$ for any $x \in \Omega$. Consider an auxiliary function $u(x) = z(x)w(x)$ where z is a positive function in $\bar{\Omega}$ to be determined. A direct calculation shows that w satisfies

$$a_{ij}D_{ij}w + B_iD_iw + \left(c + \frac{a_{ij}D_{ij}z + b_iD_iz}{z}\right)w = \frac{f}{z} \quad \text{in } \Omega,$$
$$\frac{\partial w}{\partial n} + \left(\alpha + \frac{1}{z}\frac{\partial z}{\partial n}\right)w = \frac{\varphi}{z} \quad \text{on } \partial\Omega,$$

where

$$B_i = \frac{1}{z}(a_{ij} + a_{ji})D_jz + b_i.$$

We need to choose the function $z > 0$ in $\bar{\Omega}$ such that

$$c + \frac{a_{ij}D_{ij}z + b_iD_iz}{z} \le -c_0 \quad \text{in } \Omega,$$
$$\alpha + \frac{1}{z}\frac{\partial z}{\partial n} \ge \frac{1}{2}\alpha_0 \quad \text{on } \partial\Omega,$$

or

$$\frac{a_{ij}D_{ij}z + b_iD_iz}{z} \le -c_0 \quad \text{in } \Omega,$$
$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \le \frac{1}{2}\alpha_0 \quad \text{on } \partial\Omega,$$

where c_0 is a positive constant depending only on $\lambda, \Lambda, \alpha_0$ and $\mathrm{diam}\Omega$. Suppose the domain Ω lies in $\{0 < x_1 < d\}$. Choose $z(x) = A + e^{\beta d} - e^{\beta x_1}$ for $x \in \Omega$, for some positive A and β to be determined. A direct calculation shows

$$\begin{aligned} -\frac{1}{z}\left(a_{ij}D_{ij}z + b_iD_iz\right) =& \frac{(\beta^2 a_{11} + \beta b_1)e^{\beta x_1}}{A + e^{\beta d} - e^{\beta x_1}} \\ \ge& \frac{\beta^2 a_{11} + \beta b_1}{A + e^{\beta d}} \ge \frac{1}{A + e^{\beta d}} > 0, \end{aligned}$$

if β is chosen such that $\beta^2 a_{11} + \beta b_1 \ge 1$. Then we have

$$\left|\frac{1}{z}\frac{\partial z}{\partial n}\right| \le \frac{\beta}{A}e^{\beta d} \le \frac{1}{2}\alpha_0,$$

if A is chosen large. This reduces to the special case we just discussed. The new extra first order term does not change the result. We may apply the special case to w. $\square$

REMARK 2.26. The result fails if we just assume $\alpha(x) \ge 0$ on $\partial\Omega$. In fact, we cannot even get the uniqueness.

2.4. Gradient Estimates

The basic method to derive gradient estimates, the so-called the Bernstein method, involves a differentiation of the equation with respect to $x_k, k = 1, \ldots, n$, followed by a multiplication by D_ku and summation over k. The maximum principle is then applied to the resulting equation in the function $v = |Du|^2$, possibly with some modifications. There are two kinds of gradient estimates, global gradient estimates and interior gradient estimates. We will use semi-linear equations to illustrate the idea.

Suppose Ω is a bounded and connected domain in $\mathbb{R}^n$. Consider the equation

$$a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x, u) \quad \text{in } \Omega,$$

for $u \in C^2(\Omega)$ and $f \in C(\Omega \times \mathbb{R})$. We always assume that a_{ij} and b_i are continuous and hence bounded in $\bar{\Omega}$ and that the equation is uniformly elliptic in Ω in the following sense

$$a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n,$$

for some positive constant λ.

We first derive a global gradient estimate.

THEOREM 2.27. *Suppose $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ satisfies*

$$a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x,u) \quad \text{in } \Omega, \tag{1}$$

for $a_{ij}, b_i \in C^1(\bar{\Omega})$ and $f \in C^1(\bar{\Omega} \times \mathbb{R})$. Then

$$\sup_{\Omega} |Du| \le \sup_{\partial\Omega} |Du| + C,$$

where C is a positive constant depending only on λ, diam(Ω), $|a_{ij}, b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L^\infty(\Omega)}$ and $|f|_{C^1(\bar{\Omega}\times[-M,M])}$.

PROOF. Set $L \equiv a_{ij}D_{ij} + b_iD_i$. We calculate $L(|Du|^2)$ first. Note

$$D_i(|Du|^2) = 2D_kuD_{ki}u,$$

and

$$D_{ij}(|Du|^2) = 2D_{ki}D_{kj}u + 2D_kuD_{kij}u. \tag{2}$$

Differentiating (1) with respect to x_k, multiplying by D_ku and summing over k, we have by (2)

$$\begin{aligned} &a_{ij}D_{ij}(|Du|^2) + b_iD_i(|Du|^2) = 2a_{ij}D_{ki}uD_{kj}u \\ &- 2D_ka_{ij}D_kuD_{ij}u - 2D_kb_iD_kuD_iu + 2D_zf|Du|^2 + 2D_kfD_ku. \end{aligned}$$

The ellipticity assumption implies

$$\sum_{i,j,k} a_{ij}D_{ki}uD_{kj}u \ge \lambda|D^2u|^2.$$

By the Cauchy inequality, we have

$$L(|Du|^2) \ge \lambda|D^2u|^2 - C|Du|^2 - C,$$

where C is a positive constant depending only on λ, the C^1-norms of a_{ij} and b_i in $\bar{\Omega}$ and the C^1-norm of f in $\bar{\Omega} \times [-M, M]$. We need to add another term u^2 to control $|Du|^2$ in the right hand side. By the ellipticity assumption again, we have

$$\begin{aligned} L(u^2) &= 2a_{ij}D_iuD_ju + 2u\{a_{ij}D_{ij}u + b_iD_iu\} \\ &\ge 2\lambda|Du|^2 + 2uf. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} L(|Du|^2 + \alpha u^2) &\ge \lambda|D^2u|^2 + (2\lambda\alpha - C)|Du|^2 - C \\ &\ge \lambda|D^2u|^2 + |Du|^2 - C, \end{aligned}$$

if we choose $\alpha > 0$ large, with C depending in addition on M. In order to control the constant term, we consider another function $e^{\beta x_1}$ for $\beta > 0$. Hence, we get

$$L\big(|Du|^2 + \alpha u^2 + e^{\beta x_1}\big) \ge \lambda|D^2u|^2 + |Du|^2 + \big\{\beta^2 a_{11}e^{\beta x_1} + \beta b_1 e^{\beta x_1} - C\big\}.$$

If we put the domain $\Omega \subset \{x_1 > 0\}$, then $e^{\beta x_1} \ge 1$ for any $x \in \Omega$. By choosing β large, we make the last term positive. Therefore, by taking large α, β depending only on $\lambda, \mathrm{diam}(\Omega)$, $|a_{ij}|_{C^1(\bar{\Omega})}$, $|b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L^\infty(\Omega)}, |f|_{C^1(\bar{\Omega}\times[-M,M])}$ and setting

$$w = |Du|^2 + \alpha|u|^2 + e^{\beta x_1},$$

we obtain

$$Lw \ge 0 \quad \text{in } \Omega.$$

By the maximum principle, we have

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w.$$

This finishes the proof. $\square$

Similarly, we discuss the interior gradient estimate. In this case, we just require the bound of $\sup_{\Omega} |u|$.

PROPOSITION 2.28. *Suppose $u \in C^3(\Omega)$ satisfies*

$$a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x,u) \quad \textit{in } \Omega,$$

for $a_{ij}, b_i \in C^1(\bar{\Omega})$ and $f \in C^1(\bar{\Omega} \times \mathbb{R})$. Then for any compact subset $\Omega' \subset\subset \Omega$

$$\sup_{\Omega'} |Du| \leq C,$$

where C is a positive constant depending on λ, diam(Ω), dist$(\Omega', \partial\Omega)$, $|a_{ij}|_{C^1(\bar{\Omega})}$, $|b_i|_{C^1(\bar{\Omega})}$, $M = |u|_{L^\infty(\Omega)}$ and $|f|_{C^1(\bar{\Omega}\times[-M,M])}$.

PROOF. We take a cut-off function $\gamma \in C_0^\infty(\Omega)$ with $\gamma \geq 0$ and consider an auxiliary function of the form

$$w = \gamma|Du|^2 + \alpha|u|^2 + e^{\beta x_1}.$$

Set $v = \gamma|Du|^2$. For $L = a_{ij}D_{ij} + b_iD_i$, we then have

$$Lv = (L\gamma)|Du|^2 + \gamma L(|Du|^2) + 2a_{ij}D_i\gamma D_j|Du|^2.$$

Recall in the proof of Theorem 2.27

$$L(|Du|^2) \geq \lambda|D^2u|^2 - C|Du|^2 - C.$$

Hence we have

$$Lv \geq \lambda\gamma|D^2u|^2 + 4a_{ij}D_i\gamma D_kuD_{kj}u - C|Du|^2 + (L\gamma)|Du|^2 - C.$$

The Cauchy inequality implies for any $\varepsilon > 0$

$$|4a_{ij}D_kuD_i\gamma D_{kj}u| \leq \varepsilon|D\gamma|^2|D^2u|^2 + c(\varepsilon)|Du|^2.$$

For the cut-off function γ, we require

$$|D\gamma|^2 \leq C\gamma \quad \text{in } \Omega.$$

Therefore we have by taking $\varepsilon > 0$ small

$$\begin{aligned} Lv &\geq \lambda\gamma|D^2u|^2\left(1 - \varepsilon\frac{|D\gamma|^2}{\gamma}\right) - C|Du|^2 - C \\ &\geq \frac{1}{2}\lambda\gamma|D^2u|^2 - C|Du|^2 - C. \end{aligned}$$

Now we may proceed as before. $\square$

In the rest of this section, we use barrier functions to derive boundary gradient estimates. We need to assume that the domain Ω satisfies the uniform exterior sphere property.

PROPOSITION 2.29. *Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$a_{ij}(x)D_{ij}u + b_i(x)D_iu = f(x,u) \quad \text{in } \Omega,$$

for $a_{ij}, b_i \in C(\bar{\Omega})$ and $f \in C(\bar{\Omega} \times \mathbb{R})$. Then

$$|u(x) - u(x_0)| \le C|x - x_0| \quad \text{for any } x \in \Omega \text{ and } x_0 \in \partial\Omega,$$

where C is a positive constant depending only on $\lambda, \Omega, |a_{ij}, b_i|_{L^\infty(\Omega)}, M = |u|_{L^\infty(\Omega)}$, $|f|_{L^\infty(\Omega\times[-M,M])}$ and $|\varphi|_{C^2(\bar{\Omega})}$ for some $\varphi \in C^2(\bar{\Omega})$ with $\varphi = u$ on $\partial\Omega$.

PROOF. For simplicity, we assume $u = 0$ on $\partial\Omega$. As before, we set $L = a_{ij}D_{ij} + b_iD_i$. Then we have

$$L(\pm u) = \pm f \ge -F \quad \text{in } \Omega,$$

where we denote $F = \sup_\Omega |f(\cdot, u)|$. Now we fix $x_0 \in \partial\Omega$ and construct a function w such that

$$Lw \le -F \text{ in } \Omega, \; w(x_0) = 0, \; w|_{\partial\Omega} \ge 0.$$

Then by the maximum principle we have

$$-w \le u \le w \quad \text{in } \Omega.$$

Taking normal derivatives at x_0, we obtain

$$\left|\frac{\partial u}{\partial n}(x_0)\right| \le \frac{\partial w}{\partial n}(x_0).$$

So we need to bound $\dfrac{\partial w}{\partial n}(x_0)$ independently of x_0.

Consider an exterior ball $B_R(y)$ with $\bar{B}_R(y) \cap \bar{\Omega} = \{x_0\}$. Define $d(x)$ as the distance from x to $\partial B_R(y)$. Then we have

$$0 < d(x) < D \equiv \operatorname{diam}(\Omega) \quad \text{for any } x \in \Omega.$$

In fact, $d(x) = |x - y| - R$ for any $x \in \Omega$. Consider $w = \psi(d)$ for some function ψ defined in $[0, \infty)$. Then we need

$$\begin{aligned}
&\psi(0) = 0 && (\Longrightarrow w(x_0) = 0),\\
&\psi(d) > 0 \text{ for } d > 0 && (\Longrightarrow w|_{\partial\Omega} \ge 0),\\
&\psi'(0) \text{ is controlled.}
\end{aligned}$$

From the first two inequalities, it is natural to require that $\psi'(d) > 0$. Note

$$Lw = \psi'' a_{ij}D_idD_jd + \psi' a_{ij}D_{ij}d + \psi' b_iD_id.$$

A direct calculation yields

$$\begin{aligned}
D_id(x) &= \frac{x_i - y_i}{|x - y|},\\
D_{ij}d(x) &= \frac{\delta_{ij}}{|x - y|} - \frac{(x_i - y_i)(x_i - y_i)}{|x - y|^3},
\end{aligned}$$

which imply $|Dd| = 1$ and, with $\Lambda = \sup |a_{ij}|$,

$$\begin{aligned}
a_{ij}D_{ij}d &= \frac{a_{ii}}{|x - y|} - \frac{a_{ij}}{|x - y|}\, D_idD_jd\\
&\le \frac{n\Lambda}{|x - y|} - \frac{\lambda}{|x - y|} = \frac{n\Lambda - \lambda}{|x - y|} \le \frac{n\Lambda - \lambda}{R}.
\end{aligned}$$

Therefore we obtain by the ellipticity

$$\begin{aligned} Lw &\le \psi'' a_{ij} D_i d D_j d + \psi' \left(\frac{n\Lambda - \lambda}{R} + |b| \right) \\ &\le \lambda \psi'' + \psi' \left(\frac{n\Lambda - \lambda}{R} + |b| \right), \end{aligned}$$

if we require $\psi'' < 0$. Hence in order to have $Lw \le -F$, we need

$$\lambda \psi'' + \psi' \left(\frac{n\Lambda - \lambda}{R} + |b| \right) + F \le 0.$$

To this end, we study the equation for some positive constants a and b

$$\psi'' + a\psi' + b = 0,$$

whose solution is given by

$$\psi(d) = -\frac{b}{a} d + \frac{C_1}{a} - \frac{C_2}{a} e^{-ad},$$

for some constants C_1 and C_2. For $\psi(0) = 0$, we need $C_1 = C_2$. Hence we have for some constant C

$$\psi(d) = -\frac{b}{a} d + \frac{C}{a}(1 - e^{-ad}),$$

which implies

$$\begin{aligned} \psi'(d) &= Ce^{-ad} - \frac{b}{a} = e^{-ad} \left(C - \frac{b}{a} e^{ad} \right), \\ \psi''(d) &= -Cae^{-ad}. \end{aligned}$$

In order to have $\psi'(d) > 0$, we need $C \ge \frac{b}{a} e^{aD}$. Since $\psi'(d) > 0$ for $d > 0$, so $\psi(d) > \psi(0) = 0$ for any $d > 0$. Therefore we take

$$\begin{aligned} \psi(d) &= -\frac{b}{a} d + \frac{b}{a^2} e^{aD}(1 - e^{-ad}) \\ &= \frac{b}{a} \left\{ \frac{1}{a} e^{aD}(1 - e^{-ad}) - d \right\}. \end{aligned}$$

Such a ψ satisfies all the requirements we imposed. □

2.5. Alexandroff Maximum Principle

Suppose Ω is a bounded domain in $\mathbb{R}^n$ and consider a second order elliptic operator L in Ω of the form

$$L \equiv a_{ij}(x) D_{ij} + b_i(x) D_i + c(x),$$

where coefficients a_{ij}, b_i, c are at least continuous in Ω. The ellipticity means that the coefficient matrix $A = (a_{ij})$ is positive definite everywhere in Ω. We set $D = \det(A)$ and $D^* = D^{\frac{1}{n}}$ so that D^* is the geometric mean of the eigenvalues of A. Throughout this section, we assume

$$0 < \lambda \le D^* \le \Lambda,$$

where λ and Λ are two positive constants, which denote the minimal and maximal eigenvalues of A respectively.

Before stating the main theorem, we first introduce the concept of contact sets. For $u \in C^2(\Omega)$, we define

$$\Gamma^+ = \{y \in \Omega;\ u(x) \le u(y) + Du(y) \cdot (x - y) \text{ for any } x \in \Omega\}.$$

The set Γ^+ is called the upper contact set of u and the Hessian matrix $D^2u = (D_{ij}u)$ is nonpositive on Γ^+. In fact, the upper contact set can also be defined for a continuous function u by

$$\begin{aligned}\Gamma^+ = &\{y \in \Omega;\ u(x) \le u(y) + p \cdot (x - y),\\ &\text{for any } x \in \Omega \ \text{ and some } \ p = p(y) \in \mathbb{R}^n\}.\end{aligned}$$

Clearly, u is concave if and only if $\Gamma^+ = \Omega$. If $u \in C^1(\Omega)$, then $p(y) = Du(y)$ and any support hyperplane must then be a tangent plane to the graph.

Now we consider an equation of the following form

$$Lu = f \quad \text{in } \Omega,$$

for some $f \in C(\Omega)$. The following result is referred to as the Alexandroff maximum principle.

THEOREM 2.30. *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies $Lu \ge f$ in Ω with $\frac{|b|}{D^*}, \frac{f}{D^*} \in L^n(\Omega)$ and $c \le 0$ in Ω. Then*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C\|\frac{f^-}{D^*}\|_{L^n(\Gamma^+)},$$

where Γ^+ is the upper contact set of u and C is a constant depending only on n, diam(Ω) and $\|\frac{b}{D^*}\|_{L^n(\Gamma^+)}$. In fact, C can be written as

$$d \cdot \left\{ exp\{\frac{2^{n-2}}{\omega_n n^n}\big(\|\frac{b}{D^*}\|^n_{L^n(\Gamma^+)} + 1\big)\} - 1 \right\},$$

with ω_n as the volume of the unit ball in $\mathbb{R}^n$.

REMARK 2.31. The integral domain Γ^+ can be replaced by

$$\Gamma^+ \cap \{x \in \Omega; u(x) > \sup_{\partial\Omega} u^+\}.$$

REMARK 2.32. There is no assumption on the uniform ellipticity. Compare with Theorem 2.24.

To prove Theorem 2.30, we need a lemma first.

LEMMA 2.33. *Suppose $g \in L^1_{loc}(\mathbb{R}^n)$ is nonnegative. Then for any $u \in C(\bar{\Omega}) \cap C^2(\Omega)$*

$$\int_{B_{\tilde{M}}(0)} g \le \int_{\Gamma^+} g(Du)|\det D^2u|,$$

where Γ^+ is the upper contact set of u and $\tilde{M} = (\sup_\Omega u - \sup_{\partial\Omega} u^+)/d$ with $d = diam(\Omega)$.

REMARK 2.34. For any positive definite matrix $A = (a_{ij})$, we have

$$\det(-D^2u) \le \frac{1}{D}\left(\frac{-a_{ij}D_{ij}u}{n}\right)^n \quad \text{on } \Gamma^+.$$

Hence we have another form for Lemma 2.33

$$\int_{B_{\tilde{M}}(0)} g \le \int_{\Gamma^+} g(Du)\big(\frac{-a_{ij}D_{ij}u}{nD^*}\big)^n.$$

REMARK 2.35. By taking $g = 1$, we have

$$\begin{aligned}\sup_{\Omega} u &\le \sup_{\partial\Omega} u^+ + \frac{d}{\omega_n^{\frac{1}{n}}}\left(\int_{\Gamma^+} |\det D^2u|\right)^{\frac{1}{n}} \\ &\le \sup_{\partial\Omega} u^+ + \frac{d}{\omega_n^{\frac{1}{n}}}\left(\int_{\Gamma^+}\left(-\frac{a_{ij}D_{ij}u}{nD^*}\right)^n\right)^{\frac{1}{n}}.\end{aligned}$$

This is Theorem 2.30 if $b_i \equiv 0$ and $c \equiv 0$.

PROOF OF LEMMA 2.33. Without loss of generality, we assume $u \le 0$ on $\partial\Omega$. Set $\Omega^+ = \{u > 0\}$. By the area-formula for Du in $\Gamma^+ \cap \Omega^+ \subset \Omega$, we have

$$\int_{Du(\Gamma^+\cap\Omega^+)} g \le \int_{\Gamma^+\cap\Omega^+} g(Du)|\det(D^2u)|, \tag{1}$$

where $|\det(D^2u)|$ is the Jacobian of the map $Du : \Omega \to \mathbb{R}^n$. In fact we may consider $\chi_\varepsilon = Du - \varepsilon\mathrm{Id} : \Omega \to \mathbb{R}^n$. Then $D\chi_\varepsilon = D^2u - \varepsilon I$, which is negative definite in Γ^+. Hence by the change of variable formula, we have

$$\int_{\chi_\varepsilon(\Gamma^+\cap\Omega^+)} g = \int_{\Gamma^+\cap\Omega^+} g(\chi_\varepsilon)|\det(D^2u - \varepsilon I)|,$$

which implies (1) if we let $\varepsilon \to 0$.

Now we claim $B_{\tilde{M}}(0) \subset Du(\Gamma^+ \cap \Omega^+)$, i.e., for any $a \in \mathbb{R}^n$ with $|a| < \tilde{M}$ there exists an $x \in \Gamma^+ \cap \Omega^+$ such that $a = Du(x)$. We may assume u attains its maximum $m > 0$ at $0 \in \Omega$, i.e.,

$$u(0) = m = \sup_{\Omega} u.$$

Consider an affine function for $|a| < m/d(\equiv \tilde{M})$

$$L(x) = m + a \cdot x.$$

Then $L(x) > 0$ for any $x \in \Omega$ and $L(0) = m$. Since u attains its maximum at 0, then $Du(0) = 0$. Hence there exists an x_1 close to 0 such that $u(x_1) > L(x_1) > 0$. Note that $u \le 0 < L$ on $\partial\Omega$. Hence there exists an $\tilde{x} \in \Omega$ such that $Du(\tilde{x}) = DL(\tilde{x}) = a$. Now we may translate vertically the plane $y = L(x)$ to the highest such position, i,e., the whole surface $y = u(x)$ lies below the plane. Clearly at such a point, the function u is positive. □

PROOF OF THEOREM 2.30. We should choose g appropriately in order to apply Lemma 2.33. Note $(-a_{ij}D_{ij}u)^n \le |b|^n|Du|^n$ in Ω if $f \equiv 0$ and $c \equiv 0$. This suggests that we should take $g(p) = |p|^{-n}$. However, such a function is not locally integrable (at the origin). Hence, we will choose $g(p) = (|p|^n + \mu^n)^{-1}$ and then let $\mu \to 0^+$.

First we have by the Cauchy inequality

$$\begin{aligned}
-a_{ij}D_{ij} &\le b_i D_i u + cu - f \\
&\le b_i D_i u - f \quad \text{ in } \Omega^+ = \{x; u(x) > 0\} \\
&\le |b| \cdot |Du| + f^- \\
&\le \left(|b|^n + \frac{(f^-)^n}{\mu^n}\right)^{\frac{1}{n}} \cdot (|Du|^n + \mu^n)^{\frac{1}{n}} \cdot (1+1)^{\frac{n-2}{n}},
\end{aligned}$$

and in particular

$$(-a_{ij}D_{ij}u)^n \le \left(|b|^n + (\frac{f^-}{\mu})^n\right)(|Du|^n + \mu^n) \cdot 2^{n-2}.$$

Now we choose

$$g(p) = \frac{1}{|p|^n + \mu^n}.$$

By Lemma 2.33, we have

$$\int_{B_{\tilde{M}}(0)} g \le \frac{2^{n-2}}{n^n} \int_{\Gamma^+ \cap \Omega^+} \frac{|b^n| + \mu^{-n}(f^-)^n}{D}.$$

We evaluate the integral in the left hand side in the following way

$$\int_{B_{\tilde{M}}(0)} g = \omega_n \int_0^{\tilde{M}} \frac{r^{n-1}}{r^n + \mu^n} dr = \frac{\omega_n}{n} \log \frac{\tilde{M}^n + \mu^n}{\mu^n} = \frac{\omega_n}{n} \log \left(\frac{\tilde{M}^n}{\mu^n} + 1\right).$$

Therefore we obtain

$$\tilde{M}^n \le \mu^n \left\{ exp\left\{ \frac{2^{n-2}}{\omega_n n^n} \left[\|\frac{b}{D^*}\|^n_{L^n(\Gamma^+\cap\Omega^+)} + \mu^{-n} \|\frac{f^-}{D^*}\|^n_{L^n(\Gamma^+\cap\Omega^+)} \right] \right\} - 1 \right\}.$$

If $f \not\equiv 0$, we choose $\mu = \|\frac{f^-}{D^*}\|_{L^n(\Gamma^+\cap\Omega^+)}$. If $f \equiv 0$, we may choose any $\mu > 0$ and then let $\mu \to 0$. □

In the following, we use Theorem 2.30 and Lemma 2.33 to derive a priori estimates for solutions of quasilinear equations and fully nonlinear equations. In the next result, we do not assume the uniform ellipticity.

Theorem 2.36. *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$Qu \equiv a_{ij}(x, u, Du)D_{ij}u + b(x, u, Du) = 0 \quad \text{in } \Omega,$$

where $a_{ij} \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ satisfies

$$a_{ij}(x, z, p)\xi_i\xi_j > 0 \quad \text{for any } (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n.$$

Suppose there exist non-negative functions $g \in L^n_{loc}(\mathbb{R}^n)$ and $h \in L^n(\Omega)$ such that

$$\frac{|b(x, z, p)|}{nD^*} \le \frac{h(x)}{g(p)} \quad \text{for any } (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

and

$$\int_\Omega h^n(x)dx < \int_{\mathbb{R}^n} g^n(p)dp \equiv g_\infty.$$

Then

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C\,diam(\Omega),$$

where C is a positive constant depending only on g and h.

EXAMPLE 2.37. The prescribed mean curvature equation is given by

$$(1+|Du|^2)\triangle u - D_i u D_j u D_{ij} u = nH(x)(1+|Du|^2)^{\frac{3}{2}},$$

for some $H \in C(\Omega)$. We have

$$a_{ij}(x,z,p) = (1+|p|^2)\delta_{ij} - p_i p_j,$$
$$b(x,z,p) = -nH(x)(1+|p|^2)^{\frac{3}{2}}.$$

This implies $D = (1+|p|^2)^{n-1}$ and

$$\frac{|b(x,z,p)|}{nD^*} \leq \frac{|H(x)|(1+|p|^2)^{\frac{3}{2}}}{(1+|p|^2)^{\frac{n-1}{n}}} = |H(x)|(1+|p|^2)^{\frac{n+2}{2n}},$$

and in particular

$$g_\infty = \int_{\mathbb{R}^n} g^n(p)dp = \int_{\mathbb{R}^n} \frac{dp}{(1+|p|^2)^{\frac{n+2}{2}}} = \omega_n.$$

COROLLARY 2.38. *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$(1+|Du|^2)\triangle u - D_i u D_j u D_{ij} u = nH(x)(1+|Du|^2)^{\frac{3}{2}} \quad \text{in } \Omega,$$

for some $H \in C(\Omega)$ with

$$H_0 \equiv \int_{\Omega} |H(x)|^n dx < \omega_n.$$

Then

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C\,diam(\Omega),$$

where C is a positive constant depending only on n and H_0.

PROOF OF THEOREM 2.36. We only prove for subsolutions. Assume $Qu \geq 0$ in Ω. Then we have

$$-a_{ij} D_{ij} u \leq b \quad \text{in } \Omega.$$

Note that $(D_{ij}u)$ is nonpositive in Γ^+. Hence $-a_{ij}D_{ij}u \geq 0$, which implies $b(x,u,Du) \geq 0$ in Γ^+. Then

$$\frac{b(x,z,Du)}{nD^*} \leq \frac{h(x)}{g(Du)} \quad \text{in } \Gamma^+ \cap \Omega^+.$$

We apply Lemma 2.33 to g^n and get

$$\int_{B_{\tilde{M}}(0)} g^n \leq \int_{\Gamma^+\cap\Omega^+} g^n(Du)\Big(\frac{-a_{ij}D_{ij}u}{nD^*}\Big)^n \leq \int_{\Gamma^+\cap\Omega^+} g^n(Du)\Big(\frac{b}{nD^*}\Big)^n$$
$$\leq \int_{\Gamma^+\cap\Omega^+} h^n \leq \int_{\Omega} h^n \Big(< \int_{\mathbb{R}^n} g^n\Big).$$

Therefore there exists a positive constant C, depending only on g and h, such that $\tilde{M} \leq C$. This implies

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \text{ diam}(\Omega).$$

This finishes the proof. $\square$

Next we discuss Monge-Ampére equations.

THEOREM 2.39. *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$det(D^2u) = f(x, u, Du) \quad in\ \Omega,$$

for some $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Suppose there exist nonnegative functions $g \in L^1_{loc}(\mathbb{R}^n)$ and $h \in L^1(\Omega)$ such that

$$|f(x, z, p)| \leq \frac{h(x)}{g(p)} \quad for\ any\ (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

and

$$\int_\Omega h(x)dx < \int_{\mathbb{R}^n} g(p)dp \equiv g_\infty.$$

Then

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C\ diam(\Omega),$$

where C is a positive constant depending only on g and h.

The proof is similar to that of Theorem 2.36. There are two special cases. The first case is given by $f = f(x)$. We may take $g \equiv 1$ and hence $g_\infty = \infty$.

COROLLARY 2.40. *Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy*

$$det(D^2u) = f(x) \quad in\ \Omega,$$

for some $f \in C(\bar{\Omega})$. Then

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + \frac{diam(\Omega)}{\omega_n^{\frac{1}{n}}} \left(\int_\Omega |f|^n \right)^{\frac{1}{n}}.$$

The second case is the prescribed Gauss curvature equations.

COROLLARY 2.41. *Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy*

$$det(D^2u) = K(x)(1 + |Du|^2)^{\frac{n+2}{2}} \quad in\ \Omega,$$

for some $K \in C(\bar{\Omega})$ with

$$K_0 \equiv \int_\Omega |K(x)| < \omega_n.$$

Then

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C\, diam(\Omega),$$

where C is a positive constant depending only on n and K_0.

We finish this section by proving a maximum principle in domains with small volumes, which is due to Varadham.

Consider

$$Lu \equiv a_{ij}D_{ij}u + b_iD_iu + cu \quad \text{in} \ \ \Omega,$$

where (a_{ij}) is positive definite pointwisely in Ω and

$$|b_i| + |c| \leq \Lambda, \quad \det(a_{ij}) \geq \lambda,$$

for some positive constants λ and Λ.

THEOREM 2.42. *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies $Lu \geq 0$ in Ω with $u \leq 0$ on $\partial\Omega$. Assume diam$(\Omega) \leq d$. Then there exists a positive constant δ, depending only on n, λ, Λ and d, such that if $|\Omega| \leq \delta$ then $u \leq 0$ in Ω.*

PROOF. If $c \leq 0$, then $u \leq 0$ by Theorem 2.30. In general, we write $c = c^+ - c^-$. Then

$$a_{ij}D_{ij}u + b_iD_iu - c^-u \geq -c^+u (\equiv f).$$

By Theorem 2.30, we have

$$\sup_\Omega u \leq C\|c^+u^+\|_{L^n(\Omega)} \leq C\|c^+\|_{L^\infty}|\Omega|^{\frac{1}{n}} \cdot \sup_\Omega u \leq \frac{1}{2}\sup_\Omega u,$$

if $|\Omega|$ is small, where C is a positive constant, depending only on n, λ, Λ and d. Hence we get $u \leq 0$ in Ω. □

Compare this with Proposition 2.15, the maximum principle for narrow domains.

2.6. The Moving Plane Method

In this section, we revisit the moving plane method to discuss the symmetry of solutions.

THEOREM 2.43. *Suppose $u \in C(\bar{B}_1) \cap C^2(B_1)$ is a positive solution of*

$$\begin{aligned} \Delta u + f(u) &= 0 \quad \text{in } B_1 \\ u &= 0 \quad \text{on } \partial B_1, \end{aligned}$$

where f is locally Lipschitz in $\mathbb{R}$. Then u is radially symmetric in B_1 and $\dfrac{\partial u}{\partial r}(x) < 0$ for $x \neq 0$.

We already proved Theorem 2.43 if u is C^2 up to the boundary. Here we give a method which does not depend on the smoothness of domains nor the smoothness of solutions up to the boundary.

LEMMA 2.44. *Suppose that the bounded domain Ω is convex in x_1 direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$ and that $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ is a positive solution of*

$$\begin{aligned} \Delta u + f(u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where f is locally Lipschitz in $\mathbb{R}$. Then u is symmetric with respect to x_1 and $D_{x_1}u(x) < 0$ for any $x \in \Omega$ with $x_1 > 0$.

PROOF. Write $x = (x_1, y) \in \Omega$ for $y \in \mathbb{R}^{n-1}$. We will prove

(1) $\quad u(x_1, y) < u(x_1^*, y)$ for any $x_1 > 0$ and $x_1^* < x_1$ with $x_1^* + x_1 > 0$.

Then by letting $x_1^* \to -x_1$, we have $u(x_1, y) \leq u(-x_1, y)$ for any x_1. By changing the direction $x_1 \to -x_1$, we get the symmetry.

Let $a = \sup x_1$ for $(x_1, y) \in \Omega$. For $0 < \lambda < a$, define

$$\begin{aligned}
&\Sigma_\lambda = \{x \in \Omega; x_1 > \lambda\},\\
&T_\lambda = \{x_1 = \lambda\},\\
&\Sigma'_\lambda = \text{the reflection of } \Sigma_\lambda \text{ with respect to } T_\lambda,\\
&x_\lambda = (2\lambda - x_1, x_2, \cdots x_n) \text{ for } x = (x_1, x_2, \cdots x_n).
\end{aligned}$$

In Σ_λ, we define

$$w_\lambda(x) = u(x) - u(x_\lambda) \quad \text{for } x \in \Sigma_\lambda.$$

Then we have by the mean value theorem

$$\begin{aligned}
&\Delta w_\lambda + c(x, \lambda) w_\lambda = 0 \quad \text{in } \Sigma_\lambda,\\
&w_\lambda \le 0 \text{ and } w_\lambda \not\equiv 0 \quad \text{on } \partial\Sigma_\lambda,
\end{aligned}$$

where $c(x, \lambda)$ is a bounded function in Σ_λ.

We need to show $w_\lambda < 0$ in Σ_λ for any $\lambda \in (0, a)$. This implies in particular that w_λ assumes along $\partial\Sigma_\lambda \cap \Omega$ its maximum in Σ_λ. By Theorem 2.6, Hopf Lemma, we have for any such $\lambda \in (0, a)$

$$D_{x_1} w_\lambda \Big|_{x_1=\lambda} = 2 D_{x_1} u \Big|_{x_1=\lambda} < 0.$$

For any λ close to a, we have $w_\lambda < 0$ by Proposition 2.15, the maximum principle for narrow domains, or Theorem 2.42, the maximum principle for domains with small volumes. Let (λ_0, a) be the largest interval of values of λ such that $w_\lambda < 0$ in Σ_λ. We want to show $\lambda_0 = 0$. If $\lambda_0 > 0$, by the continuity, $w_{\lambda_0} \le 0$ in Σ_{λ_0} and $w_{\lambda_0} \not\equiv 0$ on $\partial\Sigma_{\lambda_0}$. Then Theorem 2.8, the strong maximum principle, implies $w_{\lambda_0} < 0$ in Σ_{λ_0}. We will show that for any small $\varepsilon > 0$

$$w_{\lambda_0 - \varepsilon} < 0 \quad \text{in } \Sigma_{\lambda_0 - \varepsilon}.$$

Fix a $\delta > 0$ to be determined. Let K be a closed subset in Σ_{λ_0} such that $|\Sigma_{\lambda_0} \setminus K| < \delta/2$. The fact $w_{\lambda_0} < 0$ in Σ_{λ_0} implies

$$w_{\lambda_0}(x) \le -\eta < 0 \quad \text{for any } x \in K.$$

By the continuity, we have

$$w_{\lambda_0 - \varepsilon} < 0 \quad \text{in } K.$$

For any $\varepsilon > 0$ small, $|\Sigma_{\lambda_0 - \varepsilon} \setminus K| < \delta$. We choose δ in such a way that we apply Theorem 2.42 to $w_{\lambda_0 - \varepsilon}$ in $\Sigma_{\lambda_0 - \varepsilon} \setminus K$. Hence we get

$$w_{\lambda_0 - \varepsilon}(x) \le 0 \quad \text{in } \Sigma_{\lambda_0 - \varepsilon} \setminus K,$$

and then by Theorem 2.12

$$w_{\lambda_0 - \varepsilon}(x) < 0 \quad \text{in } \Sigma_{\lambda_0 - \varepsilon} \setminus K.$$

Therefore we obtain for any small $\varepsilon > 0$

$$w_{\lambda_0 - \varepsilon}(x) < 0 \quad \text{in } \Sigma_{\lambda_0 - \varepsilon}.$$

This contradicts the choice of λ_0 and hence finishes the proof. $\square$

CHAPTER 3

Weak Solutions, Part I

In this chapter and the next, we discuss the regularity of weak solutions of elliptic equations of divergence form. In order to explain ideas clearly, we will discuss equations of the following form only

$$-D_j\big(a_{ij}(x)D_iu\big) + c(x)u = f(x).$$

We assume Ω is a domain in $\mathbb{R}^n$. The function $u \in H^1(\Omega)$ is a *weak solution* if it satisfies

$$\int_\Omega \big(a_{ij}D_iuD_j\varphi + cu\varphi\big) = \int_\Omega f\varphi \quad \text{for any } \varphi \in H_0^1(\Omega),$$

where we assume

(i) the leading coefficients $a_{ij} \in L^\infty(\Omega)$ are *uniformly elliptic*, i.e., for some positive constant λ there holds

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n;$$

(ii) the coefficient $c \in L^{\frac{n}{2}}(\Omega)$ and the nonhomogeneous term $f \in L^{\frac{2n}{n+2}}(\Omega)$.
By the Sobolev embedding theorem, (ii) is the least assumption on c and f to have a meaningful equation.

We will prove various interior regularity results concerning the solution u if we have better assumptions on coefficients a_{ij} and c and on the nonhomogeneous term f. Basically there are two class of regularity results, perturbation results and non-perturbation results. The first is based on regularity assumptions on the leading coefficients a_{ij}, which are assumed to be at least continuous. Under such assumptions, we compare solutions of the underlying equations with harmonic functions, or solutions of constant coefficient equations. Then, the regularity of solutions depends on how close they are to harmonic functions or how close the leading coefficients a_{ij} are to constant coefficients. In this direction, we have Schauder estimates and $W^{2,p}$ estimates. In this chapter, we only discuss the Schauder estimates. For the second kind of regularity, there is no continuity assumption on the leading coefficients a_{ij}. Hence the result is not based on perturbations. The iteration methods introduced by DeGiorgi and Moser are successful in dealing with the non-perturbation situation. Results proved by them are fundamental for the discussion of quasilinear equations, where coefficients depend on solutions. In fact, the linearlity has no bearing in their arguments. This permits an extension of these results to quasilinear equations with appropriate structure conditions.

Boundary regularities can be discussed in a similar way. We leave details to readers.

3.1. Growth of Local Integrals

Let $B_R(x_0)$ be the ball in $\mathbb{R}^n$ of radius R centered at x_0. The well-known Sobolev theorem asserts that, if $u \in W^{1,p}(B_R(x_0))$ with $p > n$, then u is Hölder continuous with exponent $\alpha = 1 - n/p$.

In the first part of this section, we prove a general result, due to S. Campanato, which characterizes Hölder continuous functions by the growth of their local integrals. This result is very useful for studying the regularity of solutions of elliptic differential equations. In the second part of this section we prove a result, due to John and Nirenberg, which gives an equivalent definition of functions of the bounded mean oscillation.

Let Ω be a bounded connected open set in $\mathbb{R}^n$ and let $u \in L^1(\Omega)$. For any ball $B_r(x_0) \subset \Omega$, define

$$u_{x_0,r} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u.$$

We first prove Campanato's characterization of Hölder continuity.

THEOREM 3.1. *Suppose $u \in L^2(\Omega)$ satisfies for some $\alpha \in (0,1)$*

$$\int_{B_r(x)} |u - u_{x,r}|^2 \le M^2 r^{n+2\alpha} \quad \textit{for any } B_r(x) \subset \Omega.$$

Then $u \in C^\alpha(\Omega)$, and for any $\Omega' \subset\subset \Omega$

$$\sup_{\Omega'} |u| + \sup_{x,y\in\Omega', x\ne y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le c\big(M + \|u\|_{L^2(\Omega)}\big),$$

where c is a positive constant depending only on n, α, Ω and Ω'.

PROOF. Denote $R_0 = \operatorname{dist}(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$ and $0 < r_1 < r_2 \le R_0$, we have

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \le 2\big(|u(x) - u_{x_0,r_1}|^2 + |u(x) - u_{x_0,r_2}|^2\big).$$

By integrating with respect to x in $B_{r_1}(x_0)$, we obtain

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \le \frac{2}{\omega_n r_1^n}\Big(\int_{B_{r_1}(x_0)} |u - u_{x_0,r_1}|^2 + \int_{B_{r_2}(x_0)} |u - u_{x_0,r_2}|^2\Big),$$

and hence

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \le c(n) M^2 r_1^{-n}\big(r_1^{n+2\alpha} + r_2^{n+2\alpha}\big). \tag{1}$$

For any $R \le R_0$, with $r_1 = R/2^{i+1}, r_2 = R/2^i$, we get

$$|u_{x_0,2^{-(i+1)}R} - u_{x_0,2^{-i}R}| \le c(n) 2^{-(i+1)\alpha} M R^\alpha,$$

and therefore for $h < k$

$$|u_{x_0,2^{-h}R} - u_{x_0,2^{-k}R}| \le \frac{c(n)}{2^{(h+1)\alpha}} M R^\alpha \sum_{i=0}^{k-h-1} \frac{1}{2^{i\alpha}} \le \frac{c(n,\alpha)}{2^{h\alpha}} M R^\alpha.$$

This shows that $\{u_{x_0,2^{-i}R}\} \subset \mathbb{R}$ is a Cauchy sequence, hence a convergent one. Its limit $\hat{u}(x_0)$ is independent of the choice of R, since (1) can be applied with $r_1 = 2^{-i}R$ and $r_2 = 2^{-i}\bar{R}$ whenever $0 < R < \bar{R} \le R_0$. Thus we get

$$\hat{u}(x_0) = \lim_{r\to 0} u_{x_0,r},$$

with

$$|u_{x_0,r} - \hat{u}(x_0)| \le c(n,\alpha) M r^\alpha \quad \text{for any } 0 < r \le R_0. \tag{2}$$

Note that $\{u_{x,r}\}$ converges, as $r \to 0+$, in $L^1(\Omega)$ to the function u, by the Lebesgue theorem. Then we have $u = \hat{u}$ a.e., and (2) implies that $\{u_{x,r}\}$ converges uniformly to $u(x)$ in Ω'. Since $x \mapsto u_{x,r}$ is continuous for any $r > 0$, $u(x)$ is continuous. By (2) we get

$$|u(x)| \le CMR^\alpha + |u_{x,R}|,$$

for any $x \in \Omega'$ and $R \le R_0$. Hence u is bounded in Ω' with

$$\sup_{\Omega'} |u| \le c(MR_0^\alpha + \|u\|_{L^2(\Omega)}).$$

Now we prove that u is Hölder continuous. Let $x, y \in \Omega'$ with $R = |x-y| < R_0/2$. Then we have

$$|u(x) - u(y)| \le |u(x) - u_{x,2R}| + |u(y) - u_{y,2R}| + |u_{x,2R} - u_{y,2R}|.$$

The first two terms on the right hand side are already estimated in (2). For the last term we write

$$|u_{x,2R} - u_{y,2R}| \le |u_{x,2R} - u(\zeta)| + |u_{y,2R} - u(\zeta)|.$$

Integrating with respect to ζ over $B_{2R}(x) \cap B_{2R}(y)$, which contains $B_R(x)$, yields

$$\begin{aligned} |u_{x,2R} - u_{y,2R}|^2 &\le \frac{2}{|B_R(x)|}\Big(\int_{B_{2R}(x)} |u - u_{x,2R}|^2 + \int_{B_{2R}(y)} |u - u_{y,2R}|^2\Big) \\ &\le c(n,\alpha) M^2 R^{2\alpha}. \end{aligned}$$

Therefore we have

$$|u(x) - u(y)| \le c(n,\alpha) M |x-y|^\alpha.$$

For $|x-y| > R_0/2$ we obtain

$$|u(x) - u(y)| \le 2 \sup_{\Omega'} |u| \le c\big(M + \frac{1}{R_0^\alpha}\|u\|_{L^2}\big)|x-y|^\alpha.$$

This finishes the proof. □

The Sobolev theorem is an easy consequence of Theorem 3.1. In fact, we have the following result due to Morrey.

Corollary 3.2. *Suppose $u \in H^1_{loc}(\Omega)$ satisfies for some $\alpha \in (0,1)$*

$$\int_{B_r(x)} |Du|^2 \le M^2 r^{n-2+2\alpha} \quad \textit{for any } B_r(x) \subset \Omega.$$

Then $u \in C^\alpha(\Omega)$, and for any $\Omega' \subset\subset \Omega$

$$\sup_{\Omega'} |u| + \sup_{x,y \in \Omega', x \ne y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le c\big(M + \|u\|_{L^2(\Omega)}\big),$$

where c is a positive constant depending only on n, α, Ω and Ω'.

Proof. By the Poincaré inequality, we obtain

$$\int_{B_r(x)} |u - u_{x,r}|^2 \le c(n) r^2 \int_{B_r(x)} |Du|^2 \le c(n) M^2 r^{n+2\alpha}.$$

With Theorem 3.1, we have the desired result. □

The following result will be needed in the next section.

LEMMA 3.3. *Suppose $u \in H^1(\Omega)$ satisfies for some $\mu \in [0, n)$*

$$\int_{B_r(x_0)} |Du|^2 \le Mr^\mu \quad \text{for any } B_r(x_0) \subset \Omega.$$

Then for any $\Omega' \subset\subset \Omega$ and any $B_r(x_0) \subset \Omega$ with $x_0 \in \Omega'$

$$\int_{B_r(x_0)} |u|^2 \le c\big(M + \int_\Omega u^2\big) r^\lambda,$$

where $\lambda = \mu + 2$ if $\mu < n-2$ and λ is any number in $[0, n)$ if $n-2 \le \mu < n$, and c is a positive constant depending only on n, λ, μ, Ω and Ω'.

PROOF. As before denote $R_0 = dist(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$ and $0 < r \le R_0$, the Poincaré inequality yields

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \le cr^2 \int_{B_r(x_0)} |Du|^2 dx \le c(n) M r^{\mu+2}.$$

This implies

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \le c(n) M r^\lambda,$$

where λ is as in Lemma 3.3. For any $0 < \rho < r \le R_0$, we have

$$\begin{aligned}
\int_{B_\rho(x_0)} u^2 &\le 2\int_{B_\rho(x_0)} |u_{x_0,r}|^2 + 2\int_{B_\rho(x_0)} |u - u_{x_0,r}|^2 \\
&\le c(n)\rho^n |u_{x_0,r}|^2 + 2\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \\
&\le c(n)\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} u^2 + Mr^\lambda,
\end{aligned}$$

where we used

$$|u_{x_0,r}|^2 \le \frac{c(n)}{r^n} \int_{B_r(x_0)} u^2.$$

Hence we have

$$\int_{B_\rho(x_0)} u^2 \le c\Big(\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} u^2 + Mr^\lambda\Big) \quad \text{for any } 0 < \rho < r \le R_0. \tag{1}$$

We note $\lambda \in (0, n)$. Now we claim

$$\int_{B_\rho(x_0)} u^2 \le c\Big(\left(\frac{\rho}{r}\right)^\lambda \int_{B_r(x_0)} u^2 + M\rho^\lambda\Big) \quad \text{for any } 0 < \rho < r \le R_0. \tag{2}$$

By choosing $r = R_0$, we obtain

$$\int_{B_\rho(x_0)} u^2 \le c\rho^\lambda\Big(\int_\Omega u^2 + M\Big) \quad \text{for any } \rho \le R_0.$$

In order to get (2) from (1), we apply the following technical lemma to the function $\phi(r) = \int_{B_r(x_0)} u^2$. □

LEMMA 3.4. *Let $\phi(t)$ be a nonnegative and nondecreasing function on $[0, R]$. Suppose for some nonnegative constants A, B, α, β with $\beta < \alpha$*

$$\phi(\rho) \le A\Big(\Big(\frac{\rho}{r}\Big)^{\alpha} + \varepsilon\Big)\phi(r) + Br^{\beta} \quad \text{for any } 0 < \rho \le r \le R.$$

Then for any $\gamma \in (\beta, \alpha)$, there exists a constant ε_0 such that, if $\varepsilon < \varepsilon_0$,

$$\phi(\rho) \le c\Big(\Big(\frac{\rho}{r}\Big)^{\gamma} \phi(r) + B\rho^{\beta}\Big) \quad \text{for any } 0 < \rho \le r \le R,$$

where ε_0 and c are positive constants depending only on A, α, β and γ. In particular,

$$\phi(r) \le c\Big(\frac{\phi(R)}{R^{\gamma}} r^{\gamma} + Br^{\beta}\Big) \quad \text{for any } 0 < r \le R.$$

PROOF. For any $\tau \in (0, 1)$ and $r < R$, we have

$$\phi(\tau r) \le A\tau^{\alpha}(1 + \varepsilon\tau^{-\alpha})\phi(r) + Br^{\beta}.$$

Choose $\tau < 1$ such that $2A\tau^{\alpha} = \tau^{\gamma}$ and assume $\varepsilon_0\tau^{-\alpha} < 1$. Then we get for every $r < R$

$$\phi(\tau r) \le \tau^{\gamma}\phi(r) + Br^{\beta},$$

and therefore for any integer $k > 0$

$$\begin{aligned} \phi(\tau^{k+1} r) &\le \tau^{\gamma}\phi(\tau^{k} r) + B\tau^{k\beta} r^{\beta} \\ &\le \tau^{(k+1)\gamma}\phi(r) + B\tau^{k\beta} r^{\beta} \sum_{j=0}^{k} \tau^{j(\gamma-\beta)} \\ &\le \tau^{(k+1)\gamma}\phi(r) + \frac{B\tau^{k\beta} r^{\beta}}{1 - \tau^{\gamma-\beta}}. \end{aligned}$$

Choosing k such that $\tau^{k+2} r < \rho \le \tau^{k+1} r$, we obtain

$$\phi(\rho) \le \frac{1}{\tau^{\gamma}} \Big(\frac{\rho}{r}\Big)^{\gamma} \phi(r) + \frac{B\rho^{\beta}}{\tau^{2\beta}(1 - \tau^{\gamma-\beta})}.$$

This finishes the proof. □

In the rest of this section, we discuss functions of the bounded mean oscillation (BMO). The following result is due to John and Nirenberg and referred to as the John-Nirenberg lemma.

THEOREM 3.5. *Suppose $u \in L^1(\Omega)$ satisfies*

$$\int_{B_r(x)} |u - u_{x,r}| \le Mr^n \quad \text{for any } \ B_r(x) \subset \Omega.$$

Then for any $B_r(x) \subset \Omega$

$$\int_{B_r(x)} e^{\frac{p_0}{M}|u - u_{x,r}|} \le Cr^n,$$

where p_0 and C are positive constants depending only on n.

REMARK 3.6. Functions satisfying the condition of Theorem 3.5 are called functions of the bounded mean oscillation (BMO). We note that

$$L^{\infty} \text{ is a proper subset of BMO.}$$

The function $u(x) = \log(x)$ in $(0, 1) \subset \mathbb{R}$ is in BMO but not in L^{∞}.

For convenience, we use cubes instead of balls. We need the Calderon-Zygmund decomposition in the proof of Theorem 3.5. First, we introduce some terminology.

Take the unit cube Q_0. Cut it equally into 2^n cubes, which we take as the first generation. Do the same cutting for these small cubes to get the second generation. Continue this process. These cubes (from all generations) are called *dyadic cubes*. Any $(k+1)$-generation cube Q arises from some k-generation cube $\tilde{Q}$, which is called the *predecessor* of Q.

LEMMA 3.7. *Suppose $f \in L^1(Q_0)$ is nonnegative and $\alpha > |Q_0|^{-1} \int_{Q_0} f$ is a fixed constant. Then there exists a sequence of (nonoverlapping) dyadic cubes $\{Q_j\}$ in Q_0 such that*

$$f(x) \leq \alpha \quad a.e. \ in \ \ Q_0 \setminus \cup_j Q_j,$$

and

$$\alpha \leq \frac{1}{|Q_j|} \int_{Q_j} f dx < 2^n \alpha.$$

PROOF. Cut Q_0 into 2^n dyadic cubes and keep the cube Q if $|Q|^{-1} \int_Q f \geq \alpha$. Continue cutting for others, and always keep the cube Q if $|Q|^{-1} \int_Q f \geq \alpha$ and cut the rest. Let $\{Q_j\}$ be the cubes we have kept during this infinite process. We only need to verify

$$f(x) \leq \alpha \ \text{ a.e. in } \ Q_0 \setminus \cup_j Q_j.$$

Let $F = Q_0 \setminus \cup_j Q_j$. For any $x \in F$, from the way we collect $\{Q_j\}$, there exists a sequence of cubes Q^i containing x such that

$$\frac{1}{|Q^i|} \int_{Q^i} f < \alpha,$$

and

$$\operatorname{diam}(Q^i) \to 0 \text{ as } i \to \infty.$$

The Lebesgue density theorem implies

$$f \leq \alpha \text{ a.e. in } \ F.$$

This finishes the proof. □

PROOF OF THEOREM 3.5. Assume $\Omega = Q_0$. We may rewrite the assumption in terms of cubes as follows

$$\int_Q |u - u_Q| < M|Q| \quad \text{for any } Q \subset Q_0.$$

We prove that there exist two positive constants c_1 and c_2, depending only on n, such that for any $Q \subset Q_0$

$$|\{x \in Q; |u - u_Q| > t\}| \leq c_1 |Q| \exp\left(-\frac{c_2}{M} t\right).$$

Then Theorem 3.5 follows easily.

Without loss of generality, we assume $M = 1$. Choose

$$\alpha > 1 \geq |Q_0|^{-1} \int_{Q_0} |u - u_{Q_0}| dx.$$

Apply the Calderon-Zygmund decomposition to $f = |u - u_{Q_0}|$. There exists a sequence of (nonoverlapping) cubes $\{Q_j^{(1)}\}_{j=1}^\infty$ such that

$$\alpha \le \frac{1}{|Q_j^{(1)}|}\int_{Q_j^{(1)}} |u - u_{Q_0}| < 2^n\alpha,$$

and

$$|u(x) - u_{Q_0}| \le \alpha \quad a.e.\ x \in Q_0 \setminus \cup_{j=1}^\infty Q_j^{(1)}.$$

This implies

$$\sum_j |Q_j^{(1)}| \le \frac{1}{\alpha}\int_{Q_0} |u - u_{Q_0}| \le \frac{1}{\alpha}|Q_0|,$$

and

$$|u_{Q_j^{(1)}} - u_{Q_0}| \le \frac{1}{|Q_j^{(1)}|}\int_{Q_j^{(1)}} |u - u_{Q_0}|dx \le 2^n\alpha.$$

The definition of the BMO norm implies for each j

$$\frac{1}{|Q_j^{(1)}|}\int_{Q_j^{(1)}} |u - u_{Q_j{(1)}}|dx \le 1 < \alpha.$$

Apply the decomposition procedure above to $f = |u - u_{Q_j^{(1)}}|$ in $Q_j^{(1)}$. There exists a sequence of (nonoverlapping) cubes $\{Q_j^{(2)}\}$ in $\cup_j Q_j^{(1)}$ such that

$$\sum_{j=1}^\infty |Q_j^{(2)}| \le \frac{1}{\alpha}\sum_j \int_{Q_j^{(1)}} |u - u_{Q_j^{(1)}}| \le \frac{1}{\alpha}\sum_j |Q_j^{(1)}| \le \frac{1}{\alpha^2}|Q_0|,$$

and

$$|u(x) - u_{Q_j^{(1)}}| \le \alpha \quad \text{for a.e. } x \in Q_j^{(1)} \setminus \cup Q_j^{(2)},$$

which implies

$$|u(x) - u_{Q_0}| \le 2\cdot 2^n\alpha \quad \text{for a.e. } x \in Q_0 \setminus \cup_j Q_j^{(2)}.$$

Continue this process. For any integer $k \ge 1$, there exists a sequence of disjoint cubes $\{Q_j^{(k)}\}$ such that

$$\sum_j |Q_j^{(k)}| \le \frac{1}{\alpha^k}|Q_0|,$$

and

$$|u(x) - u_{Q_0}| \le k2^n\alpha \quad \text{for a.e. } x \in Q_0 \setminus \cup_j Q_j^{(k)}.$$

Thus, we obtain

$$|\{x \in Q_0; |u - u_{Q_0}| > 2^n k\alpha\}| \le \sum_{j=1}^\infty |Q_j^{(k)}| \le \frac{1}{\alpha^k}|Q_0|.$$

For any t, there exists an integer k such that $t \in [2^n k\alpha, 2^n(k+1)\alpha)$. This implies

$$\alpha^{-k} = \alpha\alpha^{-(k+1)} = \alpha e^{-(k+1)\log\alpha} \le \alpha\ \mathrm{epx}\big(-\frac{\log\alpha}{2^n\alpha}t\big).$$

This finishes the proof. □

3.2. Hölder Continuity of Solutions

In this section, we prove the Hölder regularity for solutions. The basic idea is to freeze the leading coefficients and then to compare solutions with harmonic functions. The regularity of solutions depends on how close solutions are to harmonic functions. Hence, we need some regularity assumption on the leading coefficients.

Suppose $a_{ij} \in L^\infty(B_1)$ is uniformly elliptic in B_1, i.e.,

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any } x \in B_1, \xi \in \mathbb{R}^n.$$

In the following, we assume that a_{ij} is at least continuous and that $u \in H^1(B_1)$ satisfies

$$(*) \qquad \int_{B_1} a_{ij}D_iuD_j\varphi + cu\varphi = \int_{B_1} f\varphi \quad \text{for any } \varphi \in H_0^1(B_1).$$

The main theorem in this section is the following Hölder estimates for solutions.

THEOREM 3.8. *Let $u \in H^1(B_1)$ solve $(*)$. Assume $a_{ij} \in C^0(\bar{B}_1)$, $c \in L^n(B_1)$ and $f \in L^q(B_1)$ for some $q \in (n/2,\ n)$. Then $u \in C^\alpha(B_1)$ with $\alpha = 2-n/q \in (0,1)$. Moreover, there exists an R_0 such that, for any $x \in B_{\frac{1}{2}}$ and $r \leq R_0$,*

$$\int_{B_r(x)} |Du|^2 \leq Cr^{n-2+2\alpha}\big(\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)}\big),$$

where R_0 and C are positive constants depending only on λ, Λ, τ and $\|c\|_{L^n}$, with τ denoting the modulus of the continuity of a_{ij}, i.e.,

$$|a_{ij}(x) - a_{ij}(y)| \leq \tau(|x-y|) \quad \textit{for any } x, y \in B_1.$$

If $c \equiv 0$, we may replace $\|u\|_{H^1(B_1)}$ with $\|Du\|_{L^2(B_1)}$.

The idea of the proof is to compare the solution u with harmonic functions and use the perturbation argument. First, we prove a basic estimate for harmonic functions.

LEMMA 3.9. *Suppose $\{a_{ij}\}$ is a constant positive definite matrix satisfying for some $0 < \lambda \leq \Lambda$*

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \textit{for any } \xi \in \mathbb{R}^n.$$

Suppose $w \in H^1\big(B_r(x_0)\big)$ is a weak solution of

$$(1) \qquad a_{ij}D_{ij}w = 0 \quad \textit{in } B_r(x_0).$$

Then, for any $0 < \rho \leq r$

$$\int_{B_\rho(x_0)} |Dw|^2 \leq C\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Dw|^2,$$

and

$$\int_{B_\rho(x_0)} |Dw - (Dw)_{x_0,\rho}|^2 \leq C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Dw - (Dw)_{x_0,r}|^2,$$

where C is a positive constant depending only on n and Λ/λ.

PROOF. Note that if w is a solution of (1) so is any of its derivatives. We may apply Lemma 1.42 in Chapter 1 to Dw. □

Now we compare any functions with harmonic functions.

Corollary 3.10. *Suppose w is as in Lemma 2.2 and u is an arbitrary H^1-function in $B_r(x_0)$. Then, for any $0<\rho\le r$*

$$\int_{B_\rho(x_0)}|Du|^2\le C\Big\{\Big(\frac{\rho}{r}\Big)^n\int_{B_r(x_0)}|Du|^2+\int_{B_r(x_0)}|D(u-w)|^2\Big\},$$

and

$$\begin{aligned}\int_{B_\rho(x_0)}|Du-(Du)_{x_0,\rho}|^2\le& C\Big\{\Big(\frac{\rho}{r}\Big)^{n+2}\int_{B_r(x_0)}|Du-(Du)_{x_0,r}|^2\\&+\int_{B_r(x_0)}|D(u-w)|^2\Big\},\end{aligned}$$

where C is a positive constant depending only on n and Λ/λ.

Proof. With $v=u-w$, we have for any $0<\rho\le r$

$$\begin{aligned}\int_{B_\rho(x_0)}|Du|^2&\le 2\int_{B_\rho(x_0)}|Dw|^2+2\int_{B_\rho(x_0)}|Dv|^2\\&\le C\Big(\frac{\rho}{r}\Big)^n\int_{B_r(x_0)}|Dw|^2+2\int_{B_r(x_0)}|Dv|^2\\&\le C\Big(\frac{\rho}{r}\Big)^n\int_{B_r(x_0)}|Du|^2+c\Big[1+\Big(\frac{\rho}{r}\Big)^n\Big]\int_{B_r(x_0)}|Dv|^2,\end{aligned}$$

and

$$\begin{aligned}&\int_{B_\rho(x_0)}|Du-(Du)_{x_0,\rho}|^2\\\le&2\int_{B_\rho(x_0)}|Du-(Dw)_{x_0,\rho}|^2+2\int_{B_\rho(x_0)}|Dv|^2\\\le&4\int_{B_\rho(x_0)}|Dw-(Dw)_{x_0,\rho}|^2+6\int_{B_\rho(x_0)}|Dv|^2\\\le&C\Big(\frac{\rho}{r}\Big)^{n+2}\int_{B_r(x_0)}|Dw-(Dw)_{x_0,r}|^2+6\int_{B_r(x_0)}|Dv|^2\\\le&C\Big(\frac{\rho}{r}\Big)^{n+2}\int_{B_r(x_0)}|Du-(Du)_{x_0,r}|^2+c\Big[1+\Big(\frac{\rho}{r}\Big)^{n+2}\Big]\int_{B_r(x_0)}|Dv|^2.\end{aligned}$$

This finishes the proof. □

The regularity of u depends on how close u is to w, the solution of the constant coefficient equation.

We now prove Theorem 3.8.

Proof of Theorem 3.8. We decompose u into a sum $v+w$ where w satisfies a homogeneous equation and v has estimates in terms of nonhomogeneous terms.

For any $B_r(x_0)\subset B_1$, we write the equation in the following form

$$\int_{B_1}a_{ij}(x_0)D_iuD_j\varphi=\int_{B_1}f\varphi-cu\varphi+\big(a_{ij}(x_0)-a_{ij}(x)\big)D_iuD_j\varphi.$$

In $B_r(x_0)$, the Dirichlet problem

$$\int_{B_r(x_0)}a_{ij}(x_0)D_iwD_j\varphi=0\quad\text{for any }\varphi\in H_0^1\big(B_r(x_0)\big)$$

has a unique solution w with $w - u \in H_0^1\big(B_r(x_0)\big)$. Obviously, the function $v = u - w \in H_0^1(B_r(x_0))$ satisfies the equation

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i v D_j \varphi = \int_{B_r(x_0)} f\varphi - cu\varphi + \big(a_{ij}(x_0) - a_{ij}(x)\big) D_i u D_j \varphi$$
$$\text{for any } \varphi \in H_0^1\big(B_r(x_0)\big).$$

By taking the test function $\varphi = v$, we obtain

$$\int_{B_r(x_0)} |Dv|^2 \le c \Bigg\{ \tau^2(r) \int_{B_r(x_0)} |Du|^2 + \left(\int_{B_r(x_0)} |c|^n \right)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \Bigg\},$$

where we used the Sobolev inequality

$$\left(\int_{B_r(x_0)} v^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \le c(n) \left(\int_{B_r(x_0)} |Dv|^2 \right)^{\frac{1}{2}},$$

for $v \in H_0^1(B_r(x_0))$. Therefore Corollary 3.10 implies for any $0 < \rho \le r$

$$\tag{1} \begin{aligned} \int_{B_\rho(x_0)} |Du|^2 \le C \Bigg\{ &\left[\left(\frac{\rho}{r}\right)^n + \tau^2(r) \right] \int_{B_r(x_0)} |Du|^2 \\ &+ \left(\int_{B_r(x_0)} |c|^n \right)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \Bigg\}, \end{aligned}$$

where C is a positive constant depending only on λ and Λ. By the Hölder inequality, we have

$$\left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \le \left(\int_{B_r(x_0)} |f|^q \right)^{\frac{2}{q}} r^{n-2+2\alpha},$$

where $\alpha = 2 - n/q \in (0,1)$ if $n/2 < q < n$. Hence (1) implies for any $B_r(x_0) \subset B_1$ and any $0 < \rho \le r$

$$\begin{aligned} \int_{B_\rho(x_0)} |Du|^2 \le C \Bigg\{ &\left[\left(\frac{\rho}{r}\right)^n + \tau^2(r) \right] \int_{B_r(x_o)} |Du|^2 + r^{n-2+2\alpha} \|f\|^2_{L^q(B_1)} \\ &+ \left(\int_{B_r(x_0)} |c|^n \right)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 \Bigg\}. \end{aligned}$$

Case 1. $c \equiv 0$. We have for any $B_r(x_0) \subset B_1$ and for any $0 < \rho \le r$

$$\int_{B_\rho(x_0)} |Du|^2 \le C \left\{ \left[\left(\frac{\rho}{r}\right)^n + \tau^2(r) \right] \int_{B_r(x_0)} |Du|^2 + r^{n-2+2\alpha} \|f\|^2_{L^q(B_1)} \right\}.$$

Now the result would follow if in the above inequality we could write $\rho^{n-2+2\alpha}$ instead of $r^{n-2+2\alpha}$. This is in fact true and is stated in Lemma 3.4. By Lemma

3.4, there exists an $R_0 > 0$ such that, for any $x_0 \in B_{\frac{1}{2}}$ and any $0 < \rho < r \le R_0$,

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{\left(\frac{\rho}{r}\right)^{n-2+2\alpha}\int_{B_r(x_0)} |Du|^2 + \rho^{n-2+2\alpha}\|f\|^2_{L^q(B_1)}\right\}.$$

In particular, taking $r = R_0$, we obtain for any $\rho < R_0$

$$\int_{B_\rho(x_0)} |Du|^2 \le C\rho^{n-2+2\alpha}\left\{\int_{B_1} |Du|^2 + \|f\|^2_{L^q(B_1)}\right\}.$$

Case 2. General case. We have for any $B_r(x_0) \subset B_1$ and any $0 < \rho \le r$

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2 \le C\Big\{ &\left[\left(\frac{\rho}{r}\right)^n + \tau^2(r)\right]\int_{B_r(x_0)} |Du|^2 \\ &+ r^{n-2+2\alpha}\chi(F) + \int_{B_r(x_0)} u^2\Big\},\end{aligned} \tag{2}$$

where $\chi(F) = \|f\|^2_{L^q(B_1)}$. We will prove for any $x_0 \in B_{1/2}$ and any $0 < \rho < r \le 1/2$

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2 \le C\Big\{ &\left[\left(\frac{\rho}{r}\right)^n + \tau^2(r)\right]\int_{B_r(x_0)} |Du|^2 \\ &+ r^{n-2+2\alpha}\left[\chi(F) + \int_{B_1} u^2 + \int_{B_1} |Du|^2\right]\Big\}.\end{aligned} \tag{3}$$

We need a bootstrap argument. First by Lemma 3.3, there exists an $R_1 \in (1/2, 1)$ such that for any $x_0 \in B_{R_1}$ and any $0 < r \le 1 - R_1$

$$\int_{B_r(x_0)} u^2 \le Cr^{\delta_1}\left\{\int_{B_1} |Du|^2 + \int_{B_1} u^2\right\}, \tag{4}$$

where $\delta_1 = 2$ if $n > 2$ and δ_1 is arbitrary in (0,2) if $n = 2$. This, with (2), yields

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2 \le C\Big\{ &\left[\left(\frac{\rho}{r}\right)^n + \tau^2(r)\right]\int_{B_r(x_0)} |Du|^2 \\ &+ r^{n-2+2\alpha}\chi(F) + r^{\delta_1}\|u\|^2_{H^1(B_1)}\Big\}.\end{aligned}$$

Then (3) holds in the following cases: (i) $n = 2$, by choosing $\delta_1 = 2\alpha$; (ii) $n > 2$ while $n - 2 + 2\alpha \le 2$, by choosing $\delta_1 = 2$. For $n > 2$ and $n - 2 + 2\alpha > 2$, we have

$$\int_{B_\rho(x_0)} |Du|^2 \le c\left\{\left[\left(\frac{\rho}{r}\right)^n + \tau^2(r)\right]\int_{B_r(x_0)} |Du|^2 + r^2[\chi(F) + \|u\|^2_{H^1(B_1)}]\right\}.$$

Lemma 3.4 again yields for any $x_0 \in B_{R_1}$ and any $0 < r \le 1 - R_1$

$$\int_{B_r(x_0)} |Du|^2 \le Cr^2\left\{\chi(F) + \|u\|^2_{H^1(B_1)}\right\}.$$

Hence by Lemma 3.3, there exists an $R_2 \in (1/2, R_1)$ such that for any $x_0 \in B_{R_2}$ and any $0 < r \le R_1 - R_2$

$$\int_{B_r(x_0)} u^2 \le Cr^{\delta_2}\left\{\chi(F) + \|u\|^2_{H^1(B_1)}\right\}, \tag{5}$$

where $\delta_2 = 4$ if $n > 4$ and δ_2 is arbitrary in $(2, n)$ if $n = 3$ or 4. Notice (5) is an improvement compared with (4). Substitute (5) in (2) and continue the process. After finitely many steps, we obtain (3). $\square$

3.3. Hölder Continuity of Gradients

In this section, we prove the Hölder regularity for gradients of solutions. We follow the same idea used to prove Theorem 3.8.

Suppose $a_{ij} \in L^\infty(B_1)$ is uniformly elliptic in B_1, i.e.,

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } x \in B_1, \xi \in \mathbb{R}^n.$$

We assume that $u \in H^1(B_1)$ satisfies

$$(*) \qquad \int_{B_1} a_{ij} D_i u D_j \varphi + cu\varphi = \int_{B_1} f\varphi \quad \text{for any } \varphi \in H_0^1(B_1).$$

The main theorem in this section is the following Hölder estimate for gradients.

Theorem 3.11. *Let $u \in H^1(B_1)$ solve $(*)$. Assume $a_{ij} \in C^\alpha(\bar{B}_1)$, $c \in L^q(B_1)$ and $f \in L^q(B_1)$ for some $q > n$ and $\alpha = 1 - n/q \in (0,1)$. Then $Du \in C^\alpha(B_1)$. Moreover, there exists an $R_0 \in (0,1)$ such that, for any $x \in B_{\frac{1}{2}}$ and $r \le R_0$,*

$$\int_{B_r(x)} |Du - (Du)_{x,r}|^2 \le Cr^{n+2\alpha}\big\{\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)}\big\},$$

where R_0 and C are positive constants depending only on λ, $|a_{ij}|_{C^\alpha}$ and $|c|_{L^q}$.

Proof. As in the proof of Theorem 3.8, we decompose u into a sum $v+w$ where w satisfies a homogeneous equation and v has estimates in terms of nonhomogeneous terms.

For any $B_r(x_0) \subset B_1$, we write the equation in the following form

$$\int_{B_1} a_{ij}(x_0) D_i u D_j \varphi = \int_{B_1} f\varphi - cu\varphi + (a_{ij}(x_0) - a_{ij}(x)) D_i u D_j \varphi.$$

In $B_r(x_0)$, the Dirichlet problem

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i w D_j \varphi = 0 \quad \text{for any } \varphi \in H_0^1\big(B_r(x_0)\big),$$

has a unique solution w with $w - u \in H_0^1\big(B_r(x_0)\big)$. Obviously, the function $v = u - w \in H_0^1\big(B_r(x_0)\big)$ satisfies the equation

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i v D_j \varphi = \int_{B_r(x_0)} f\varphi - cu\varphi + (a_{ij}(x_0) - a_{ij}(x)) D_i u D_j \varphi \qquad \text{for any } \varphi \in H_0^1\big(B_r(x_0)\big).$$

By taking the test function $\varphi = v$, we obtain

$$\int_{B_r(x_0)} |Dv|^2 \le C\bigg\{\tau^2(r) \int_{B_r(x_0)} |Du|^2 + \bigg(\int_{B_r(x_0)} |c|^n\bigg)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \bigg(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\bigg)^{\frac{n+2}{n}}\bigg\}.$$

Therefore Corollary 3.10 implies for any $0 < \rho \le r$

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2 \le C\Big\{ \Big[\Big(\frac{\rho}{r}\Big)^n + \tau^2(r)\Big] \int_{B_r(x_0)} |Du|^2 \\ + \Big(\int_{B_r(x_0)} |c|^n\Big)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \Big(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{n}} \Big\},\end{aligned} \tag{1}$$

and

$$\begin{aligned}&\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \\ \le C\Big\{ &\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + \tau^2(r) \int_{B_r(x_0)} |Du|^2 \\ &+ \Big(\int_{B_r(x_0)} |c|^n\Big)^{\frac{2}{n}} \int_{B_r(x_0)} u^2 + \Big(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{n}} \Big\},\end{aligned} \tag{2}$$

where C is a positive constant depending only on λ and Λ.

By the Hölder inequality, we have for any $B_r(x_0) \subset B_1$

$$\Big(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\Big)^{\frac{n+2}{n}} \le \Big(\int_{B_r(x_0)} |f|^q\Big)^{\frac{2}{q}} r^{n+2\alpha},$$

and

$$\Big(\int_{B_r(x_0)} |c|^n\Big)^{\frac{2}{n}} \le r^{2\alpha} \Big(\int_{B_r(x_0)} |c|^q\Big)^{\frac{2}{q}},$$

with $\alpha = 1 - n/q$.

Case 1. $a_{ij} \equiv$ const., $c \equiv 0$. In this case, $\tau(r) \equiv 0$. By (2), there holds for any $B_r(x_0) \subset B_1$ and $0 < \rho \le r$

$$\begin{aligned}&\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \\ \le &C\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha} \|f\|^2_{L^q(B_1)} \Big\}.\end{aligned}$$

By Lemma 3.4, we replace $r^{n+2\alpha}$ by $\rho^{n+2\alpha}$ to get the result.

Case 2. $c \equiv 0$. By (1) and (2), we have for any $B_r(x_0) \subset B_1$ and any $\rho < r$

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{ \Big[\Big(\frac{\rho}{r}\Big)^n + r^{2\alpha}\Big] \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha} \|f\|^2_{L^q(B_1)} \right\}, \tag{3}$$

and

$$\begin{aligned}\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 \\ + r^{2\alpha} \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha} \|f\|^2_{L^q(B_1)} \Big\}.\end{aligned} \tag{4}$$

We need to estimate the integral

$$\int_{B_r(x_0)} |Du|^2.$$

Write $\chi(F) = \|f\|^2_{L^q(B_1)}$.

Take small $\delta > 0$. Then (3) implies

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{\left[\left(\frac{\rho}{r}\right)^n + r^{2\alpha}\right]\int_{B_r(x_0)} |Du|^2 + r^{n-2\delta}\chi(F)\right\}.$$

Hence Lemma 3.4 implies the existence of an $R_1 \in (3/4, 1)$ with $r_1 = 1 - R_1$ such that, for any $x_0 \in B_{R_1}$ and any $0 < r \le r_1$,

$$\int_{B_r(x_0)} |Du|^2 \le Cr^{n-2\delta}\left\{\chi(F) + \|Du\|^2_{L^2(B_1)}\right\}. \tag{5}$$

Therefore, by substituting (5) in (4) we obtain for any $0 < \rho < r \le r_1$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\left\{\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha-2\delta}\left[\chi(F) + \|Du\|^2_{L^2(B_1)}\right]\right\}.$$

By Lemma 3.4 again, we have for any $x_0 \in B_{R_1}$ and any $0 < \rho < r \le r_1$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\left\{\left(\frac{\rho}{r}\right)^{n+2\alpha-2\delta}\int_{B_r} |Du - (Du)_{x_0,r}|^2 + \rho^{n+2\alpha-2\delta}[\chi(F) + \|Du\|^2_{L^2(B_1)}]\right\}.$$

With $r = r_1$ this implies that for any $x_0 \in B_{R_1}$ and any $0 < r \le r_1$

$$\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 \le Cr^{n+2\alpha-2\delta}\left\{\chi(F) + \|Du\|^2_{L^2(B_1)}\right\}.$$

Hence $Du \in C^{\alpha-\delta}_{\mathrm{loc}}$ for any $\delta > 0$ small. In particular, $Du \in L^\infty_{\mathrm{loc}}$ and

$$\sup_{B_{\frac{3}{4}}} |Du|^2 \le C\left\{\chi(F) + \|Du\|^2_{L^2(B_1)}\right\}. \tag{6}$$

Combining (4) and (6), we have for any $x_0 \in B_{\frac{1}{2}}$ and $0 < \rho < r \le r_1$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\left\{\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha}\left[\chi(F) + \|Du\|^2_{L^2(B_1)}\right]\right\}.$$

By Lemma 3.4 again, this implies

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\left\{\left(\frac{\rho}{r}\right)^{n+2\alpha}\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + \rho^{n+2\alpha}\left[\chi(F) + \|Du\|^2_{L^2(B_1)}\right]\right\}.$$

By choosing $r = r_1$, we have for any $x_0 \in B_{\frac{1}{2}}$ and $r \le r_1$

$$\int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 \le c r^{n+2\alpha} \left\{\chi(F) + \|Du\|^2_{L^2(B_1)}\right\}.$$

Case 3. The general case. By (1) and (2), we have for any $B_r(x_0) \subset B_1$ and $\rho < r$

$$\begin{aligned}\int_{B_\rho(x_0)} |Du|^2 \le C\Big\{ & \Big[\Big(\frac{\rho}{r}\Big)^n + r^{2\alpha}\Big] \int_{B_r(x_0)} |Du|^2 \\ & + \int_{B_r(x_0)} u^2 + r^{n+2\alpha}\chi(F)\Big\},\end{aligned} \tag{7}$$

and

$$\begin{aligned}\int_{B_\rho(x_0)} |Du-(Du)_{x_0,\rho}|^2 \le C & \left\{\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 \right. \\ & \left. + r^{2\alpha}\left[\int_{B_r(x_0)} u^2 + \int_{B_r(x_0)} |Du|^2\right] + r^{n+2\alpha}\chi(F)\right\},\end{aligned} \tag{8}$$

where $\chi(F) = \|f\|^2_{L^q(B_1)}$. In (7), we replace $r^{n+2\alpha}$ by r^n. As in the proof of Theorem 3.8, we show that for any small $\delta > 0$ there exists an $R_1 \in (3/4, 1)$ such that for any $x \in B_{R_1}$ and $r < 1 - R_1$

$$\int_{B_r(x_0)} |Du|^2 \le C r^{n-2\delta} \left\{\chi(F) + \|u\|^2_{H^1(B)}\right\}. \tag{9}$$

By Lemma 3.3, we also get

$$\int_{B_r(x_0)} u^2 \le C r^{n-2\delta} \left\{\chi(F) + \|u\|^2_{H^1(B)}\right\}. \tag{10}$$

Write

$$\chi(F, u) = \|f\|^2_{L^q} + \|u\|^2_{H^1}.$$

Then, (8), (9) and (10) imply

$$\begin{aligned}&\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \\ \le & C\left\{\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha-2\delta}\chi(F,u)\right\}.\end{aligned}$$

Hence, Lemma 3.4 and Theorem 3.1 imply that $Du \in C^{\alpha-\delta}_{\text{loc}}$ for small $\delta < \alpha$. In particular, we have $u \in C^1_{\text{loc}}$ with

$$\sup_{B_{\frac{3}{4}}} |u|^2 + \sup_{B_{\frac{3}{4}}} |Du|^2 \le C\chi(F,u). \tag{11}$$

Now (8) and (11) imply

$$\begin{aligned}&\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \\ \le & C\left\{\Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha}\chi(F,u)\right\}.\end{aligned}$$

This finishes the proof. □

It is natural to ask whether $f \in L^\infty(B_1)$, with appropriate assumptions on a_{ij} and c, implies $Du \in C^1$. Consider

$$\int_{B_1} D_i u D_i \varphi = \int_{B_1} f\varphi \quad \text{for any} \quad \varphi \in H_0^1(B_1).$$

There exists an example showing that $f \in C$ and $u \in C^{1,\alpha}_{loc}$ for any $\alpha \in (0,1)$ while $D^2 u \notin C$.

EXAMPLE 3.12. In $B_R \subset \mathbb{R}^n$, with $R < 1$, consider

$$\triangle u = \frac{x_2^2 - x_1^2}{2|x|^2}\left\{\frac{n+2}{(-\log|x|)^{1/2}} + \frac{1}{2(-\log|x|)^{3/2}}\right\},$$

where the right hand side is continuous in $\bar{B}_R$ if we set it equal to zero at the origin. The function $u(x) = (x_1^2 - x_2^2)(-\log|x|)^{1/2} \in C(\bar{B}_R) \cap C^\infty(\bar{B}_R \setminus \{0\})$ satisfies the above equation in $B_R \setminus \{0\}$ and the boundary condition $u = \sqrt{-\log R}(x_1^2 - x_2^2)$ on ∂B_R. Note that u is not a classical solution of the problem since $\lim_{|x|\to 0} D_{11}u = \infty$, and therefore u is not in $C^2(B_R)$. In fact, the problem has no classical solutions (although it has a weak solution). Assume on the contrary that a classical solution v exists. Then the function $w = u - v$ is harmonic and bounded in $B_R \setminus \{0\}$. By Theorem 1.28 in Chapter 1, on removable singularities for harmonic functions, w may be redefined at the origin so that $\triangle w = 0$ in B_R and therefore belongs to $C^2(B_R)$. In particular, the (finite) limit $\lim_{|x|\to 0} D_{11}u$ exists, which is a contradiction.

CHAPTER 4

Weak Solutions, Part II

In this chapter, we continue to discuss the regularity of weak solutions of elliptic equations of the divergence form. In the first three sections, we discuss the DeGiorgi-Nash-Moser theory for linear elliptic equations of the divergence form. In the first section, we prove the local boundedness of solutions. In the second section, we prove the Hölder continuity. Then in Section 3, we discuss the Harnack inequality. For all results in these three sections, there is no regularity assumption on coefficients.

4.1. Local Boundedness

The main theorem in this section is the following boundedness result.

THEOREM 4.1. *Suppose $a_{ij} \in L^\infty(B_1)$ and $c \in L^q(B_1)$, for some $q > n/2$, satisfy the following assumptions*

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in B_1, \xi \in \mathbb{R}^n,$$

and

$$|a_{ij}|_{L^\infty} + \|c\|_{L^q} \leq \Lambda,$$

for some positive constants λ and Λ. Suppose that $u \in H^1(B_1)$ is a subsolution in the following sense

$$(*) \qquad \int_{B_1} a_{ij}D_iuD_j\varphi + cu\varphi \leq \int_{B_1} f\varphi \quad \text{for any } \varphi \in H_0^1(B_1) \text{ with } \varphi \geq 0 \text{ in } B_1.$$

If $f \in L^q(B_1)$, then $u^+ \in L^\infty_{loc}(B_1)$. Moreover, for any $\theta \in (0,1)$ and any $p > 0$

$$\sup_{B_\theta} u^+ \leq C\left\{\frac{1}{(1-\theta)^{n/p}}\|u^+\|_{L^p(B_1)} + \|f\|_{L^q(B_1)}\right\},$$

where C is a positive constant depending only on n, λ, Λ, p and q.

In the following, we use two approaches to prove this theorem, one by DeGiorgi and the other by Moser.

PROOF. We first prove for $\theta = 1/2$ and $p = 2$.

Method 1. We first present an iteration method due to DeGiorgi.

Consider $v = (u-k)^+$ for $k \geq 0$ and $\zeta \in C_0^1(B_1)$. Set $\varphi = v\zeta^2$ as the test function. Note $v = u - k, Dv = Du$ a.e. in $\{u > k\}$ and $v = 0, Dv = 0$ a.e. in $\{u \leq k\}$. Hence, if we substitute such a φ in $(*)$, we integrate in the set $\{u > k\}$.

By the Cauchy inequality, we have

$$\begin{aligned}\int a_{ij}D_iuD_j\varphi &= \int a_{ij}D_iuD_jv\zeta^2 + 2a_{ij}D_iuD_j\zeta v\zeta\\ &\geq \lambda\int|Dv|^2\zeta^2 - 2\Lambda\int|Dv||D\zeta|v\zeta\\ &\geq \frac{\lambda}{2}\int|Dv|^2\zeta^2 - \frac{2\Lambda^2}{\lambda}\int|D\zeta|^2v^2.\end{aligned}$$

Then, we obtain

$$\int|Dv|^2\zeta^2 \leq C\Big\{\int v^2|D\zeta|^2 + \int|c|v^2\zeta^2 + k^2\int|c|\zeta^2 + \int|f|v\zeta^2\Big\},$$

and hence

$$\int|D(v\zeta)|^2 \leq C\Big\{\int v^2|D\zeta|^2 + \int|c|v^2\zeta^2 + k^2\int|c|\zeta^2 + \int|f|v\zeta^2\Big\}.$$

Recall the Sobolev inequality for $v\zeta \in H_0^1(B_1)$

$$\Big(\int_{B_1}(v\zeta)^{2^*}\Big)^{\frac{2}{2^*}} \leq c(n)\int_{B_1}|D(v\zeta)|^2,$$

where $2^* = 2n/(n-2)$ for $n > 2$ and $2^* > 2$ is arbitrary if $n = 2$. The Hölder inequality implies that with $\delta > 0$ small and $\zeta \leq 1$

$$\begin{aligned}\int|f|v\zeta^2 &\leq \Big(\int|f|^q\Big)^{\frac{1}{q}}\Big(\int|v\zeta|^{2^*}\Big)^{\frac{1}{2^*}}|\{v\zeta\neq 0\}|^{1-\frac{1}{2^*}-\frac{1}{q}}\\ &\leq c(n)\|f\|_{L^q}\Big(\int|D(v\zeta)|^2\Big)^{\frac{1}{2}}|\{v\zeta\neq 0\}|^{\frac{1}{2}+\frac{1}{n}-\frac{1}{q}}\\ &\leq \delta\int|D(v\zeta)|^2 + c(n,\delta)\|f\|_{L^q}^2|\{v\zeta\neq 0\}|^{1+\frac{2}{n}-\frac{2}{q}}.\end{aligned}$$

Note for $q > n/2$

$$1 + \frac{2}{n} - \frac{2}{q} > 1 - \frac{1}{q}.$$

Therefore, we have

$$\int|D(v\zeta)|^2 \leq C\Big\{\int v^2|D\zeta|^2 + \int|c|v^2\zeta^2 + k^2\int|c|\zeta^2 + F^2|\{v\zeta\neq 0\}|^{1-\frac{1}{q}}\Big\},$$

where $F = \|f\|_{L^q(B_1)}$.

We claim

$$\int|D(v\zeta)|^2 \leq C\Big\{\int v^2|D\zeta|^2 + (k^2+F^2)|\{v\zeta\neq 0\}|^{1-\frac{1}{q}}\Big\}, \tag{1}$$

if $|\{v\zeta\neq 0\}|$ is small.

It is obvious if $c \equiv 0$. In fact, in this special case there is no restriction on the set $\{v\zeta \neq 0\}$. In general, the Hölder inequality implies

$$\begin{aligned}\int |c|v^2\zeta^2 &\le \left(\int |c|^q\right)^{\frac{1}{q}} \left(\int (v\zeta)^{2^*}\right)^{\frac{2}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}-\frac{1}{q}} \\ &\le c(n)\int |D(v\zeta)|^2 \left(\int |c|^q\right)^{\frac{1}{q}} |\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{q}},\end{aligned}$$

and

$$\int |c|\zeta^2 \le \left(\int |c|^q\right)^{\frac{1}{q}} |\{v\zeta \neq 0\}|^{1-\frac{1}{q}}.$$

Therefore, we have

$$\begin{aligned}\int |D(v\zeta)|^2 \le C\bigg\{ \int v^2|D\zeta|^2 &+ \int |D(v\zeta)|^2 |\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{q}} \\ &+ (k^2+F^2)|\{v\zeta \neq 0\}|^{1-\frac{1}{q}}\bigg\}.\end{aligned}$$

This implies (1) if $|\{v\zeta \neq 0\}|$ is small.

To continue, we note by the Sobolev inequality

$$\begin{aligned}\int (v\zeta)^2 &\le \left(\int (v\zeta)^{2^*}\right)^{\frac{2}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}} \\ &\le c(n)\int |D(v\zeta)|^2 |\{v\zeta \neq 0\}|^{\frac{2}{n}}.\end{aligned}$$

Therefore, we have

$$\int (v\zeta)^2 \le C\bigg\{ \int v^2|D\zeta|^2|\{v\zeta \neq 0\}|^{\frac{2}{n}} + (k+F)^2|\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{1}{q}}\bigg\},$$

if $|\{v\zeta \neq 0\}|$ is small. Hence there exists an $\varepsilon > 0$ such that

$$\int (v\zeta)^2 \le C\bigg\{ \int v^2|D\zeta|^2|\{v\zeta \neq 0\}|^{\varepsilon} + (k+F)^2|\{v\zeta \neq 0\}|^{1+\varepsilon}\bigg\},$$

if $|\{v\zeta \neq 0\}|$ is small. For any fixed $0 < r < R \le 1$, choose $\zeta \in C_0^\infty(B_R)$ such that $\zeta \equiv 1$ in B_r, and $0 \le \zeta \le 1$ and $|D\zeta| \le 2(R-r)^{-1}$ in B_1. Set

$$A(k,r) = \{x \in B_r; u \ge k\}.$$

We conclude that for any $0 < r < R \le 1$ and $k > 0$

$$\begin{aligned}\int_{A(k,r)} (u-k)^2 \le C\bigg\{ &\frac{1}{(R-r)^2}|A(k,R)|^{\varepsilon}\int_{A(k,R)} (u-k)^2 \\ &+ (k+F)^2|A(k,R)|^{1+\varepsilon}\bigg\},\end{aligned} \tag{2}$$

if $|A(k,R)|$ is small. Note

$$|A(k,R)| \le \frac{1}{k}\int_{A(k,R)} u^+ \le \frac{1}{k}\|u^+\|_{L^2}.$$

Hence (2) holds if $k \ge k_0 = C\|u^+\|_{L^2}$ for some large C depending only on λ and Λ.

Next, we claim for $k = C(k_0 + F)$

$$\int_{A(k,1/2)} (u-k)^2 = 0.$$

To this end, we take any $h > k \geq k_0$ and any $0 < r < 1$. It is obvious that $A(k,r) \supset A(h,r)$. Hence we have

$$\int_{A(h,r)} (u-h)^2 \leq \int_{A(k,r)} (u-k)^2,$$

and

$$|A(h,r)| = |B_r \cap \{u - k > h - k\}| \leq \frac{1}{(h-k)^2} \int_{A(k,r)} (u-k)^2.$$

Therefore, by (2) we have for any $h > k \geq k_0$ and $1/2 \leq r < R \leq 1$

$$\begin{aligned}
&\int_{A(h,r)} (u-h)^2 \\
\leq& C\left\{ \frac{1}{(R-r)^2} \int_{A(h,R)} (u-h)^2 + (h+F)^2 |A(h,R)| \right\} |A(h,R)|^{\varepsilon} \\
\leq& C\left\{ \frac{1}{(R-r)^2} + \frac{(h+F)^2}{(h-k)^2} \right\} \frac{1}{(h-k)^{2\varepsilon}} \left(\int_{A(k,R)} (u-k)^2 \right)^{1+\varepsilon},
\end{aligned}$$

or

$$\|(u-h)^+\|_{L^2(B_r)} \leq C\left\{ \frac{1}{R-r} + \frac{h+F}{h-k} \right\} \frac{1}{(h-k)^{\varepsilon}} \|(u-k)^+\|^{1+\varepsilon}_{L^2(B_R)}. \tag{3}$$

Now we carry out the iteration. Set

$$\varphi(k,r) = \|(u-k)^+\|_{L^2(B_r)}.$$

For $\tau = 1/2$ and some $k > 0$ to be determined, define for $\ell = 0, 1, 2, \cdots$,

$$\begin{aligned}
k_\ell &= k_0 + k\left(1 - \frac{1}{2^\ell}\right) \quad (\leq k_0 + k), \\
r_\ell &= \tau + \frac{1}{2^\ell}(1-\tau).
\end{aligned}$$

Obviously, we have

$$k_\ell - k_{\ell-1} = \frac{k}{2^\ell}, \qquad r_{\ell-1} - r_\ell = \frac{1}{2^\ell}(1-\tau).$$

Therefore, we have for $\ell = 0, 1, 2, \cdots$

$$\begin{aligned}
\varphi(k_\ell, r_\ell) &\leq C\left\{ \frac{2^\ell}{1-\tau} + \frac{2^\ell(k_0 + F + k)}{k} \right\} \frac{2^{\varepsilon\ell}}{k^\varepsilon} [\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon} \\
&\leq \frac{C}{1-\tau} \cdot \frac{k_0 + F + k}{k^{1+\varepsilon}} \cdot 2^{(1+\varepsilon)\ell} \cdot [\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon}.
\end{aligned}$$

Next, we prove inductively for any $\ell = 0, 1, \cdots$

$$\varphi(k_\ell, r_\ell) \leq \frac{\varphi(k_0, r_0)}{\gamma^\ell} \quad \text{for some } \gamma > 1, \tag{4}$$

if k is sufficiently large. Obviously, it is true for $\ell = 0$. Suppose it is true for $\ell - 1$. We write

$$[\varphi(k_{\ell-1}, r_{\ell-1})]^{1+\varepsilon} \leq \left\{ \frac{\varphi(k_0, r_0)}{\gamma^{\ell-1}} \right\}^{1+\varepsilon} = \frac{\varphi(k_0, r_0)^\varepsilon}{\gamma^{\ell\varepsilon - (1+\varepsilon)}} \cdot \frac{\varphi(k_0, r_0)}{\gamma^\ell}.$$

Then, we obtain

$$\varphi(k_\ell, r_\ell) \leq \frac{C\gamma^{1+\varepsilon}}{1-\tau} \cdot \frac{k_0 + F + k}{k^{1+\varepsilon}} \cdot [\varphi(k_0, r_0)]^\varepsilon \cdot \frac{2^{\ell(1+\varepsilon)}}{\gamma^{\ell\varepsilon}} \cdot \frac{\varphi(k_0, r_0)}{\gamma^\ell}.$$

Choose γ first such that $\gamma^\varepsilon = 2^{1+\varepsilon}$ and note $\gamma > 1$. Next, we need

$$\frac{C\gamma^{1+\varepsilon}}{1-\tau} \cdot \left(\frac{\varphi(k_0, r_0)}{k}\right)^\varepsilon \cdot \frac{k_0 + F + k}{k} \leq 1.$$

Therefore, we choose

$$k = C_*\{k_0 + F + \varphi(k_0, r_0)\},$$

for C_* large. Let $\ell \to +\infty$ in (4). We conclude

$$\varphi(k_0 + k, \tau) = 0.$$

Hence we have

$$\sup_{B_{\frac{1}{2}}} u^+ \leq (C_* + 1)\{k_0 + F + \varphi(k_0, r_0)\}.$$

Recall $k_0 = C\|u^+\|_{L^2(B_1)}$ and $\varphi(k_0, r_0) \leq \|u^+\|_{L^2(B_1)}$. This finishes the proof by the approach due to DeGiorgi.

Method 2. We will present an iteration argument due to Moser. First we explain the idea. By choosing a test function appropriately, we estimate L^{p_1} norm of u in a small ball by L^{p_2} norm of u in a large ball for $p_1 > p_2$, i.e.,

$$\|u\|_{L^{p_1}(B_{r_1})} \leq C\|u\|_{L^{p_2}(B_{r_2})},$$

for $p_1 > p_2$ and $r_1 < r_2$. This is a reversed Hölder inequality. As a sacrifice, C behaves like $\dfrac{1}{r_2 - r_1}$. We then obtain the desired result by an iteration and a careful choice of $\{r_i\}$ and $\{p_i\}$.

For some $k > 0$ and $m > 0$, set $\bar{u} = u^+ + k$ and

$$\bar{u}_m = \begin{cases} \bar{u} & \text{if } u < m \\ k + m & \text{if } u \geq m. \end{cases}$$

Then we have $D\bar{u}_m = 0$ in $\{u < 0\}$ and $\{u > m\}$ and $\bar{u}_m \leq \bar{u}$. Consider the test function

$$\varphi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in H_0^1(B_1),$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. A direct calculation yields

$$\begin{aligned} D\varphi &= \beta\eta^2\bar{u}_m^{\beta-1}D\bar{u}_m\bar{u} + \eta^2\bar{u}_m^\beta D\bar{u} + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}) \\ &= \eta^2\bar{u}_m^\beta(\beta D\bar{u}_m + D\bar{u}) + 2\eta D\eta(\bar{u}_m^\beta\bar{u} - k^{\beta+1}). \end{aligned}$$

We should emphasize that later on we will begin the iteration with $\beta = 0$. Note $\varphi = 0$ and $D\varphi = 0$ in $\{u \leq 0\}$. Hence, if we substitute such a φ in $(*)$, we integrate in the set $\{u>0\}$. Note also that $u^+\le \bar u$ and $\bar u_m^\beta \bar u - k^{\beta+1}\le \bar u_m^\beta \bar u$ for $k>0$. First, we have by the Hölder inequality

$$\begin{aligned}
&\int a_{ij}D_iuD_j\varphi\\
=&\int a_{ij}D_i\bar u(\beta D_j\bar u_m + D_j\bar u)\eta^2\bar u_m^\beta + 2\int a_{ij}D_i\bar u D_j\eta(\bar u_m^\beta\bar u - k^{\beta+1})\eta\\
\ge&\lambda\beta\int \eta^2\bar u_m^\beta|D\bar u_m|^2 + \lambda\int\eta^2\bar u_m^\beta|D\bar u|^2 - \Lambda\int|D\bar u||D\eta|\bar u_m^\beta\bar u\eta\\
\ge&\lambda\beta\int \eta^2\bar u_m^\beta|D\bar u_m|^2 + \frac{\lambda}{2}\int\eta^2\bar u_m^\beta|D\bar u|^2 - \frac{2\Lambda^2}{\lambda}\int|D\eta|^2\bar u_m^\beta\bar u^2.
\end{aligned}$$

Hence, we obtain by noting $\bar u\ge k$

$$\begin{aligned}
&\beta\int\eta^2\bar u_m^\beta|D\bar u_m|^2 + \int\eta^2\bar u_m^\beta|D\bar u|^2\\
\le& C\Big\{\int|D\eta|^2\bar u_m^\beta\bar u^2 + \int\big(|c|\eta^2\bar u_m^\beta\bar u^2 + |f|\eta^2\bar u_m^\beta\bar u\Big\}\\
\le& C\Big\{\int|D\eta|^2\bar u_m^\beta\bar u^2 + \int c_0\eta^2\bar u_m^\beta\bar u^2\Big\},
\end{aligned}$$

where c_0 is defined by

$$c_0 = |c| + \frac{|f|}{k}.$$

Choose $k=\|f\|_{L^q}$ if f is not identically zero. Otherwise choose an arbitrary $k>0$ and eventually let $k\to 0+$. By the assumption, we have

$$\|c_0\|_{L^q}\le \Lambda+1.$$

Set $w=\bar u_m^{\frac{\beta}{2}}\bar u$. Note

$$|Dw|^2\le(1+\beta)\{\beta\bar u_m^\beta|D\bar u_m|^2 + \bar u_m^\beta|D\bar u|^2\}.$$

Therefore, we have

$$\int|Dw|^2\eta^2\le C\Big\{(1+\beta)\int w^2|D\eta|^2 + (1+\beta)\int c_0w^2\eta^2\Big\},$$

or

$$\int|D(w\eta)|^2\le C\Big\{(1+\beta)\int w^2|D\eta|^2 + (1+\beta)\int c_0w^2\eta^2\Big\}.$$

The Hölder inequality implies

$$\int c_0w^2\eta^2\le\left(\int c_0^q\right)^{\frac1q}\left(\int(\eta w)^{\frac{2q}{q-1}}\right)^{1-\frac1q}\le(\Lambda+1)\left(\int(\eta w)^{\frac{2q}{q-1}}\right)^{1-\frac1q}.$$

By the interpolation inequality and the Sobolev inequality with

$$2^*=\frac{2n}{n-2}>\frac{2q}{q-1}>2 \text{ if } q>n/2,$$

we have

$$\begin{aligned}
\|\eta w\|_{L^{\frac{2q}{q-1}}}&\le\varepsilon\|\eta w\|_{L^{2^*}} + C(n,q)\varepsilon^{-\frac{n}{2q-n}}\|\eta w\|_{L^2}\\
&\le\varepsilon\|D(\eta w)\|_{L^2} + C(n,q)\varepsilon^{-\frac{n}{2q-n}}\|\eta w\|_{L^2},
\end{aligned}$$

for any small $\varepsilon > 0$. Therefore, we obtain

$$\int |D(w\eta)|^2 \leq C\Big\{(1+\beta)\int w^2|D\eta|^2 + (1+\beta)^{\frac{2q}{2q-n}}\int w^2\eta^2\Big\},$$

and in particular

$$\int |D(w\eta)|^2 \leq C(1+\beta)^\alpha \int (|D\eta|^2+\eta^2)w^2,$$

where α in a positive number depending only on n and q. The Sobolev inequality then implies

$$\left(\int |\eta w|^{2\chi}\right)^{\frac{1}{\chi}} \leq C(1+\beta)^\alpha \int (|D\eta|^2+\eta^2)w^2,$$

where $\chi = \dfrac{n}{n-2} > 1$ for $n > 2$ and $\chi > 2$ for $n = 2$. For any $0 < r < R \leq 1$, consider an $\eta \in C_0^1(B_R)$ with the property

$$\eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\eta| \leq \frac{2}{R-r}.$$

Then, we obtain

$$\left(\int_{B_r} w^{2\chi}\right)^{\frac{1}{\chi}} \leq C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} w^2.$$

Recalling the definition of w, we have

$$\left(\int_{B_r} \bar{u}^{2\chi}\bar{u}_m^{\beta\chi}\right)^{\frac{1}{\chi}} \leq C\frac{(1+\beta)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}^2\bar{u}_m^\beta.$$

Set $\gamma = \beta + 2 \geq 2$. Then we obtain

$$\left(\int_{B_r} \bar{u}_m^{\gamma\chi}\right)^{\frac{1}{\chi}} \leq C\frac{(\gamma-1)^\alpha}{(R-r)^2}\int_{B_R} \bar{u}^\gamma,$$

provided the integral in the right hand side is bounded. By letting $m \to +\infty$ we conclude that

$$\|\bar{u}\|_{L^{\gamma\chi}(B_r)} \leq \left(C\frac{(\gamma-1)^\alpha}{(R-r)^2}\right)^{\frac{1}{\gamma}}\|\bar{u}\|_{L^\gamma(B_R)},$$

provided $\|\bar{u}\|_{L^\gamma(B_R)} < +\infty$, where C is a positive constant depending only on n, q, λ, Λ, and independent of γ. The above estimate suggests that we iterate, beginning with $\gamma = 2$, as 2, 2χ, $2\chi^2$, $\cdots$. Now we set for $i = 0, 1, 2, \cdots$,

$$\gamma_i = 2\chi^i \quad \text{and} \quad r_i = \frac{1}{2} + \frac{1}{2^{i+1}}.$$

By $\gamma_i = \chi\gamma_{i-1}$ and $r_{i-1} - r_i = 1/2^{i+1}$, we have for $i = 1, 2, \cdots$,

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \leq C(n,q,\lambda,\Lambda)^{\frac{i}{\chi^i}}\|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})},$$

provided $\|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})} < +\infty$. Hence, by an iteration we obtain

$$\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} \leq C^{\sum \frac{i}{\chi^i}}\|\bar{u}\|_{L^2(B_1)},$$

and in particular

$$\left(\int_{B_{\frac{1}{2}}} \bar{u}^{2\chi^i}\right)^{\frac{1}{2\chi^i}} \leq C\left(\int_{B_1} \bar{u}^2\right)^{\frac{1}{2}}.$$

Letting $i \to +\infty$, we get

$$\sup_{B_{\frac{1}{2}}} \bar{u} \le C\|\bar{u}\|_{L^2(B_1)}$$

or

$$\sup_{B_{\frac{1}{2}}} u^+ \le C\{\|u^+\|_{L^2(B_1)} + k\}.$$

Recall the definition of k. This finishes the proof for $p = 2$ by the approach due to Moser.

REMARK 4.2. If the subsolution u is bounded, we simply take the test function

$$\varphi = \eta^2(\bar{u}^{\beta+1} - k^{\beta+1}) \in H_0^1(B_1),$$

for some $\beta \ge 0$ and some nonnegative function $\eta \in C_0^1(B_1)$.

Next, we discuss the general case of Theorem 4.1. This is based on a dilation argument. Take any $R \le 1$. Define

$$\tilde{u}(y) = u(Ry) \quad \text{for } y \in B_1.$$

It is easy to see that $\tilde{u}$ satisfies

$$\int_{B_1} \tilde{a}_{ij} D_i \tilde{u} D_j \varphi + \tilde{c}\tilde{u}\varphi \le \int_{B_1} \tilde{f}\varphi$$
$$\text{for any } \varphi \in H_0^1(B_1) \text{ and } \phi \ge 0 \text{ in } B_1,$$

where

$$\tilde{a}(y) = a(Ry), \quad \tilde{c}(y) = R^2 c(Ry), \quad \tilde{f}(y) = R^2 f(Ry),$$

for any $y \in B_1$. A direct calculation shows

$$|\tilde{a}_{ij}|_{L^\infty(B_1)} + \|\tilde{c}\|_{L^q(B_1)} = |a_{ij}|_{L^\infty(B_R)} + R^{2-\frac{n}{q}}\|c\|_{L^q(B_R)} \le \Lambda.$$

We may apply what we just proved to $\tilde{u}$ in B_1 and rewrite the result in terms of u. Hence, we obtain for $p \ge 2$

$$\sup_{B_{\frac{R}{2}}} u^+ \le C\{\frac{1}{R^{n/p}}\|u^+\|_{L^p(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\},$$

where C is a positive constant depending only on n, λ, Λ, p and q.

The estimate in $B_{\theta R}$ can be obtained by applying the above result to $B_{(1-\theta)R}(y)$ for any $y \in B_{\theta R}$. By taking $R = 1$, we have Theorem 4.1 for any $\theta \in (0, 1)$ and $p \ge 2$.

Now we prove the statement for $p \in (0, 2)$. We showed that there holds for any $\theta \in (0, 1)$ and $0 < R \le 1$

$$\begin{aligned}\|u^+\|_{L^\infty(B_{\theta R})} &\le C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{2}}}\|u^+\|_{L^2(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\right\}\\ &\le C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{2}}}\|u^+\|_{L^2(B_R)} + \|f\|_{L^q(B_1)}\right\}.\end{aligned}$$

For $p \in (0, 2)$, we have

$$\int_{B_R} (u^+)^2 \le \|u^+\|_{L^\infty(B_R)}^{2-p} \int_{B_R} (u^+)^p,$$

and hence by the Hölder inequality

$$
\begin{aligned}
&\|u^+\|_{L^\infty(B_{\theta R})}\\
\leq &C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{2}}}\|u^+\|^{1-\frac{p}{2}}_{L^\infty(B_R)}\left(\int_{B_R}(u^+)^p dx\right)^{\frac{1}{2}}+\|f\|_{L^q(B_R)}\right\}\\
\leq &\frac{1}{2}\|u^+\|_{L^\infty(B_R)}+C\left\{\frac{1}{[(1-\theta)R]^{\frac{n}{p}}}\left(\int_{B_R}(u^+)^p\right)^{\frac{1}{p}}+\|f\|_{L^q(B_R)}\right\}.
\end{aligned}
$$

Set $f(t)=\|u^+\|_{L^\infty(B_t)}$ for $t\in(0,1]$. Then, we obtain for any $0<r<R\leq 1$

$$f(r)\leq \frac{1}{2}f(R)+\frac{C}{(R-r)^{\frac{n}{p}}}\|u^+\|_{L^p(B_1)}+C\|f\|_{L^q(B_1)}.$$

We apply the following lemma to get for any $0<r<R<1$

$$f(r)\leq \frac{C}{(R-r)^{\frac{n}{p}}}\|u^+\|_{L^p(B_1)}+C\|f\|_{L^q(B_1)}.$$

By letting $R\to 1-$, we get for any $\theta<1$

$$\|u^+\|_{L^\infty(B_\theta)}\leq \frac{C}{(1-\theta)^{\frac{n}{p}}}\|u^+\|_{L^p(B_1)}+C\|f\|_{L^q(B_1)}.$$

This finishes the proof. □

We need the following simple lemma.

LEMMA 4.3. *Let $f(t)\geq 0$ be bounded in $[\tau_0,\tau_1]$ with $\tau_0\geq 0$. Suppose for any $\tau_0\leq t<s\leq\tau_1$*

$$f(t)\leq \theta f(s)+\frac{A}{(s-t)^\alpha}+B,$$

for some $\theta\in[0,1)$. Then there holds for any $\tau_0\leq t<s\leq\tau_1$

$$f(t)\leq c\left\{\frac{A}{(s-t)^\alpha}+B\right\},$$

where c is a positive constant depending only on α and θ.

PROOF. Fix $\tau_0\leq t<s\leq\tau_1$. For some $0<\tau<1$, we consider the sequence $\{t_i\}$ defined by

$$t_0=t \text{ and } t_{i+1}=t_i+(1-\tau)\tau^i(s-t).$$

Note $t_\infty=s$. By an iteration, we get

$$f(t)=f(t_0)\leq \theta^k f(t_k)+\left[\frac{A}{(1-\tau)^\alpha}(s-t)^{-\alpha}+B\right]\sum_{i=0}^{k-1}\theta^i\tau^{-i\alpha}.$$

Choose $\tau<1$ such that $\theta\tau^{-\alpha}<1$, i.e., $\theta<\tau^\alpha<1$. As $k\to\infty$, we have

$$f(t)\leq c(\alpha,\theta)\left\{\frac{A}{(1-\tau)^\alpha}(s-t)^{-\alpha}+B\right\}.$$

This finishes the proof. □

In the rest of this section, we use the Moser's iteration to prove a high integrability result, which is closely related to Theorem 4.1. For the next result, we require $n\geq 3$.

THEOREM 4.4. *Suppose $a_{ij} \in L^\infty(B_1)$ and $c \in L^{n/2}(B_1)$ satisfy the following assumption*

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any } x \in B_1, \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ. Suppose that $u \in H^1(B_1)$ is a subsolution in the following sense

$$\int_{B_1} a_{ij} D_i u D_j \varphi + cu\varphi \leq \int_{B_1} f\varphi$$
$$\text{for any } \varphi \in H_0^1(B_1) \text{ and } \varphi \geq 0 \text{ in } B_1.$$

If $f \in L^q(B_1)$ for some $q \in [\frac{2n}{n+2}, \frac{n}{2})$, then $u^+ \in L^{q^*}_{loc}(B_1)$ for $\frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}$. Moreover,

$$\|u^+\|_{L^{q^*}(B_{\frac{1}{2}})} \leq C\Big\{\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\Big\},$$

where C is a positive constant depending only on n, λ, Λ, q and $\varepsilon(K)$, with

$$\varepsilon(K) = \Big(\int_{\{|c|>K\}} |c|^{\frac{n}{2}}\Big)^{\frac{2}{n}}.$$

PROOF. For $m > 0$, set $\bar u = u^+$ and

$$\bar u_m = \begin{cases} \bar u & \text{if } u < m \\ m & \text{if } u \geq m. \end{cases}$$

Then consider a test function

$$\varphi = \eta^2 \bar u_m^\beta \bar u \in H_0^1(B_1),$$

for some $\beta \geq 0$ and some nonnegative function $\eta \in C_0^1(B_1)$. By similar calculations as in the proof of Theorem 4.1, we obtain

$$\Big(\int \eta^{2\chi}\bar u_m^{\beta\chi}\bar u^{2\chi}\Big)^{\frac{1}{\chi}} \leq C(1+\beta)\Big\{\int |D\eta|^2 \bar u_m^\beta \bar u^2 + \int |c|\eta^2 \bar u_m^\beta \bar u^2 + \int |f|\eta^2 \bar u_m^\beta \bar u\Big\},$$

where $\chi = \dfrac{n}{n-2} > 1$. The Hölder inequality implies for any $K > 0$

$$\begin{aligned}\int |c|\eta^2\bar u_m^\beta \bar u^2 &\leq K\int_{\{|c|\leq K\}} \eta^2 \bar u_m^\beta \bar u^2 + \int_{\{|c|>K\}} |c|\eta^2\bar u_m^\beta \bar u^2 \\ &\leq K\int \eta^2\bar u_m^\beta \bar u^2 + \Big(\int_{\{|c|>K\}} |c|^{\frac{n}{2}}\Big)^{\frac{2}{n}}\Big(\int (\eta^2 \bar u_m^\beta \bar u^2)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} \\ &\leq K\int \eta^2 \bar u_m^\beta \bar u^2 + \varepsilon(K)\Big(\int \eta^{2\chi}\bar u_m^{\beta\chi}\bar u^{2\chi}\Big)^{\frac{1}{\chi}}.\end{aligned}$$

Note $\varepsilon(K) \to 0$ as $K \to +\infty$, since $c \in L^{n/2}(B_1)$. Hence, for bounded β we obtain by choosing large $K = K(\beta)$

$$\Big(\int \eta^{2\chi}\bar u_m^{\beta\chi}\bar u^{2\chi}\Big)^{\frac{1}{\chi}} \leq C(1+\beta)\Big\{\int (|D\eta|^2 + \eta^2)\bar u_m^\beta \bar u^2 + \int |f|\eta^2 \bar u_m^\beta \bar u\Big\}.$$

Note

$$\bar u_m^\beta \bar u \leq \bar u_m^{\beta - \frac{\beta}{\beta+2}} \bar u^{1+\frac{\beta}{\beta+2}} = (\bar u_m^\beta \bar u^2)^{\frac{\beta+1}{\beta+2}}.$$

Therefore, by the Hölder inequality again we have for $\eta \le 1$

$$\int |f|\eta^2 \bar{u}_m^\beta \bar{u} \le \Big(\int |f|^q\Big)^{\frac{1}{q}} \Big(\int (\eta^2 \bar{u}_m^\beta \bar{u}^2)^\chi\Big)^{\frac{\beta+1}{(\beta+2)\chi}} |\mathrm{supp}\ \eta|^{1-\frac{1}{q}-\frac{\beta+1}{(\beta+2)\chi}}$$

$$\le \varepsilon\Big(\int \eta^{2\chi}\bar{u}^\chi \bar{u}_m^{\beta\chi}\Big)^{\frac{1}{\chi}} + C(\varepsilon,\beta)\Big(\int |f|^q\Big)^{\frac{\beta+2}{q}},$$

provided

$$1 - \frac{1}{q} - \frac{\beta+1}{(\beta+2)\chi} \ge 0,$$

which is equivalent to

$$\beta + 2 \le \frac{q(n-2)}{n-2q}.$$

Hence, β is required to be bounded, depending only on n and q. Then we obtain

$$\Big(\int \eta^{2\chi}\bar{u}_m^{\beta\chi}\bar{u}^{2\chi}\Big)^{\frac{1}{\chi}} \le C\Big\{\int (|D\eta|^2 + \eta^2)\bar{u}_m^\beta \bar{u}^2 + \|f\|_{L^q}^{\beta+2}\Big\}.$$

By setting $\gamma = \beta + 2$, we have by the definition of q^*

$$2 \le \gamma \le \frac{q(n-2)}{n-2q} = \frac{q^*}{\chi}. \tag{1}$$

We conclude, as before, for any such γ in (1) and any $0 < r < R \le 1$

$$\|\bar{u}\|_{L^{\chi\gamma}(B_r)} \le C\Big\{\frac{1}{(R-r)^{\frac{2}{\gamma}}}\|\bar{u}\|_{L^\gamma(B_R)} + \|f\|_{L^q(B_1)}\Big\}, \tag{2}$$

provided $\|\bar{u}\|_{L^\gamma(B_R)} < +\infty$. Again this suggests an iteration $2, 2\chi, 2\chi^2, \cdots$.

For a given $q \in [\frac{2n}{n+2}, \frac{n}{2})$, there exists a positive integer k such that

$$2\chi^{k-1} \le \frac{q(n-2)}{n-2q} < 2\chi^k.$$

For such a k, we get by finitely many iterations of (2)

$$\|\bar{u}\|_{L^{2\chi^k}(B_{\frac{3}{4}})} \le C\Big\{\|\bar{u}\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\Big\},$$

and in particular

$$\|\bar{u}\|_{L^{\frac{q^*}{\chi}}(B_{\frac{3}{4}})} \le C\Big\{\|\bar{u}\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\Big\}.$$

With $\gamma = \dfrac{q^*}{\chi}$ in (2), we obtain

$$\|\bar{u}\|_{L^{q^*}(B_{\frac{1}{2}})} \le C\Big\{\|\bar{u}\|_{L^{\frac{q^*}{\chi}}(B_{\frac{3}{4}})} + \|f\|_{L^q(B_1)}\Big\}.$$

This finishes the proof. □

4.2. The Hölder Continuity

We first discuss homogeneous equations with no lower order terms. Consider

$$Lu \equiv -D_i(a_{ij}(x)D_j u) \quad \text{in } B_1 \subset \mathbb{R}^n,$$

where $a_{ij} \in L^\infty(B_1)$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any } x \in B_1 \text{ and } \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ.

Definition 4.5. The function $u \in H^1_{loc}(B_1)$ is called a subsolution (supersolution) of the equation $Lu = 0$, if

$$\int_{B_1} a_{ij} D_i u D_j \varphi \leq 0 \ (\geq 0),$$

for any $\varphi \in H^1_0(B_1)$ and $\varphi \geq 0$.

Lemma 4.6. *Let $\Phi \in C^{0,1}_{loc}(\mathbb{R})$ be convex.*

(i) If u is a subsolution and $\Phi' \geq 0$, then $v = \Phi(u)$ is a subsolution provided $v \in H^1_{loc}(B_1)$;

(ii) If u is a supersolution and $\Phi' \leq 0$, then $v = \Phi(u)$ is a subsolution provided $v \in H^1_{loc}(B_1)$.

Remark 4.7. If u is a subsolution, then $(u-k)^+$ is also a subsolution, where $(u-k)^+ = \max\{0, u-k\}$. In this case, $\Phi(s) = (s-k)^+$.

Proof. We only prove (i). Assume first $\Phi \in C^2_{loc}(\mathbb{R})$. Then we have $\Phi'(s) \geq 0$ and $\Phi''(s) \geq 0$. Consider a $\varphi \in C^1_0(B_1)$ with $\varphi \geq 0$. A direct calculation yields

$$\int_{B_1} a_{ij} D_i v D_j \varphi = \int_{B_1} a_{ij} \Phi'(u) D_i u D_j \varphi$$
$$= \int_{B_1} a_{ij} D_i u D_j(\Phi'(u)\varphi) - \int_{B_1} (a_{ij} D_i u D_j u)\varphi\Phi''(u) \leq 0,$$

where $\Phi'(u)\varphi \in H^1_0(B_1)$ is nonnegative. In general, set $\Phi_\epsilon(s) = \rho_\epsilon * \Phi(s)$ with ρ_ϵ as the standard mollifier. Then $\Phi'_\epsilon(s) = \rho_\epsilon * \Phi'(s) \geq 0$ and $\Phi''_\epsilon(s) \geq 0$. Hence $\Phi_\epsilon(u)$ is a subsolution by what we just proved. Note $\Phi'_\epsilon(s) \to \Phi'(s)$ a.e. as $\epsilon \to 0^+$. The Lebesgue dominant convergence theorem implies the desired result. $\square$

We also need the following Poincaré-Sobolev inequality.

Lemma 4.8. *Let ϵ be a positive constant. For any $u \in H^1(B_1)$, if*

$$|\{x \in B_1; u = 0\}| \geq \epsilon|B_1|,$$

then

$$\int_{B_1} u^2 \leq C \int_{B_1} |Du|^2,$$

where C is a positive constant depending only on ϵ and n.

Proof. We prove by contradiction. If not, there exists a sequence $\{u_m\} \subset H^1(B_1)$ such that

$$|\{x \in B_1; u_m = 0\}| \geq \epsilon|B_1|,$$

and

$$\int_{B_1} u_m^2 = 1, \quad \int_{B_1} |Du_m|^2 \to 0 \text{ as } m \to \infty.$$

We may assume $u_m \to u_0$ strongly in $L^2(B_1)$ and weakly in $H^1(B_1)$, with $u_0 \in H^1(B_1)$. Clearly u_0 is a nonzero constant. Then we have

$$\begin{aligned} 0 = \lim_{m\to\infty} \int_{B_1} |u_m - u_0|^2 &\geq \lim_{m\to\infty} \int_{\{u_m=0\}} |u_m - u_0|^2 \\ &\geq |u_0|^2 \inf_m |\{u_m = 0\}| > 0. \end{aligned}$$

This is a contradiction. □

Now, we begin to discuss the Hölder continuity. We first prove the following result, which is often referred to as the density theorem.

THEOREM 4.9. *Suppose $\epsilon \in (0,1)$ is a constant and u is a positive supersolution in B_2 with*

$$|\{x \in B_1; u \geq 1\}| \geq \epsilon |B_1|.$$

Then

$$\inf_{B_{\frac{1}{2}}} u \geq C,$$

where C is a positive constant depending only on ϵ, n and Λ/λ.

PROOF. We may assume $u \geq \delta > 0$ and then let $\delta \to 0+$ at the end.

By Lemma 4.6, $v = (\log u)^-$ is a subsolution, bounded by $\log \delta^{-1}$. Theorem 4.1 yields

$$\sup_{B_{\frac{1}{2}}} v \leq C \left(\int_{B_1} |v|^2 \right)^{\frac{1}{2}}.$$

Note $|\{x \in B_1; v = 0\}| = |\{x \in B_1; u \geq 1\}| \geq \epsilon |B_1|$. Lemma 4.8 implies

$$\sup_{B_{\frac{1}{2}}} v \leq C \left(\int_{B_1} |Dv|^2 \right)^{\frac{1}{2}}. \tag{1}$$

We will prove that the right-hand side is bounded. To this end, consider a test function $\varphi = \dfrac{\zeta^2}{u}$ for $\zeta \in C_0^1(B_2)$. Then we obtain

$$0 \leq \int a_{ij} D_i u D_j \left(\frac{\zeta^2}{u} \right) = -\int \zeta^2 \frac{a_{ij} D_i u D_j u}{u^2} + 2 \int \frac{\zeta a_{ij} D_i u D_j \zeta}{u},$$

and hence

$$\int \zeta^2 |D \log u|^2 \leq C \int |D\zeta|^2.$$

So for fixed $\zeta \in C_0^1(B_2)$ with $\zeta \equiv 1$ in B_1, we have

$$\int_{B_1} |D \log u|^2 \leq C.$$

Combining with (1), we obtain

$$\sup_{B_{\frac{1}{2}}} \ v = \sup_{B_{\frac{1}{2}}} \ (\log u)^- \leq C,$$

or

$$\inf_{B_{\frac{1}{2}}} u \geq e^{-C} > 0.$$

This finishes the proof. □

The next result controls the oscillation of solutions.

THEOREM 4.10. *Suppose u is a bounded solution of $Lu = 0$ in B_2. Then*

$$osc_{B_{\frac{1}{2}}} u \leq \gamma \ osc_{B_1} u,$$

where $\gamma \in (0, 1)$ is a constant depending only on n and Λ/λ.

PROOF. The local boundedness is proved in the previous section. Set

$$\alpha_1 = \sup_{B_1} u \quad \text{and} \quad \beta_1 = \inf_{B_1} u.$$

Consider

$$\frac{u - \beta_1}{\alpha_1 - \beta_1} \quad \text{and} \quad \frac{\alpha_1 - u}{\alpha_1 - \beta_1}.$$

Obviously, they are solutions of $Lv = 0$. Note the following equivalence

$$u \geq \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{2},$$

$$u \leq \frac{1}{2}(\alpha_1 + \beta_1) \Longleftrightarrow \frac{\alpha_1 - u}{\alpha_1 - \beta_1} \geq \frac{1}{2}.$$

Case 1. Suppose

$$|\{x \in B_1; \frac{2(u - \beta_1)}{\alpha_1 - \beta_1} \geq 1\}| \geq \frac{1}{2}|B_1|.$$

Apply Theorem 4.9 to $\frac{u - \beta_1}{\alpha_1 - \beta_1} \geq 0$ in B_1. We have for some $C > 1$

$$\inf_{B_{\frac{1}{2}}} \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{C},$$

and hence

$$\inf_{B_{\frac{1}{2}}} u \geq \beta_1 + \frac{1}{C}(\alpha_1 - \beta_1).$$

Case 2. Suppose

$$|\{x \in B_1; \frac{2(\alpha_1 - u)}{\alpha_1 - \beta_1} \geq 1\}| \geq \frac{1}{2}|B_1|.$$

Similarly, we obtain

$$\sup_{B_{\frac{1}{2}}} u \leq \alpha_1 - \frac{1}{C}(\alpha_1 - \beta_1).$$

Now, we set

$$\alpha_2 = \sup_{B_{\frac{1}{2}}} u \quad \text{and} \quad \beta_2 = \inf_{B_{\frac{1}{2}}} u.$$

Note that $\beta_2 \geq \beta_1$ and $\alpha_2 \leq \alpha_1$. In both cases, we have

$$\alpha_2 - \beta_2 \leq (1 - \frac{1}{C})(\alpha_1 - \beta_1).$$

This finishes the proof. □

The DeGiorgi theorem on Hölder continuity is an easy consequence of above results.

THEOREM 4.11. *Suppose u is an $H^1(B_1)$ solution of $Lu = 0$ in B_1. Then $u \in C^\alpha(B_1)$ for some $\alpha \in (0,1)$ depending only on n and Λ/λ. Moreover,*

$$\sup_{B_{\frac{1}{2}}} |u(x)| + \sup_{x,y \in B_{\frac{1}{2}}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le C\|u\|_{L^2(B_1)},$$

where C is a positive constant depending only on n and Λ/λ.

In the rest of this section, we discuss the Hölder continuity of solutions of general linear equations. We need the following lemma.

LEMMA 4.12. *Suppose that $a_{ij} \in L^\infty(B_r)$ satisfies*

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } x \in B_r,\ \xi \in \mathbb{R}^n,$$

for some $0 < \lambda \le \Lambda < +\infty$. Suppose $u \in H^1(B_r)$ satisfies

$$\int_{B_r} a_{ij} D_i u D_j \varphi = 0 \quad \text{for any } \varphi \in H_0^1(B_r).$$

Then there exists an $\alpha \in (0,1)$ such that for any $\rho < r$

$$\int_{B_\rho} |Du|^2 \le C\left(\frac{\rho}{r}\right)^{n-2+2\alpha} \int_{B_r} |Du|^2,$$

where C and α depend only on n and Λ/λ.

PROOF. By a dilation, we consider $r = 1$. We restrict our consideration to the range $\rho \in (0, \frac{1}{4}]$, since it is trivial for $\rho \in (\frac{1}{4}, 1]$. We may further assume $\int_{B_1} u = 0$, since the function $u - |B_1|^{-1}\int_{B_1} u$ solves the same equation. The Poincaré inequality yields

$$\int_{B_1} u^2 \le c(n) \int_{B_1} |Du|^2.$$

Hence Theorem 4.11 implies for $|x| \le 1/2$

$$|u(x) - u(0)|^2 \le C|x|^{2\alpha} \int_{B_1} |Du|^2,$$

where $\alpha \in (0,1)$ is as determined in Theorem 4.11. For any $0 < \rho \le 1/4$, take a cut-off function $\zeta \in C_0^\infty(B_{2\rho})$ with $\zeta \equiv 1$ in B_ρ, and $0 \le \zeta \le 1$ and $|D\zeta| \le 2/\rho$. Then set $\varphi = \zeta^2\big(u - u(0)\big)$. Now the equation yields

$$\begin{aligned}
0 &= \int_{B_1} a_{ij} D_i u \left(\zeta^2 D_j u + 2\zeta D_j \zeta (u - u(0))\right) \\
&\ge \frac{\lambda}{2} \int_{B_{2\rho}} \zeta^2 |Du|^2 - C \sup_{B_{2\rho}} |u - u(0)|^2 \int_{B_{2\rho}} |D\zeta|^2.
\end{aligned}$$

Therefore, we have

$$\int_{B_\rho} |Du|^2 \le C\rho^{n-2} \sup_{B_{2\rho}} |u - u(0)|^2.$$

The conclusion follows easily. $\square$

Now, we prove the following result in the same way we proved Theorem 3.8 in Chapter 3, with Lemma 3.9 in Chapter 3 replaced by Lemma 4.12.

THEOREM 4.13. *Assume* $a_{ij} \in L^\infty(B_1)$ *and* $c \in L^n(B_1)$ *satisfy*

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \textit{for any } x \in B_1, \xi \in \mathbb{R}^n,$$

for some $0 < \lambda \le \Lambda < +\infty$. *Suppose that* $u \in H^1(B_1)$ *satisfies*

$$\int_{B_1} a_{ij} D_j u D_i \varphi + cu\varphi = \int_{B_1} f\varphi \quad \textit{for any } \varphi \in H_0^1(B_1).$$

If $f \in L^q(B_1)$ *for some* $q > n/2$, *then* $u \in C^\alpha(B_1)$ *for some* $\alpha \in (0,1)$, *depending only on* n, q, λ *and* Λ. *Moreover, there exists an* $R_0 \in (0,1)$ *such that for any* $x \in B_{\frac{1}{2}}$ *and* $r \le R_0$

$$\int_{B_r(x)} |Du|^2 \le Cr^{n-2+2\alpha}\Big\{\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)}\Big\},$$

where R_0 *and* C *are constants depending only on* n, q, λ, Λ *and* $\|c\|_{L^n}$.

4.3. Moser's Harnack Inequality

In this section, we only discuss equations without lower order terms. Suppose Ω is a domain in $\mathbb{R}^n$. We always assume that $a_{ij} \in L^\infty(\Omega)$ satisfies

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ.

We first state a result of local boundedness, which is simply a dilated version of Theorem 4.1.

THEOREM 4.14. *Let* $u \in H^1(\Omega)$ *be a nonnegative subsolution in* Ω *in the following sense*

$$\int_\Omega a_{ij} D_i u D_j \varphi \le \int_\Omega f\varphi \quad \textit{for any } \varphi \in H_0^1(\Omega) \textit{ with } \phi \ge 0 \textit{ in } \Omega.$$

Suppose $f \in L^q(\Omega)$ *for some* $q > n/2$. *Then for any* $B_R \subset \Omega$, *any* $0 < r < R$ *and any* $p > 0$

$$\sup_{B_r} u \le C\big(\frac{1}{(R-r)^{n/p}}\|u^+\|_{L^p(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\big),$$

where C *is a positive constant depending only on* n, λ, Λ, p *and* q.

The next result is referred to as the weak Harnack inequality.

THEOREM 4.15. *Let* $u \in H^1(\Omega)$ *be a nonnegative supersolution in* Ω *in the following sense*

$$(*) \qquad \int_\Omega a_{ij} D_i u D_j \varphi \ge \int_\Omega f\varphi \quad \textit{for any } \varphi \in H_0^1(\Omega) \textit{ with } \varphi \ge 0 \textit{ in } \Omega.$$

Suppose $f \in L^q(\Omega)$ *for some* $q > n/2$. *Then for any* $B_R \subset \Omega$, *any* $0 < p < n/(n-2)$ *and any* $0 < \theta < \tau < 1$

$$\inf_{B_{\theta R}} u + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)} \ge C\left(\frac{1}{R^n}\int_{B_{\tau R}} u^p\right)^{\frac{1}{p}},$$

where C *is a positive constant depending only on* n, p, q, λ, Λ, θ *and* τ.

PROOF. We only prove for $R = 1$.

Step 1. We prove that the result holds for some $p_0 > 0$.

Set $\bar{u} = u + k > 0$, for some $k > 0$ to be determined and $v = \bar{u}^{-1}$. First, we derive an equation for v. For any $\varphi \in H_0^1(B_1)$ with $\varphi \geq 0$ in B_1, consider $\bar{u}^{-2}\varphi$ as the test function in $(*)$. We have

$$\int_{B_1} a_{ij} D_i u \frac{D_j \varphi}{\bar{u}^2} - 2\int_{B_1} a_{ij} D_i u D_j \bar{u} \frac{\varphi}{\bar{u}^3} \geq \int_{B_1} f \frac{\varphi}{\bar{u}^2}.$$

Note $D\bar{u} = Du$ and $Dv = -\bar{u}^2 D\bar{u}$. Therefore, we obtain

$$\int_{B_1} a_{ij} D_j v D_i \varphi + \tilde{f} v \varphi \leq 0,$$

where

$$\tilde{f} = \frac{f}{\bar{u}}.$$

In other words, v is a nonnegative subsolution of some homogeneous equation. Choose $k = \|f\|_{L^q}$ if f is not identical zero. Otherwise, choose an arbitrary $k > 0$ and then let $k \to 0+$. Note

$$\|\tilde{f}\|_{L^q(B_1)} \leq 1.$$

Theorem 4.1 implies that for any $\tau \in (\theta, 1)$ and any $p > 0$

$$\sup_{B_\theta} \bar{u}^{-p} \leq C \int_{B_\tau} \bar{u}^{-p},$$

or,

$$\begin{aligned} \inf_{B_\theta} \bar{u} &\geq C \left(\int_{B_\tau} \bar{u}^{-p} dx \right)^{-\frac{1}{p}} \\ &= C \left(\int_{B_\tau} \bar{u}^{-p} \int_{B_\tau} \bar{u}^{p} \right)^{-\frac{1}{p}} \left(\int_{B_\tau} \bar{u}^{p} \right)^{\frac{1}{p}}, \end{aligned}$$

where C is a positive constant depending only on n, q, p, λ, Λ, τ and θ.

The key point next is to show that there exists a $p_0 > 0$ such that

$$\int_{B_\tau} \bar{u}^{-p_0} \cdot \int_{B_\tau} \bar{u}^{p_0} \leq C,$$

where C is a positive constant depending only on n, q, λ, Λ and τ. We will show for any $\tau < 1$

$$\int_{B_\tau} e^{p_0|w|} \leq C, \tag{1}$$

where $w = \log \bar{u} - \beta$ with $\beta = |B_\tau|^{-1} \int_{B_\tau} \log \bar{u}$.

We have two methods to proceed:

(i) To prove directly.

(ii) To prove that w is BMO, i.e., for any $B_r(y) \subset B_1$,

$$\frac{1}{r^n} \int_{B_r} |w - w_{y,r}| dx \leq C.$$

Then (1) follows from Theorem 3.5 in Chapter 3 (John-Nirenberg Lemma).

We first prove (1) directly. Recall $\bar{u} = u + k \geq k > 0$. Note that

$$e^{p_0|w|} = 1 + p_0|w| + \frac{(p_0|w|)^2}{2!} + \cdots + \frac{(p_0|w|)^n}{n!} + \cdots .$$

Hence we need to estimate

$$\int_{B_\tau} |w|^\beta,$$

for each positive integer β.

We first derive an equation for w. Consider $\bar{u}^{-1}\varphi$ as the test function in $(*)$. Here we need $\varphi \in L^\infty(B_1) \cap H_0^1(B_1)$ with $\varphi \geq 0$. By a direct calculation as before and by the fact $Dw = \bar{u}^{-1}D\bar{u}$, we have

$$\int_{B_1} a_{ij}D_iwD_jw\varphi \leq \int_{B_1} a_{ij}D_iwD_j\varphi + \int_{B_1}(-\tilde{f}\varphi) \tag{2}$$
$$\text{for any } \varphi \in L^\infty(B_1) \cap H_0^1(B_1) \text{ with } \varphi \geq 0.$$

Replace φ by φ^2 in (2). Then the Cauchy inequality implies

$$\int_{B_1} |Dw|^2\varphi^2 \leq C\Big\{\int_{B_1} |D\varphi|^2 + \int_{B_1} |\tilde{f}|\varphi^2\Big\}.$$

By the Hölder inequality and the Sobelev inequality, we obtain

$$\int_{B_1} |\tilde{f}|\varphi^2 \leq \|\tilde{f}\|_{L^{n/2}}\|\varphi\|^2_{L^{2n/(n-2)}} \leq c(n,q)\|D\varphi\|^2_{L^2}.$$

Therefore, we have

$$\int_{B_1} |Dw|^2\varphi^2 \leq C\int_{B_1} |D\varphi|^2, \tag{3}$$

where C is a positive constant depending only on n, q, λ and Λ. Take $\varphi \in C_0^1(B_1)$ with $\varphi \equiv 1$ in B_τ. Then we obtain

$$\int_{B_\tau} |Dw|^2 \leq C, \tag{4}$$

where C is a positive constant depending only on n, q, λ, Λ and τ. Hence the Poincaré inequality implies

$$\int_{B_\tau} w^2 \leq c(n,\tau)\int_{B_\tau} |Dw|^2 \leq C,$$

since $\int_{B_\tau} w = 0$. By (3), we conclude for any $\tau' \in (\tau, 1)$

$$\int_{B_{\tau'}} w^2 \leq C, \tag{5}$$

where C is a positive constant depending only on n, q, λ, Λ, τ and τ'.

Next, we estimate $\int_{B_\tau} |w|^\beta$ for any $\beta \geq 2$. Choose $\varphi = \zeta^2|w_m|^{2\beta} \in H_0^1(B_1) \cap L^\infty(B_1)$ with

$$w_m = \begin{cases} -m & w \leq -m \\ w & |w| < m \\ m & w \geq m. \end{cases}$$

Substitute such a φ in (2) to get

$$\int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w D_j w \leq (2\beta) \int_{B_1} \zeta^2 a_{ij} D_i w D_j |w_m| |w_m|^{2\beta-1} + \int_{B_1} 2\zeta |w_m|^{2\beta} a_{ij} D_i w D_j \zeta + \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}.$$

Note

$$a_{ij} D_i w D_j |w_m| = a_{ij} D_i w_m D_j |w_m| \leq a_{ij} D_i w_m D_j w_m \quad \text{a.e. in } B_1.$$

The Hölder inequality implies

$$\begin{aligned}(2\beta)|w_m|^{2\beta-1} &\leq \frac{2\beta-1}{2\beta}|w_m|^{2\beta} + \frac{1}{2\beta}(2\beta)^{2\beta} \\ &= (1-\frac{1}{2\beta})|w_m|^{2\beta} + (2\beta)^{2\beta-1}.\end{aligned}$$

We obtain

$$\begin{aligned}\int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w D_j w \leq{}& (1-\frac{1}{2\beta}) \int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w_m D_j w_m \\ &+ (2\beta)^{2\beta-1} \int_{B_1} \zeta^2 a_{ij} D_i w_m D_j w_m \\ &+ \int_{B_1} 2\zeta |w_m|^{2\beta} a_{ij} D_i w D_j \zeta + \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta},\end{aligned}$$

and hence

$$\begin{aligned}\int_{B_1} \zeta^2 |w_m|^{2\beta} a_{ij} D_i w D_j w \leq{}& (2\beta)^{2\beta} \int_{B_1} \zeta^2 a_{ij} D_i w_m D_j w_m \\ &+ 4\beta \int_{B_1} \zeta |w_m|^{2\beta} a_{ij} D_i w D_j \zeta + 2\beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_{B_1} \zeta^2 |w_m|^{2\beta} |Dw|^2 \leq C\Big\{&(2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw_m|^2 \\ &+ \beta \int_{B_1} \zeta |w_m|^{2\beta} |Dw||D\zeta| + \beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}\Big\}.\end{aligned}$$

Note that the first term in the right hand side is bounded in (4). Applying the Cauchy inequality to the second term in the right hand side, we conclude

$$\begin{aligned}\int_{B_1} \zeta^2 |w_m|^{2\beta} |Dw|^2 \leq C\Big\{&(2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw_m|^2 \\ &+ \beta^2 \int_{B_1} |w_m|^{2\beta} |D\zeta|^2 + \beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}\Big\}.\end{aligned}$$

Note $Dw = Dw_m$ for $|w| < m$ and $Dw_m = 0$ for $|w| > m$. Hence we have

$$\begin{aligned}\int_{B_1} \zeta^2 |w_m|^{2\beta} |Dw_m|^2 \leq C\{&(2\beta)^{2\beta} \int_{B_1} \zeta^2 |Dw_m|^2 \\ &+ \beta^2 \int_{B_1} |w_m|^{2\beta} |D\zeta|^2 + \beta \int_{B_1} |\tilde{f}| \zeta^2 |w_m|^{2\beta}\}.\end{aligned}$$

In the following, we write $w = w_m$ and then let $m \to +\infty$ at the end. By the Hölder inequality, we obtain

$$\begin{aligned}|D(\zeta|w|^\beta)|^2 &\le 2|D\zeta|^2|w|^{2\beta} + 2\beta^2\zeta^2|w|^{2\beta-2}|Dw|^2\\ &\le 2|D\zeta|^2|w|^{2\beta} + 2\zeta^2|Dw|^2(\frac{\beta-1}{\beta}|w|^{2\beta} + \frac{1}{\beta}\beta^{2\beta}),\end{aligned}$$

and hence

$$\begin{aligned}\int_{B_1}|D(\zeta|w|^\beta)|^2 \le C\Big\{(2\beta)^{2\beta}\int_{B_1}\zeta^2|Dw|^2\\ + \beta^2\int|D\zeta|^2|w|^{2\beta} + \beta\int_{B_1}|\tilde f|\zeta^2|w|^{2\beta}\Big\}.\end{aligned}$$

Then the Hölder inequality implies

$$\int_{B_1}|\tilde f|\zeta^2|w|^{2\beta} \le \left(\int_{B_1}|\tilde f|^q\right)^{\frac{1}{q}}\left(\int_{B_1}(\zeta|w|^\beta)^{\frac{2q}{q-1}}\right)^{1-\frac{1}{q}}.$$

Note $2^* = \dfrac{2n}{n-2} > \dfrac{2q}{q-1} > 2$ if $q > n/2$. By the interpolation inequality and the Sobolev inequality, we have for any small $\varepsilon > 0$

$$\begin{aligned}\|\zeta|w|^\beta\|_{L^{\frac{2q}{q-1}}} &\le \varepsilon\|\zeta|w|^\beta\|_{L^{2^*}} + C(n,q)\varepsilon^{-\frac{n}{2q-n}}\|\zeta|w|^\beta\|_{L^2}\\ &\le \varepsilon\|D(\zeta|w|^\beta)\|_{L^2} + C(n,q)\varepsilon^{-\frac{n}{2q-n}}\|\zeta|w|^\beta\|_{L^2}.\end{aligned}$$

Therefore, we obtain by (3)

$$\begin{aligned}\int_{B_1}|D(\zeta|w|^\beta)|^2 &\le C\Big\{(2\beta)^{2\beta}\int_{B_1}\zeta^2|Dw|^2 + \beta^\alpha\int_{B_1}(|D\zeta|^2+\zeta^2)|w|^{2\beta}\Big\}\\ &\le C\Big\{(2\beta)^{2\beta}\int_{B_1}|D\zeta|^2 + \beta^\alpha\int_{B_1}(|D\zeta|^2+\zeta^2)|w|^{2\beta}\Big\},\end{aligned}$$

for some positive constant α depending only on n and q. Apply the Sobolev inequality for $\zeta|w|^\beta \in W_0^{1,2}(\mathbb{R}^n)$ with $\chi = \dfrac{n}{n-2}$ to get

$$\left(\int_{B_1}\zeta^{2\chi}|w|^{2\beta\chi}\right)^{\frac{1}{\chi}} \le C\beta^\alpha\Big\{(2\beta)^{2\beta}\int_{B_1}|D\zeta|^2 + \int_{B_1}(|D\zeta|^2+\zeta^2)|w|^{2\beta}\Big\}.$$

Choose a cut-off function ζ as follows. For $\tau \le r < R \le 1$, set $\zeta \equiv 1$ on B_r, $\zeta \equiv 0$ in $B_1 \setminus B_R$ and $|D\zeta| \le \dfrac{2}{R-r}$. Therefore, we have

$$\left(\int_{B_r}|w|^{2\beta\chi}\right)^{\frac{1}{\chi}} \le \frac{C\beta^\alpha}{(R-r)^2}\{(2\beta)^{2\beta} + \int_{B_R}|w|^{2\beta}\}.$$

For some $\tau' \in (\tau, 1)$, set for any $i = 1, 2, \cdots$

$$\beta_i = \chi^{i-1}, \quad \text{and} \quad r_i = \tau + \frac{1}{2^{i-1}}(\tau' - \tau).$$

Then for each $i = 1, 2, \cdots$, we have

$$\left(\int_{B_{r_i}}|w|^{2\chi^i}\right)^{\frac{1}{\chi}} \le \frac{C\chi^{(i-1)\alpha}2^{2(i-1)}}{(\tau'-\tau)^2}\Big\{(2\chi^{i-1})^{2\chi^{i-1}} + \int_{B_{r_{i-1}}}|w|^{2\chi^{i-1}}\Big\}.$$

Set

$$I_j = \|w\|_{L^{2\chi^j}(B_{r_j})}.$$

Then, we have for $j = 1, 2, \cdots$,

$$I_j \leq C^{\frac{j}{2\chi^j}} \{2\chi^{j-1} + I_{j-1}\},$$

where C is a positive constant depending only on n, q, λ, Λ, τ and τ'. Iterating the above inequality and observing that

$$\sum_{i=0}^{\infty} \frac{i}{\chi^i} < \infty,$$

we obtain

$$I_j \leq C \sum_{i=1}^{j} \chi^{i-1} + CI_0,$$

or,

$$I_j \leq C\chi^j + CI_0.$$

Now for $\beta \geq 2$, there exists a j such that $2\chi^{j-1} \leq \beta < 2\chi^j$. Hence, we have

$$\begin{aligned} I_\beta(B_\tau) \equiv \left(\int_{B_\tau} |w|^\beta\right)^{\frac{1}{\beta}} &\leq CI_j \leq C\chi^j + CI_0 \\ &\leq C\beta + CI_0 \leq C_0\beta, \end{aligned}$$

since I_0 is bounded in (5). Then, we obtain for $\beta \geq 1$

$$\int_{B_\tau} |w|^\beta dx \leq C_0^\beta \beta^\beta \leq C_0^\beta e^\beta \beta!,$$

where we used the Sterling formula for the integer β. Hence, for any integer $\beta \geq 1$, we conclude

$$\int_{B_\tau} \frac{(p_0|w|)^\beta}{\beta!} \leq p_0^\beta (C_0 e)^\beta \leq \frac{1}{2^\beta},$$

by choosing $p_0 = (2C_0 e)^{-1}$. This proves that

$$\begin{aligned} \int e^{p_0|w|} &= \int 1 + p_0|w| + \frac{(p_0|w|)^2}{2!} + \cdots \\ &\leq 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots \leq 2. \end{aligned}$$

We remark that the above method is elementary in nature.

Now we give the second proof of the estimate (1) by using BMO. The estimate (3) yields

$$\int_{B_1} |Dw|^2 \zeta^2 \leq C \int_{B_1} |D\zeta|^2 \quad \text{for any } \zeta \in C_0^1(B_1).$$

Then for any $B_{2r}(y) \subset B_1$, choose ζ with

$$\operatorname{supp} \zeta \subset B_{2r}(y), \ \zeta \equiv 1 \text{ in } B_r(y), \ |D\zeta| \leq \frac{2}{r}.$$

We obtain

$$\int_{B_r(y)} |Dw|^2 \leq Cr^{n-2}.$$

Hence the Poincaré inequality implies

$$\frac{1}{r^n}\int_{B_r(y)}|w-w_{y,r}| \le \frac{1}{r^{\frac{n}{2}}}\left(\int_{B_r(y)}|w-w_{y,r}|^2\right)^{\frac{1}{2}}$$

$$\le \frac{1}{r^{\frac{n}{2}}}\left(r^2\int_{B_r(y)}|Dw|^2\right)^{\frac{1}{2}} \le C,$$

or, $w \in BMO$. Then the John-Nirenberg Lemma implies

$$\int_{B_\tau} e^{p_0|w|} \le C.$$

Step 2. The result holds for any positive $p < n/(n-2)$.

We need to prove, for any $0 < r_1 < r_2 < 1$ and $0 < p_2 < p_1 < n/(n-2)$,

$$\left(\int_{B_{r_1}} \bar{u}^{p_1}\right)^{\frac{1}{p_1}} \le C\left(\int_{B_{r_2}} \bar{u}^{p_2}\right)^{\frac{1}{p_2}}, \tag{6}$$

where C is positive constant depending only on n, q, λ, Λ, r_1, r_2, p_1 and p_2. A similar calculation may be found before. Here, we just point out some key steps.

Take $\varphi = \bar{u}^{-\beta}\eta^2$ for $\beta \in (0,1)$ as the test function in $(*)$. Then we have

$$\int_{B_1}|D\bar{u}|^2\bar{u}^{-\beta-1}\eta^2 \le C\left\{\frac{1}{\beta^2}\int_{B_1}|D\eta|^2\bar{u}^{1-\beta} + \frac{1}{\beta}\int_{B_1}\frac{|f|}{k}\eta^2\bar{u}^{1-\beta}\right\}.$$

Set $\gamma = 1-\beta \in (0,1)$ and $w = \bar{u}^{\frac{\gamma}{2}}$. Then, we obtain

$$\int |Dw|^2\eta^2 \le \frac{C}{(1-\gamma)^\alpha}\int w^2(|D\eta|^2+\eta^2),$$

or

$$\int |D(w\eta)|^2 \le \frac{C}{(1-\gamma)^\alpha}\int w^2(|D\eta|^2+\eta^2),$$

for some positive $\alpha > 0$. The Sobolev embedding theorem and an appropriate choice of cut-off functions imply, with $\chi = n/n-2$, for any $0 < r < R < 1$

$$\left(\int_{B_r} w^{2\chi}\right)^{\frac{1}{\chi}} \le \frac{C}{(1-\gamma)^\alpha}\cdot\frac{1}{(R-r)^2}\int_{B_R} w^2,$$

or

$$\left(\int_{B_r}\bar{u}^{\gamma\chi}\right)^{\frac{1}{\gamma\chi}} \le \left(\frac{C}{(1-\gamma)^\alpha}\frac{1}{(R-r)^2}\right)^{\frac{1}{\gamma}}\left(\int_{B_R}\bar{u}^\gamma\right)^{\frac{1}{\gamma}}.$$

This holds for any $\gamma \in (0,1)$. Now (6) follows after finitely many iterations. □

Now the Moser's Harnack inequality is an easy consequence of above results.

THEOREM 4.16. *Let $u \in H^1(\Omega)$ be a nonnegative solution in Ω*

$$\int_\Omega a_{ij}D_iuD_j\varphi = \int_\Omega f\varphi \quad \textit{for any } \varphi \in H_0^1(\Omega).$$

Suppose $f \in L^q(\Omega)$ for some $q > n/2$. Then for any $B_R \subset \Omega$,

$$\sup_{B_{\frac{R}{2}}} u \le C\left\{\inf_{B_{\frac{R}{2}}} u + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\right\},$$

where C is a positive constant depending only on n, λ, Λ and q.

The Hölder continuity follows easily from Theorem 4.16.

Corollary 4.17. *Let $u \in H^1(\Omega)$ be a solution in Ω, i.e.,*

$$\int_\Omega a_{ij} D_i u D_j \varphi = \int_\Omega f\varphi \quad \textit{for any } \varphi \in H_0^1(\Omega).$$

Suppose $f \in L^q(\Omega)$ for some $q > n/2$. Then $u \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$ depending only on n, q, λ and Λ. Moreover, for any $B_R \subset \Omega$

$$|u(x) - u(y)| \le C\Big(\frac{|x-y|}{R}\Big)^\alpha \Big\{\Big(\frac{1}{R^n}\int_{B_R} u^2\Big)^{\frac{1}{2}} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\Big\}$$
$$\textit{for any } x, y \in B_{\frac{R}{2}},$$

where C is a positive constant depending only on n, λ, Λ and q.

Proof. We prove the estimate for $R = 1$. Set for $r \in (0,1)$

$$M(r) = \sup_{B_r} u \quad \text{and} \quad m(r) = \inf_{B_r} u.$$

Then $M(r) < +\infty$ and $m(r) > -\infty$. It suffices to show

$$\omega(r) \equiv M(r) - m(r) \le Cr^\alpha\big\{\|u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\big\} \quad \text{for any } r < \frac{1}{2}.$$

Set $\delta = 2 - \dfrac{n}{q}$. Apply Theorem 4.16 to $M(r) - u \ge 0$ in B_r to get

$$\sup_{B_{\frac{r}{2}}}(M(r) - u) \le C\Big\{\inf_{B_{\frac{r}{2}}}\big(M(r) - u\big) + r^\delta\|f\|_{L^q(B_r)}\Big\},$$

i.e.,

$$M(r) - m(\frac{r}{2}) \le C\Big\{\big(M(r) - M(\frac{r}{2})\big) + r^\delta\|f\|_{L^q(B_r)}\Big\}. \tag{1}$$

Similarly, apply Theorem 4.16 to $u - m(r) \ge 0$ in B_r to get

$$M(\frac{r}{2}) - m(r) \le C\Big\{\big(m(\frac{r}{2}) - m(r)\big) + r^\delta\|f\|_{L^q(B_r)}\Big\}. \tag{2}$$

Then by adding (1) and (2) together, we obtain

$$\omega(r) + \omega(\frac{r}{2}) \le C\Big\{\big(\omega(r) - \omega(\frac{r}{2})\big) + r^\delta\|f\|_{L^q(B_r)}\Big\},$$

or

$$\omega(\frac{r}{2}) \le \gamma\omega(r) + Cr^\delta\|f\|_{L^q(B_r)},$$

for some $\gamma = \dfrac{C-1}{C+1} < 1$.

Apply Lemma 4.18 below with μ chosen such that

$$\alpha = (1-\mu)\log\gamma/\log\tau < \mu\delta.$$

We obtain

$$\omega(\rho) \le C\rho^\alpha\Big\{\omega(\frac{1}{2}) + \|f\|_{L^q(B_1)}\Big\} \quad \text{for any } \rho \in (0, \frac{1}{2}].$$

While Theorem 4.14 implies

$$\omega(\frac{1}{2}) \leq C\{\|u\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\}.$$

This finishes the proof. □

LEMMA 4.18. *Let ω and σ be non-decreasing functions in an interval $(0, R]$. Suppose for some $0 < \gamma, \tau < 1$*

$$\omega(\tau r) \leq \gamma\omega(r) + \sigma(r) \quad \text{for any } r \leq R.$$

Then, for any $\mu \in (0,1)$ and $r \leq R$,

$$\omega(r) \leq C\Big\{(\frac{r}{R})^{\alpha}\omega(R) + \sigma(r^{\mu}R^{1-\mu})\Big\},$$

where C is a positive constant depending only on γ, τ and $\alpha = (1-\mu)\log\gamma/\log\tau$.

PROOF. Fix some $r_1 \leq R$. Then for any $r \leq r_1$, we have

$$\omega(\tau r) \leq \gamma\omega(r) + \sigma(r_1),$$

since σ is nondecreasing. We now iterate this inequality to get for any positive integer k

$$\omega(\tau^k r_1) \leq \gamma^k\omega(r_1) + \sigma(r_1)\sum_{i=0}^{k-1}\gamma^i \leq \gamma^k\omega(R) + \frac{\sigma(r_1)}{1-\gamma}.$$

For any $r \leq r_1$, we choose k such that

$$\tau^k r_1 < r \leq \tau^{k-1} r_1.$$

Hence, we have

$$\begin{aligned}\omega(r) &\leq \omega(\tau^{k-1}r_1) \leq \gamma^{k-1}\omega(R) + \frac{\sigma(r_1)}{1-\gamma}\\ &\leq \frac{1}{\gamma}(\frac{r}{r_1})^{\log\gamma/\log\tau}\omega(R) + \frac{\sigma(r_1)}{1-\gamma}.\end{aligned}$$

By letting $r_1 = r^{\mu}R^{1-\mu}$, we obtain

$$\omega(r) \leq \frac{1}{\gamma}(\frac{r}{R})^{(1-\mu)(\log\gamma/\log\tau)}\omega(R) + \frac{\sigma(r^{\mu}R^{1-\mu})}{1-\gamma}.$$

This finishes the proof. □

We also have the following Liouville theorem.

COROLLARY 4.19. *Suppose u is a solution of a homogeneous equation in $\mathbb{R}^n$, i.e.,*

$$\int_{\mathbb{R}^n} a_{ij}D_iuD_j\varphi = 0 \quad \text{for any } \varphi \in H_0^1(\mathbb{R}^n).$$

If u is bounded, then u is a constant.

PROOF. As in the proof of Corollary 4.17, we have for some $\gamma < 1$

$$\omega(r) \leq \gamma\omega(2r) \quad \text{for any } r > 0.$$

By an iteration, we obtain

$$\omega(r) \leq \gamma^k\omega(2^k r) \leq C\gamma^k,$$

since u is bounded. Hence by letting $k \to \infty$, we conclude

$$\omega(r) = 0 \quad \text{for any } r > 0.$$

Therefore, u is constant. □

4.4. Nonlinear Equations

Up to now, we have been discussing linear equations of the following form

$$-D_j(a_{ij}(x)D_i u) = f(x) \quad \text{in } B_1.$$

It is natural to ask how these results generalize to nonlinear equations. To answer this question, let us consider an equation for a solution v of the form

$$v(x) = \Phi\big(u(x)\big),$$

for some smooth function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi' \neq 0$. Any estimates for u can be translated to those for v. To find the equation for v, we write

$$u = \Psi(v),$$

with $\Psi = \Phi^{-1}$. Then by setting $\eta = \Psi'(v)\xi$ for $\xi \in C_0^\infty(B_1)$, we have

$$\begin{aligned}\int a_{ij} D_i u D_j \xi &= \int a_{ij} \Psi'(v) D_i v D_j \xi \\ &= \int a_{ij} D_i v D_j \eta - \int \frac{\Psi''(v)}{\Psi'(v)} a_{ij} D_i v D_j v \eta.\end{aligned}$$

Therefore if u satisfies

$$\int a_{ij} D_i u D_j \xi = \int f(x)\xi \quad \text{for any } \ \xi \in H_0^1(B_1),$$

then v satisfies

$$\int a_{ij} D_i v D_j \eta = \int \left(\frac{\Psi''(v)}{\Psi'(v)} a_{ij} D_i v D_j v + \frac{1}{\Psi'(v)} f \right) \eta \ \text{ for any } \eta \in C_0^\infty(B_1).$$

Note that the nonlinear term has a quadratic growth in terms of Dv. Hence, we may extend the space of test functions to $H_0^1(B_1) \cap L^\infty(B_1)$. It turns out that $H^1(B_1) \cap L^\infty(B_1)$ is also the right space for solutions. The following example illustrates the boundedness of solutions is essential.

Example 4.20. Consider the equation

$$-\Delta u = |Du|^2 \quad \text{in } B_R \subset \mathbb{R}^2,$$

with $R < 1$. It is easy to check that $u(x) = \log\log|x|^{-1} - \log\log R^{-1} \in H^1(B_R)$ is a weak solution with zero boundary data. Note that $u(x) \equiv 0$ is also a solution.

In this section, we assume $a_{ij} \in L^\infty(B_1)$ satisfies

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } x \in B_1 \text{ and } \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ. We consider the nonlinear equation of the form

$$(*) \qquad \int a_{ij}(x) D_i u D_j \varphi = \int b(x, u, Du)\varphi \quad \text{for any } \varphi \in H_0^1(B_1) \cap L^\infty(B_1).$$

We say the nonlinear term b satisfies *the natural growth condition* if

$$|b(x,u,p)| \le C(u)\big(f(x)+|p|^2\big) \quad \text{for any } (x,u,p)\in B_1\times\mathbb{R}\times\mathbb{R}^n,$$

for some constant $C(u)$ depending only on u, and $f\in L^q(B_1)$ for some $q\ge \dfrac{2n}{n+2}$. We always assume

$$u\in H^1(B_1)\cap L^\infty(B_1).$$

LEMMA 4.21. *Suppose $u\in H^1(B_1)$ is a nonnegative solution of $(*)$ with $|u|\le M$ in B_1 and that b satisfies the natural growth condition with $f(x)\in L^q(B_1)$ for some $q>\dfrac{n}{2}$. Then for any $B_R\subset B_1$*

$$\sup_{B_{\frac{R}{2}}} u \le C\big\{\inf_{B_{\frac{R}{2}}} u + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\big\},$$

where C is a positive constant depending only on n,λ,Λ,M and q.

PROOF. Let $v=\frac{1}{\alpha}(e^{\alpha u}-1)$ for some $\alpha>0$. Then for $\varphi\in H_0^1(B_1)\cap L^\infty(B_1)$ with $\varphi\ge 0$, we have

$$\begin{aligned}\int a_{ij}D_ivD_j\varphi &= \int a_{ij}e^{\alpha u}D_iuD_j\varphi\\ &= \int a_{ij}D_iuD_j(e^{\alpha u}\varphi) - \alpha\int a_{ij}e^{\alpha u}D_iuD_ju\varphi\\ &= \int b(x,u,Du)e^{\alpha u}\varphi - \alpha\int a_{ij}e^{\alpha u}D_iuD_ju\varphi\\ &\le C(M)\int (f(x)+|Du|^2)e^{\alpha u}\varphi - \alpha\lambda\int |Du|^2e^{\alpha u}\varphi.\end{aligned}$$

Hence by taking α large, we have

$$\begin{aligned}&\int a_{ij}D_ivD_j\varphi \le C\int f(x)\varphi \\ &\qquad \text{for any } \varphi\in H_0^1(B_1)\cap L^\infty(B_1) \text{ with } \varphi\ge 0,\end{aligned} \tag{1}$$

for some positive constant C depending only on n,λ,Λ and M. Observe that u and v are compatible. Therefore by Theorem 4.14, we obtain for any $p>0$

$$\begin{aligned}\sup_{B_{\frac{R}{2}}} u &\le C(M,\alpha)\sup_{B_{\frac{R}{2}}} v\\ &\le C\left\{\left(\frac{1}{R^n}\int_{B_R} v^p\right)^{\frac{1}{p}} + R^{2-\frac{n}{q}}\left(\int_{B_R} f^q\right)^{\frac{1}{q}}\right\}\\ &\le C\left\{\left(\frac{1}{R^n}\int_{B_R} u^p\right)^{\frac{1}{p}} + R^{2-\frac{n}{q}}\left(\int_{B_R} f^q\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

For the lower bound, we let $w=\dfrac{1}{\alpha}(1-e^{-\alpha u})$. As before, by choosing $\alpha>0$ large we have

$$\int a_{ij}D_iwD_j\varphi \ge C\int f(x)\varphi \quad \text{for any } \varphi\in H_0^1(B_1)\cap L^\infty(B_1) \text{ with } \varphi\ge 0.$$

Hence by Theorem 4.15, we obtain for any $p \in (0, \dfrac{n}{n-2})$

$$\left(\frac{1}{R^n}\int_{B_R} u^p\right)^{\frac{1}{p}} \le C\left\{\inf_{B_{\frac{R}{2}}} u + R^{2-\frac{n}{q}}\left(\int_{B_R} f^q\right)^{\frac{1}{q}}\right\}.$$

We finish the proof by combining the above inequalities. $\square$

Remark 4.22. In estimate (1) in the above proof, take $\varphi = (u+M)\eta^2$ for some $\eta \in C_0^1(B_1)$. Then by the Hölder inequality, we conclude

$$\int |Du|^2\eta^2 \le C\left\{\int \left(|D\eta|^2 + |f|\eta^2\right)\right\},$$

for some positive constant C depending only on n, λ, Λ and M. This implies the interior L^2-estimate of the gradient Du in terms of these constants together with $\|f\|_{L^1(B_1)}$. This fact will be used in the proof of Theorem 4.24.

Corollary 4.23. *Suppose $u \in H^1(B_1)$ is a bounded solution of $(*)$ and that b satisfies the natural growth condition with $f(x) \in L^q(B_1)$ for some $q > \dfrac{n}{2}$. Then $u \in C^\alpha_{loc}(B_1)$, for some $\alpha \in (0,1)$ depending only on n, λ, Λ, q and $|u|_{L^\infty}$. Moreover,*

$$|u(x) - u(y)| \le C|x-y|^\alpha \quad \text{for any } x, y \in B_{\frac{1}{2}},$$

where C is a positive constant depending only on n, λ, Λ, q, $|u|_{L^\infty(B_1)}$ and $\|f\|_{L^q(B_1)}$.

Proof. The proof is identical to that of Theorem 4.17, with Theorem 4.16 replaced by Lemma 4.21. $\square$

Theorem 4.24. *Suppose $u \in H^1(B_1)$ is a bounded solution of $(*)$ and that b satisfies the natural growth condition with $f \in L^q(B_1)$ for some $q > n$. Assume further that $a_{ij} \in C^\alpha(B_1)$ for $\alpha = 1 - \dfrac{n}{q}$. Then $Du \in C^\alpha_{loc}(B_1)$. Moreover,*

$$|Du|_{C^\alpha(B_{\frac{1}{2}})} \le C,$$

where C is a positive constant depending only on n, λ, Λ, q, $|u|_{L^\infty(B_1)}$ and $\|f\|_{L^q(B_1)}$.

Proof. We only need to prove $Du \in L^\infty_{loc}$. Then the Hölder continuity is implied by Theorem 3.11 in Chapter 3. For any $B_r(x_0) \subset B_1$, solve for w such that

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i w D_j \varphi = 0 \quad \text{for any } \varphi \in H_0^1\big(B_r(x_0)\big),$$

with $w - u \in H_0^1\big(B_r(x_0)\big)$. Then the maximum principle implies

$$\inf_{B_r(x_0)} u \le w \le \sup_{B_r(x_0)} u \quad \text{in } B_r(x_0),$$

or

$$\sup_{B_r(x_0)} |u - w| \le osc_{B_r(x_0)} u. \tag{1}$$

By Lemma 3.9 in Chapter 3, we have for any $0 < \rho \le r$

$$\int_{B_\rho(x_0)} |Du|^2 \le c\left\{\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Du|^2 + \int_{B_r(x_0)} |D(u-w)|^2\right\}, \tag{2}$$

and

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le c\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + \int_{B_r(x_0)} |D(u-w)|^2 \Big\}. \tag{3}$$

Note that the function $v = u - w \in H_0^1\big(B_r(x_0)\big)$ satisfies

$$\int_{B_r(x_0)} a_{ij}(x_0) D_i v D_j \varphi = \int_{B_r(x_0)} b(x, u, Du)\varphi + \int_{B_r(x_0)} \big(a_{ij}(x_0) - a_{ij}(x)\big) D_i u D_j \varphi$$
$$\text{for any } \varphi \in H_0^1\big(B_r(x_0)\big) \cap L^\infty\big(B_r(x_0)\big).$$

Taking $\varphi = v$ and by the Sobolev inequality, we obtain

$$\int_{B_r(x_0)} |Dv|^2 \le C\Big\{ \int_{B_r(x_0)} |Du|^2|v| + r^{2\alpha} \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q(B_1)} \Big\}.$$

Hence with (1), we conclude

$$\int_{B_r(x_0)} |Dv|^2 \le C\Big\{ \Big(r^{2\alpha} + osc_{B_r(x_0)}\, u \Big) \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q} \Big\}. \tag{4}$$

Corollary 4.23 implies $u \in C^{\delta_0}$ for some $\delta_0 > 0$. Therefore, we have by (2) and (4)

$$\int_{B_\rho(x_0)} |Du|^2 \le C\left\{ \Big[\Big(\frac{\rho}{r}\Big)^n + r^{2\alpha} + r^{\delta_0} \Big] \int_{B_r(x_0)} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q} \right\}.$$

By Lemma 3.4 in Chapter 3, we obtain for any $\delta < 1$ and any $B_r(x_0) \subset B_{\frac{7}{8}}$

$$\int_{B_r(x_0)} |Du|^2 \le C r^{n-2+2\delta} \Big\{ \int_{B_{\frac{7}{8}}} |Du|^2 + \|f\|^2_{L^q(B_1)} \Big\}.$$

This implies $u \in C^\delta_{\mathrm{loc}}$ for any $\delta < 1$. Moreover, for any $B_r(x_0) \subset B_{\frac{3}{4}}$

$$\mathrm{osc}_{B_r(x_0)} u \le C r^\delta,$$

where C is a positive constant depending only on n, λ, Λ, q, $|u|_{L^\infty(B_1)}$ and $\|f\|_{L^q(B_1)}$, by the Remark 4.22. With (4), we have for any $B_r(x_0) \subset B_{\frac{2}{3}}$

$$\begin{aligned}\int_{B_r(x_0)} |Dv|^2 \le& C\Big\{ (r^{2\alpha} + r^\delta) r^{n-2+2\delta} \int_{B_{\frac{7}{8}}} |Du|^2 + r^{n+2\alpha}\|f\|^2_{L^q} \Big\} \\ \le& C r^{n+2\alpha'},\end{aligned}$$

for some $\alpha' < \alpha$ if $\delta \in (0,1)$ is chosen such that $3\delta > 2$ and $\alpha + \delta > 1$. Hence, with (3) we obtain for any $B_r(x_0) \subset B_{\frac{2}{3}}$ and any $0 < \rho \le r$

$$\int_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \le C\Big\{ \Big(\frac{\rho}{r}\Big)^{n+2} \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + r^{n+2\alpha'} \Big\}.$$

By Lemma 3.4 and Theorem 3.1 in Chapter 3 again, we conclude that $Du \in C^{\alpha'}_{loc}$ for some $\alpha' < \alpha$, and in particular $Du \in L^\infty_{loc}$. This finishes the proof. $\square$

CHAPTER 5

Viscosity Solutions

In this chapter, we generalize the notion of classical solutions to viscosity solutions and study their regularities. We define viscosity solutions by comparing them with quadratic polynomials and thus remove the requirement that solutions be at least C^2. The main tool to study viscosity solutions is the maximum principle due to Alexandroff. We first generalize such a maximum principle to viscosity solutions and then use the resulting estimate to discuss the regularity of viscosity solutions. We use it to control the distribution functions of solutions and obtain Harnack inequality, which generalizes a result by Krylov and Safonov, and hence C^α regularity. We also use it to approximate solutions in L^∞ by quadratic polynomials and get Schauder ($C^{2,\alpha}$) estimates. The methods are basically nonlinear, in the sense that they do not rely on differentiating equations. The results obtained in this chapter hold for general fully nonlinear equations, although in this note we focus only on linear equations.

5.1. Alexandroff Maximum Principle

We begin this section with the definition of viscosity solutions. This very weak concept of solutions enables us to define a class of functions containing all classical solutions of linear and nonlinear elliptic equations with fixed ellipticity constants and bounded measurable coefficients.

Suppose that Ω is a bounded and connected domain in $\mathbb{R}^n$ and that $a_{ij} \in C(\Omega)$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ. Consider the operator L in Ω defined by

$$Lu \equiv a_{ij}(x)D_{ij}u,$$

for $u \in C^2(\Omega)$.

Suppose $u \in C^2(\Omega)$ is a supersolution in Ω, i.e., $Lu \leq 0$. Then for any $\varphi \in C^2(\Omega)$ with $L\varphi > 0$, we have

$$L(u - \varphi) < 0 \quad \text{in } \Omega.$$

This implies by the maximum principle that $u - \varphi$ cannot have local interior minimums in Ω. In other words, if $u - \varphi$ has a local minimum at $x_0 \in \Omega$, then

$$L\varphi(x_0) \leq 0.$$

Geometrically, $u - \varphi$ having a local minimum at x_0 means that φ touches u from below at x_0 if we adjust φ appropriately by adding a constant. This suggests the following definition. Here, we always assume $f \in C(\Omega)$.

DEFINITION 5.1. $u \in C(\Omega)$ is a viscosity supersolution (resp. subsolution) of $Lu = f$ in Ω if, for any $x_0 \in \Omega$ and any function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum (resp. maximum) at x_0, there holds

$$L\varphi(x_0) \le f(x_0) \qquad (\text{resp. } L\varphi(x_0) \ge f(x_0)).$$

We say that u is a viscosity solution if it is a viscosity subsolution and a viscosity supersolution.

REMARK 5.2. By an approximation, we may replace the C^2 function φ by a quadratic polynomial Q.

REMARK 5.3. The above analysis shows that a classical supersolution is a viscosity supersolution. It is straightforward to prove that a C^2 viscosity supersolution is a classical supersolution. Similar statements hold for subsolutions and solutions.

The notion of viscosity solutions can be generalized to nonlinear elliptic equations accordingly.

Now we define in a weak way the class of "all solutions to all elliptic equations". For any function φ, which is C^2 at x_0, we have the following equivalence

$$\begin{aligned}
&\sum_{i,j=1}^{n} a_{ij}(x_0) D_{ij}\varphi(x_0) \le 0 \\
\Longleftrightarrow &\sum_{k=1}^{n} \alpha_k e_k \le 0 \text{ with } \lambda \le \alpha_k \le \Lambda, e_k = e_k(D^2\varphi(x_0)) \\
\Longleftrightarrow &\sum_{e_i>0} \alpha_i e_i + \sum_{e_i<0} \alpha_i e_i \le 0 \\
\Longleftrightarrow &\sum_{e_i>0} \alpha_i e_i \le \sum_{e_i<0} \alpha_i (-e_i),
\end{aligned}$$

which implies

$$\lambda \sum_{e_i>0} e_i \le \Lambda \sum_{e_i<0} (-e_i),$$

where $e_1, \cdots, e_n$ are eigenvalues of the Hessian matrix $D^2\varphi(x_0)$. This means that positive eigenvalues of $D^2\varphi(x_0)$ are controlled by negative eigenvalues.

DEFINITION 5.4. Suppose f is a continuous function in Ω and that λ and Λ are two positive constants. We define $u \in C(\Omega)$ to belong to $\mathcal{S}^+(\lambda, \Lambda, f)$ (resp. $\mathcal{S}^-(\lambda, \Lambda, f)$) if, for any $x_0 \in \Omega$ and any function $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum (resp. maximum) at x_0, there holds

$$\begin{aligned}
&\lambda \sum_{e_i>0} e_i(x_0) + \Lambda \sum_{e_i<0} e_i(x_0) \le f(x_0) \\
&(\text{resp.} \quad \Lambda \sum_{e_i>0} e_i(x_0) + \lambda \sum_{e_i<0} e_i(x_0) \ge f(x_0)),
\end{aligned}$$

where $e_1(x_0), \cdots, e_n(x_0)$ are eigenvalues of the Hessian matrix $D^2\varphi(x_0)$.

We set $\mathcal{S}(\lambda, \Lambda, f) = \mathcal{S}^+(\lambda, \Lambda, f) \cap \mathcal{S}^-(\lambda, \Lambda, f)$.

REMARK 5.5. Any viscosity supersolutions of

$$a_{ij}D_{ij}u = f \quad \text{in } \Omega$$

belong to the class $\mathcal{S}^+(\lambda, \Lambda, f)$, if

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^n.$$

The class $\mathcal{S}^+(\lambda, \Lambda, f)$ and $\mathcal{S}^-(\lambda, \Lambda, f)$ also include solutions of fully nonlinear equations. Among them are the Pucci's equations.

EXAMPLE 5.6. For any two positive constants $\lambda \le \Lambda$, let A be a symmetric matrix whose eigenvalues belong to $[\lambda, \Lambda]$, i.e.,

$$\lambda|\xi|^2 \le A_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n.$$

Denote by $\mathcal{A}_{\lambda,\Lambda}$ the class of all such matrices. For any symmetric matrix M, we define the Pucci's extremal operators

$$\mathcal{M}^-(M) = \mathcal{M}^-(\lambda, \Lambda, M) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} A_{ij}M_{ij},$$
$$\mathcal{M}^+(M) = \mathcal{M}^+(\lambda, \Lambda, M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} A_{ij}M_{ij}.$$

Pucci's equations are given by

$$\mathcal{M}^-(\lambda, \Lambda, M) = f, \quad \mathcal{M}^+(\lambda, \Lambda, M) = g,$$

for continuous functions f and g in Ω. It is easy to see

$$\mathcal{M}^-(\lambda, \Lambda, M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$
$$\mathcal{M}^+(\lambda, \Lambda, M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $e_1, \cdots, e_n$ are eigenvalues of M. Therefore $u \in \mathcal{S}^+(\lambda, \Lambda, f)$ if and only if $\mathcal{M}^-(\lambda, \Lambda, D^2u) \le f$ in the viscosity sense, i.e., for any $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$ there holds

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0)) \le f(x_0).$$

By the definition of $\mathcal{M}^-$ and $\mathcal{M}^+$, it is easy to check for any two symmetric matrices M and N

$$\begin{aligned} \mathcal{M}^-(M) + \mathcal{M}^-(N) &\le \mathcal{M}^-(M+N) \le \mathcal{M}^+(M) + \mathcal{M}^-(N) \\ &\le \mathcal{M}^+(M+N) \le \mathcal{M}^+(M) + \mathcal{M}^+(N). \end{aligned}$$

This property will be needed in the next section.

Next, we derive the Alexandroff maximum principle for viscosity solutions. It has the role of energy inequalities for solutions of equations of divergence form.

Let v be a continuous function in an open convex set Ω. The convex envelope of v in Ω is defined by

$$\Gamma(v)(x) = \sup_L \{L(x); L \le v \text{ in } \Omega, L \text{ is an affine function}\},$$

for any $x \in \Omega$. It is easy to see that $\Gamma(v)$ is a convex function in Ω. The set

$$\{v = \Gamma(v)\} = \{x \in \Omega; v(x) = \Gamma(v)(x)\}$$

is called the (lower) contact set of v. Points in the contact set are called contact points.

The following is the classical version of the Alexandroff maximum principle. We do not require that functions be solutions to elliptic equations. See Lemma 2.33 in Chapter 2.

LEMMA 5.7. *Suppose u is a $C^{1,1}$ function in B_1 with $u \geq 0$ on ∂B_1. Then*

$$\sup_{B_1} u^- \leq c\Big(\int_{B_1 \cap \{u=\Gamma_u\}} \det D^2 u\Big)^{\frac{1}{n}},$$

where Γ_u is the convex envelope of $-u^- = \min\{u, 0\}$ and c is a positive constant depending only on n.

Now we state a version for viscosity solutions.

THEOREM 5.8. *Suppose u belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in B_1 with $u \geq 0$ on ∂B_1 for some $f \in C(\Omega)$. Then*

$$\sup_{B_1} u^- \leq c\Big(\int_{B_1 \cap \{u=\Gamma_u\}} (f^+)^n\Big)^{\frac{1}{n}},$$

where Γ_u is the convex envelope of $-u^- = \min\{u, 0\}$ and c is a positive constant depending only on n, λ and Λ.

PROOF. We prove that Γ_u is a $C^{1,1}$ function in B_1 and that at any contact point x_0

$$f(x_0) \geq 0, \tag{1}$$

and

$$L(x) \leq \Gamma_u(x) \leq L(x) + C\big\{f(x_0) + \varepsilon(x)\big\}|x - x_0|^2, \tag{2}$$

for some affine function L and any x close to x_0, where $\varepsilon(x) \to 0$ as $x \to x_0$ and C is a positive constant depending only on n, λ and Λ. We then obtain by (2)

$$\det D^2\Gamma_u(x) \leq C\big(f(x)\big)^n \quad \text{for a.e. } x \in \{u = \Gamma_u\}.$$

Now we apply Lemma 5.7 to Γ_u to get the desired result.

Suppose x_0 is a contact point, i.e., $u(x_0) = \Gamma_u(x_0)$. We may assume $x_0 = 0$. We also assume, by subtracting a supporting plane at $x_0 = 0$, that $u \geq 0$ in B_1 and that $u(0) = 0$.

In order to prove (1), we take $h(x) = -\varepsilon|x|^2/2$ in B_1. Obviously, $u - h$ has a minimum at 0. Note that the eigenvalues of $D^2h(0)$ are $-\varepsilon, \cdots, -\varepsilon$. By the definition of $\mathcal{S}^+(\lambda, \Lambda, f)$, we have

$$-n\Lambda\varepsilon \leq f(0).$$

We get (1) by letting $\varepsilon \to 0$.

For (2), we prove

$$0 \leq \Gamma_u(x) \leq C(n, \lambda, \Lambda)\big\{f(0) + \varepsilon(x)\big\}|x|^2 \quad \text{for any } x \in B_1,$$

where $\varepsilon(x) \to 0$ as $x \to 0$. By setting $w = \Gamma_u$, we need to estimate for any small $r > 0$

$$C_r = \frac{1}{r^2} \max_{B_r} w.$$

Fix an $r > 0$. By the convexity, w attains its maximum in $\bar{B}_r$ at some point on the boundary, say, $(0, \cdots, 0, r)$. The set $\{x \in B_1; w(x) \le w(0, \cdots, 0, r)\}$ is convex and contains B_r. It follows easily that

$$w(x', r) \ge w(0, \cdots, 0, r) = C_r r^2 \quad \text{for any } x = (x', r) \in B_1.$$

Take a positive number N to be determined and set

$$R_r = \{(x', x_n); |x'| \le Nr, |x_n| \le r\}.$$

We construct a quadratic polynomial that touches u from below in R_r and curves upward very much. Set for some $b > 0$

$$h(x) = (x_n + r)^2 - b|x'|^2.$$

Then we have

(i) for $x_n = -r$, $h \le 0$;

(ii) for $|x'| = Nr$, $h \le (4 - bN^2)r^2 \le 0$ if we take $b = 4/N^2$;

(iii) for $x_n = r$, $h = 4r^2 - b|x'|^2 \le 4r^2$.

Hence if we set

$$\tilde{h}(x) = \frac{C_r}{4} h(x) = \frac{C_r}{4}\big\{(x_n + r)^2 - \frac{4}{N^2}|x'|^2\big\},$$

we obtain $\tilde{h} \le w \le u$ on ∂R_r (since w is the convex envelope of u) and $\tilde{h}(0) = C_r r^2/4 > 0 = w(0) = u(0)$. By lowering $\tilde{h}$ appropriately, we conclude that $u - \tilde{h}$ has a local minimum somewhere inside R_r. Note that the eigenvalues of $D^2\tilde{h}$ are given by $C_r/2, -2C_r/N^2, \cdots, -2C_r/N^2$. Hence by the definition of $\mathcal{S}^+(\lambda, \Lambda, f)$, we have

$$\lambda \frac{C_r}{2} - 2\Lambda(n-1)\frac{C_r}{N^2} \le \max_{R_r} f.$$

By choosing N large, depending only on n, λ and Λ, we obtain

$$C_r \le \frac{4}{\lambda} \max_{R_r} f,$$

or

$$\max_{B_r} w \le \frac{4}{\lambda} r^2 \max_{R_r} f.$$

We finish the proof by noting $\max_{R_r} f \to f(0)$ as $r \to 0$. □

We end this section with a simple consequence of the Calderon-Zygmund decomposition. We first recall some terminology. Let Q_1 be the unit cube. Cut it equally into 2^n cubes, which we take as the first generation. Do the same cutting for these small cubes to get the second generation. Continue this process. These cubes (from all generations) are called *dyadic cubes*. Any $(k+1)$-generation cube Q arises from some k-generation cube $\tilde{Q}$, which is called the *predecessor* of Q.

LEMMA 5.9. *Let $A \subset B \subset Q_1$ be measurable sets such that*

(i) $|A| < \delta$ for some $\delta \in (0, 1)$;

(ii) for any dyadic cube Q, $|A \cap Q| \ge \delta|Q|$ implies $\tilde{Q} \subset B$ for the predecessor $\tilde{Q}$ of Q.

Then

$$|A| \le \delta|B|.$$

Proof. Apply the Calderon-Zygmund decomposition, Lemma 3.7 in Chapter 3, to $f = \chi_A$. We obtain, by the assumption (i), a sequence of dyadic cubes $\{Q^j\}$ such that

$$A \subset \cup_j Q^j \text{ except for a set of measure zero,}$$

$$\delta \le \frac{|A \cap Q^j|}{|Q^j|} < 2^n \delta,$$

and

$$\frac{|A \cap \tilde{Q}^j|}{|\tilde{Q}^j|} < \delta,$$

for any predecessor $\tilde{Q}^j$ of Q^j. By the assumption (ii), we have $\tilde{Q}^j \subset B$ for each j. Hence, we obtain

$$A \subset \cup_j \tilde{Q}^j \subset B.$$

We relabel $\{\tilde{Q}^j\}$ so that they are nonoverlapping. Therefore, we get

$$|A| \le \sum_i |A \cap \tilde{Q}^i| \le \delta \sum_i |\tilde{Q}^i| \le \delta |B|.$$

This finishes the proof. $\square$

5.2. The Harnack Inequality

The main result in this section is the following Harnack inequality.

Theorem 5.10. *Suppose u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in B_1 for some $f \in C(B_1)$ with $u \ge 0$ in B_1. Then*

$$\sup_{B_{\frac{1}{2}}} u \le c\left\{ \inf_{B_{\frac{1}{2}}} u + \|f\|_{L^n(B_1)} \right\},$$

where c is a positive constant depending only on n, λ and Λ.

The interior Hölder continuity of solutions is a direct consequence, whose proof is identical to that of Theorem 4.17 in Chapter 4.

Corollary 5.11. *Suppose u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in B_1 for some $f \in C(B_1)$. Then $u \in C^\alpha(B_1)$, for some $\alpha \in (0,1)$ depending only on n, λ and Λ. Moreover,*

$$|u(x) - u(y)| \le c|x-y|^\alpha \left\{ \sup_{B_1} |u| + \|f\|_{L^n(B_1)} \right\} \quad \text{for any } x, y \in B_{\frac{1}{2}},$$

where c is a positive constant depending only on n, λ and Λ.

For convenience, we work in cubes instead of balls. We prove the following result.

Lemma 5.12. *Suppose u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ for some $f \in C(Q_{4\sqrt{n}})$, with $u \ge 0$ in $Q_{4\sqrt{n}}$. Then there exist two positive constants ε_0 and C, depending only on n, λ and Λ, such that, if $\inf_{Q_{1/4}} u \le 1$ and $\|f\|_{L^n(Q_{4\sqrt{n}})} \le \varepsilon_0$, then $\sup_{Q_{\frac{1}{4}}} u \le C$.*

Theorem 5.10 follows from Lemma 5.12 easily. For any $u \in \mathcal{S}(\lambda, \Lambda, f)$ in $Q_{4\sqrt{n}}$ with $u \geq 0$ in $Q_{4\sqrt{n}}$, consider for $\delta > 0$

$$u_\delta = \frac{u}{\inf_{Q_{1/4}} u + \delta + \frac{1}{\varepsilon_0}\|f\|_{L^n(Q_{4\sqrt{n}})}}.$$

Applying Lemma 5.12 to u_δ and then letting $\delta \to 0$, we get

$$\sup_{Q_{\frac{1}{4}}} u \leq C\Big\{\inf_{Q_{\frac{1}{4}}} u + \|f\|_{L^n(Q_{4\sqrt{n}})}\Big\}.$$

Then Theorem 5.10 follows by a standard covering argument.

Now we begin to prove Lemma 5.12. The following result plays a key role. It asserts that if a solution is small somewhere in Q_3 then it is under control in a good portion of Q_1.

LEMMA 5.13. *Suppose u belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist constants $\varepsilon_0 > 0$, $\mu \in (0,1)$ and $M > 1$, depending only on n, λ and Λ, such that if*

$$\begin{aligned} &u \geq 0 \text{ in } B_{2\sqrt{n}}, \\ &\inf_{Q_3} u \leq 1, \\ &\|f\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0, \end{aligned} \tag{1}$$

then

$$|\{u \leq M\} \cap Q_1| > \mu.$$

PROOF. We construct a function g, which is very concave outside Q_1, such that the contact set occurs in Q_1 if we correct u by g. In other words, we localize where the contact occurs by choosing suitable functions.

Note $B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{2\sqrt{n}}$. Define g in $B_{2\sqrt{n}}$ by

$$g(x) = -M\big(1 - \frac{|x|^2}{4n}\big)^\beta,$$

for large $\beta > 0$ to be determined and some $M > 0$. We choose M, according to β, such that

$$g = 0 \text{ on } \partial B_{2\sqrt{n}} \quad \text{and} \quad g \leq -2 \text{ in } Q_3. \tag{2}$$

Set $w = u + g$ in $B_{2\sqrt{n}}$. We prove by choosing β large such that

$$w \in \mathcal{S}^+(\lambda, \Lambda, f) \quad \text{in } B_{2\sqrt{n}} \setminus Q_1. \tag{3}$$

Suppose φ is a quadratic polynomial such that $w - \varphi$ has a local minimum at $x_0 \in B_{2\sqrt{n}}$. Then $u - (\varphi - g)$ has a local minimum at $x_0 \in B_{2\sqrt{n}}$. By the definition of $\mathcal{S}^+(\lambda, \Lambda, f)$ and the Pucci's extremal operator $\mathcal{M}^-$, we have

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0) - D^2 g(x_0)) \leq f(x_0),$$

or

$$\mathcal{M}^-(\lambda, \Lambda, D^2\varphi(x_0)) + \mathcal{M}^-(\lambda, \Lambda, -D^2 g(x_0)) \leq f(x_0),$$

where we used the property of $\mathcal{M}^-$. We choose β large such that

$$\mathcal{M}^-(\lambda, \Lambda, -D^2 g(x_0)) \geq 0 \quad \text{for any } x_0 \in B_{2\sqrt{n}} \setminus B_{\frac{1}{4}}.$$

To see this, we need to calculate the Hessian matrix of g. Note

$$D_{ij}g(x) = \frac{M}{2n}\beta\big(1 - \frac{|x|^2}{4n}\big)^{\beta-1}\delta_{ij} - \frac{M}{(2n)^2}\beta(\beta-1)\big(1 - \frac{|x|^2}{4n}\big)^{\beta-2}x_i x_j.$$

If we choose $x = (|x|, 0, \cdots, 0)$, then the eigenvalues of $-D^2 g(x)$ are given by

$$\frac{M}{2n}\beta\big(1 - \frac{|x|^2}{4n}\big)^{\beta-2}\big(\frac{2\beta-1}{4n}|x|^2 - 1\big),$$

with the multiplicity 1, and

$$-\frac{M}{2n}\beta\big(1 - \frac{|x|^2}{4n}\big)^{\beta-1},$$

with the multiplicity $n-1$. We choose β large so that for $|x| \geq 1/4$ the first eigenvalue is positive and the rest negative, denoted by $e^+(x)$ and $e^-(x)$ respectively. Therefore for $|x| \geq 1/4$, we have

$$\begin{aligned} &\mathcal{M}^-(\lambda, \Lambda, -D^2 g(x)) = \lambda e^+(x) + (n-1)\Lambda e^-(x) \\ =&\frac{M}{2n}\beta\big(1 - \frac{|x|^2}{4n}\big)^{\beta-2}\Big\{\lambda\big(\frac{2\beta-1}{4n}|x|^2 - 1\big) - (n-1)\Lambda\big(1 - \frac{|x|^2}{4n}\big)\Big\} \geq 0, \end{aligned}$$

if we choose β large, depending only on n, λ and Λ. This finishes the proof of (3). In fact, we obtain

$$w \in \mathcal{S}^+(\lambda, \Lambda, f + \eta) \quad \text{in } B_{2\sqrt{n}},$$

for some $\eta \in C_0^\infty(Q_1)$ and $0 \leq \eta \leq C(n, \lambda, \Lambda)$.

We now apply Theorem 5.8 to w in $B_{2\sqrt{n}}$. Note that $\inf_{Q_3} w \leq -1$ and $w \geq 0$ on $\partial B_{2\sqrt{n}}$ by (1) and (2). We obtain

$$\begin{aligned} 1 &\leq C\bigg(\int_{B_{2\sqrt{n}} \cap \{w = \Gamma_w\}} (|f| + \eta)^n\bigg)^{\frac{1}{n}} \\ &\leq C\|f\|_{L^n(B_{2\sqrt{n}})} + C|\{w = \Gamma_w\} \cap Q_1|^{\frac{1}{n}}. \end{aligned}$$

Choosing ε_0 small enough, we get

$$\frac{1}{2} \leq C|\{w = \Gamma_w\} \cap Q_1|^{\frac{1}{n}} \leq C|\{u \leq M\} \cap Q_1|^{\frac{1}{n}},$$

since $w(x) = \Gamma_w(x)$ implies $w(x) \leq 0$ and hence $u(x) \leq -g(x) \leq M$. This finishes the proof. □

Next, we prove the power decay of distribution functions.

LEMMA 5.14. *Suppose u belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist positive constants ε_0, ε and C, depending only on n, λ and Λ, such that if*

$$\begin{aligned} &u \geq 0 \text{ in } B_{2\sqrt{n}}, \\ &\inf_{Q_3} u \leq 1, \\ &\|f\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0, \end{aligned} \tag{1}$$

then

$$|\{u \geq t\} \cap Q_1| \leq Ct^{-\varepsilon} \text{ for any } t > 0.$$

Proof. We prove that, under the assumption (1), there holds

$$|\{u > M^k\} \cap Q_1| \le (1-\mu)^k \text{ for } k = 1, 2, \cdots, \tag{2}$$

where M and μ are as in Lemma 5.13.

For $k = 1$, (2) is simply Lemma 5.13. Suppose now (2) holds for $k-1$. Set

$$A = \{u > M^k\} \cap Q_1, \qquad B = \{u > M^{k-1}\} \cap Q_1.$$

We use Lemma 5.9 to prove

$$|A| \le (1-\mu)|B|. \tag{3}$$

Clearly $A \subset B \subset Q_1$ and $|A| \le |\{u > M\} \cap Q_1| \le 1 - \mu$ by Lemma 5.13. We claim that, if $Q = Q_r(x_0)$ is a cube in Q_1 such that

$$|A \cap Q| > (1-\mu)|Q|, \tag{4}$$

then $\tilde{Q} \cap Q_1 \subset B$ for $\tilde{Q} = Q_{3r}(x_0)$. We prove it by contradiction. If not, we take a $\tilde{x} \in \tilde{Q}$ such that $u(\tilde{x}) \le M^{k-1}$. Consider the transformation

$$x = x_0 + ry \quad \text{for any } y \in Q_1 \text{ and } x \in Q = Q_r(x_0),$$

and the function

$$\tilde{u}(y) = \frac{1}{M^{k-1}} u(x).$$

Then $\tilde{u} \ge 0$ in $B_{2\sqrt{n}}$ and $\inf_{Q_3} \tilde{u} \le 1$. It is easy to check that $\tilde{u} \in \mathcal{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \le \varepsilon_0$. In fact, we have

$$\tilde{f}(y) = \frac{r^2}{M^{k-1}} f(x) \text{ for any } y \in B_{2\sqrt{n}},$$

and hence

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \le \frac{r}{M^{k-1}} \|f\|_{L^n(B_{2\sqrt{n}})} \le \|f\|_{L^n(B_{2\sqrt{n}})} \le \varepsilon_0.$$

Then $\tilde{u}$ satisfies the assumption (1). We now apply Lemma 5.13 to $\tilde{u}$ to get

$$\mu < |\{\tilde{u}(y) \le M\} \cap Q_1| = r^{-n} |\{u(x) \le M^k\} \cap Q|.$$

Hence $|Q \cap A^C| > \mu |Q|$, which contradicts (4). We are in a position to apply Lemma 5.9 to get (3). □

Proof of Lemma 5.12. We prove that there exist two constants $\theta > 1$ and $M_0 >> 1$, depending only on n, λ and Λ, such that, if $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$, there exists a sequence $\{x_k\} \in B_{1/2}$ such that

$$u(x_k) \ge \theta^k P \text{ for } k = 0, 1, 2, \cdots.$$

This contradicts the boundedness of u, and hence we conclude that $\sup_{B_{1/4}} u \le M_0$.

Suppose $u(x_0) = P > M_0$ for an $x_0 \in B_{1/4}$, with M_0 and θ to be determined in the process. Consider a cube $Q_r(x_0)$, centered at x_0 with the side length r, which will be chosen later. We intend to find a point $x_1 \in Q_{4\sqrt{n}r}(x_0)$ such that $u(x_1) \ge \theta P$. To achieve this, we first choose r such that $\{u > P/2\}$ covers less than half of $Q_r(x_0)$. This can be done by using the power decay of the distribution function of u.

Note $\inf_{Q_3} u \le \inf_{Q_{1/4}} u \le 1$. Hence Lemma 5.14 implies

$$|\{u > \frac{P}{2}\} \cap Q_1| \le C\left(\frac{P}{2}\right)^{-\varepsilon}.$$

We choose r such that $r^n/2 \geq C(P/2)^{-\varepsilon}$ and $r \leq 1/4$. Then we have $Q_r(x_0) \subset Q_1$ and

$$\frac{1}{|Q_r(x_0)|}|\{u > P/2\} \cap Q_r(x_0)| \leq \frac{1}{2}. \tag{1}$$

Next, we show that for $\theta > 1$, with $\theta - 1$ small, $u \geq \theta P$ at some point in $Q_{4\sqrt{n}r}(x_0)$. We prove it by contradiction. Suppose $u \leq \theta P$ in $Q_{4\sqrt{n}r}(x_0)$. Consider the transformation

$$x = x_0 + ry \quad \text{for any } y \in Q_{4\sqrt{n}} \text{ and } x \in Q_{4\sqrt{n}r}(x_0),$$

and the function

$$\tilde{u}(y) = \frac{\theta P - u(x)}{(\theta - 1)P}.$$

Obviously $\tilde{u} \geq 0$ in $B_{2\sqrt{n}}$ and $\tilde{u}(0) = 1$, hence $\inf_{Q_3} \tilde{u} \leq 1$. It is easy to check that $\tilde{u} \in \mathcal{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \varepsilon_0$. In fact, we have

$$\tilde{f}(y) = -\frac{r^2}{(\theta - 1)P} f(x) \quad \text{for any } y \in B_{2\sqrt{n}},$$

and hence

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{(\theta - 1)P}\|f\|_{L^n(B_{2\sqrt{n}r}(x_0))} \leq \varepsilon_0,$$

if we choose P such that $r \leq (\theta - 1)P$. Hence, we may apply Lemma 5.13 to $\tilde{u}$. Note that $u(x) \leq P/2$ if and only if $\tilde{u}(y) \geq \dfrac{\theta - 1/2}{\theta - 1}$ and that $\dfrac{\theta - 1/2}{\theta - 1}$ is large if θ is close to 1. So we obtain

$$\begin{aligned}
&\frac{1}{|Q_r(x_0)|}|\{u \leq P/2\} \cap Q_r(x_0)| \\
=&|\{\tilde{u} \geq \frac{\theta - 1/2}{\theta - 1}\} \cap Q_1| \leq C\Big(\frac{\theta - 1/2}{\theta - 1}\Big)^{-\varepsilon} < \frac{1}{2},
\end{aligned}$$

if θ is chosen close to 1. This contradicts (1).

Hence, we conclude that there exists a $\theta > 1$ such that if

$$u(x_0) = P \quad \text{for an } x_0 \in B_{\frac{1}{4}},$$

then

$$u(x_1) \geq \theta P \quad \text{for an } x_1 \in Q_{4\sqrt{n}r}(x_0) \subset B_{2nr}(x_0),$$

provided

$$C(n, \lambda, \Lambda)P^{-\frac{\varepsilon}{n}} \leq r \leq (\theta - 1)P,$$

where θ and C are positive constants depending only on n, λ and Λ. We choose P such that $P \geq \big(\frac{C}{\theta - 1}\big)^{\frac{n}{n+\varepsilon}}$ and then take $r = CP^{-\frac{\varepsilon}{n}}$.

Now, we iterate the above result to get a sequence $\{x_k\}$ such that for any $k = 1, 2, \cdots$,

$$u(x_k) \geq \theta^k P \quad \text{for an } x_k \in B_{2nr_k}(x_{k-1}),$$

where

$$r_k = C(\theta^{k-1}P)^{-\frac{\varepsilon}{n}} = C\theta^{-(k-1)\frac{\varepsilon}{n}}P^{-\frac{\varepsilon}{n}}.$$

In order to have $\{x_k\} \in B_{1/2}$, we need $\sum 2nr_k < 1/4$. Hence, we choose M_0 such that

$$M_0^{\frac{\varepsilon}{n}} \geq 8nC \sum_{k=1}^{\infty} \theta^{-(k-1)\frac{\varepsilon}{n}} \quad \text{and} \quad M_0 \geq \Big(\frac{C}{\theta - 1}\Big)^{\frac{n}{n+\varepsilon}},$$

and then take $P > M_0$. This finishes the proof. $\Box$

In the rest of this section, we prove a technical lemma concerning the second order derivatives of functions in $\mathcal{S}(\lambda, \Lambda, f)$. Such a result will be needed in the discussion of $W^{2,p}$-estimates. First, we introduce some terminology.

Let Ω be a bounded domain and u be a continuous function in Ω. We define for $M > 0$

$$\begin{aligned} G_M^-(u,\Omega) = \{x_0 \in \Omega;\ & \text{there exists an affine function } L \text{ such that} \\ & L(x) - \frac{M}{2}|x - x_0|^2 \leq u(x) \text{ for } x \in \Omega \text{ with equality at } x_0\}, \\ G_M^+(u,\Omega) = \{x_0 \in \Omega;\ & \text{there exists an affine function } L \text{ such that} \\ & L(x) + \frac{M}{2}|x - x_0|^2 \geq u(x) \text{ for } x \in \Omega \text{ with equality at } x_0\}, \\ G_M(u,\Omega) = G_M^+(u,\Omega) & \cap G_M^-(u,\Omega). \end{aligned}$$

We also define

$$\begin{aligned} A_M^-(u,\Omega) =& \Omega \setminus G_M^-(u,\Omega), \\ A_M^+(u,\Omega) =& \Omega \setminus G_M^+(u,\Omega), \\ A_M(u,\Omega) =& \Omega \setminus G_M(u,\Omega). \end{aligned}$$

In other words, $G_M^-(u,\Omega)$ (resp. $G_M^+(u,\Omega)$) consists of points where there is a concave (resp. convex) paraboloid of opening M touching u from below (resp. above). Intuitively, $|A_M(u,\Omega)|$ behaves like the distribution function of D^2u. Hence for integrability of D^2u, we need to study the decay of $|A_M(u,\Omega)|$.

LEMMA 5.15. *Suppose that Ω is a bounded domain with $B_{6\sqrt{n}} \subset \Omega$ and that u belongs to $\mathcal{S}^+(\lambda,\Lambda,f)$ in $B_{6\sqrt{n}}$ for some $f \in C(B_{6\sqrt{n}})$. Then there exist positive constants δ_0, μ and C, depending only on n, λ and Λ, such that, if $|u| \leq 1$ in Ω and $\|f\|_{L^n(B_{6\sqrt{n}})} \leq \delta_0$, there holds*

$$|A_t^-(u,\Omega) \cap Q_1| \leq Ct^{-\mu} \text{ for any } t > 0.$$

If, in addition, $u \in \mathcal{S}(\lambda,\Lambda,f)$ in $B_{6\sqrt{n}}$, there holds

$$|A_t(u,\Omega) \cap Q_1| \leq Ct^{-\mu} \text{ for any } t > 0.$$

In the proof of Lemma 5.15, we need the *maximal functions* of local integrable functions. For any $g \in L^1_{loc}(\mathbb{R}^n)$, we define

$$m(g)(x) = \sup_{r>0} \frac{1}{|Q_r(x)|} \int_{Q_r(x)} |g|.$$

The maximal operator m is of weak type (1,1) and of strong type (p, p) for $1 < p \le \infty$, i.e.,

$$|\{x \in \mathbb{R}^n; m(g)(x) \ge t\}| \le \frac{c_1(n)}{t} \|g\|_{L^1(\mathbb{R}^n)} \quad \text{for any } t > 0$$
$$\|m(g)\|_{L^p(\mathbb{R}^n)} \le c_2(n, p)\|g\|_{L^p(\mathbb{R}^n)} \quad \text{for } \ 1 < p \le \infty.$$

Now we begin to prove Lemma 5.15. The following result plays an important role. It asserts that, if u has a tangent paraboloid with opening 1 from below somewhere in Q_3, then the set where u has a tangent paraboloid from below with opening M in Q_1 is large. Compare it with Lemma 5.13.

LEMMA 5.16. *Suppose that Ω is a bounded domain with $B_{6\sqrt{n}} \subset \Omega$ and that u belongs to $\mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{6\sqrt{n}}$ for some $f \in C(B_{6\sqrt{n}})$. Then there exist constants $0 < \sigma < 1$, $\delta_0 > 0$ and $M > 1$, depending only on n, λ and Λ, such that, if $\|f\|_{L^n(B_{6\sqrt{n}})} \le \delta_0$ and $G_1^-(u, \Omega) \cap Q_3 \ne \phi$, then*

$$|G_M^-(u, \Omega) \cap Q_1| \ge 1 - \sigma.$$

PROOF. Since $G_1^-(u, \Omega) \cap Q_3 \ne \phi$, there is an affine function L_1 such that

$$v \ge P_1 \quad \text{in } \Omega \text{ with equality at some point in } Q_3,$$

where

$$v(x) = \frac{u(x)}{2n} + L_1(x) \quad \text{and} \quad P_1(x) = 1 - \frac{|x|^2}{4n}.$$

This implies $v \ge 0$ in $B_{2\sqrt{n}}$ and $\inf_{Q_3} v \le 1$. Then as in the proof of Lemma 5.13, for $w = v + g$, where g in the function constructed in Lemma 5.13, we have

$$|\{w = \Gamma_w\} \cap Q_1| \ge 1 - \sigma,$$

for some $\sigma \in (0, 1)$ if δ_0 is chosen small. Now we need to prove for some $M > 1$

$$\{w = \Gamma_w\} \cap Q_1 \subset G_M^-(u, \Omega) \cap Q_1.$$

Let $x_0 \in \{w = \Gamma_w\} \cap Q_1$ and take an affine function L_2 with $L_2 < 0$ on $\partial B_{2\sqrt{n}}$ and

$$L_2 \le \Gamma_w \le v + g \quad \text{in } B_{2\sqrt{n}} \text{ with equality at } x_0.$$

It follows

$$P_2 \le L_2 - g \le v \quad \text{in } B_{2\sqrt{n}} \text{ with equality at } x_0, \tag{1}$$

for a concave paraboloid P_2 of opening M_0, a positive constant depending only on n, λ and Λ.

Next, we prove $P_2 \le v$ in $\Omega \setminus B_{2\sqrt{n}}$. Note that $P_2 < -g = 0 = P_1$ on $\partial B_{2\sqrt{n}}$ and that $P_2(x_0) = v(x_0) \ge P_1(x_0)$ with $x_0 \in Q_1 \subset B_{2\sqrt{n}}$. If we take $M_0 > 1/(2n)$, then $\{P_2 - P_1 \ge 0\}$ is convex. We conclude $P_2 - P_1 < 0$ in $\mathbb{R}^n \setminus B_{2\sqrt{n}}$. Hence, we have $P_2 \le P_1 \le v$ in $\Omega \setminus B_{2\sqrt{n}}$. By (1) and the definition of v, we get $x_0 \in G_{2nM_0}^-(u, \Omega) \cap Q_1$ with $2nM_0 > 1$. □

PROOF OF LEMMA 5.15. Recall $B_{6\sqrt{n}} \subset \Omega$, $u \in \mathcal{S}^+(\lambda, \Lambda, f)$ in $B_{6\sqrt{n}}$ and

$$|u|_{L^\infty(\Omega)} \le 1, \qquad \|f\|_{L^n(B_{6\sqrt{n}})} \le \delta_0. \tag{1}$$

We prove that there exist constants $M > 1$ and $0 < \gamma < 1$, depending only on n, λ and Λ, such that

$$|A_{M^k}^-(u, \Omega) \cap Q_1| \le \gamma^k \quad \text{for any } k = 0, 1, \cdots.$$

Step 1. There exist constants $M > 1$ and $0 < \sigma < 1$ such that

$$|G_M^-(u,\Omega) \cap Q_1| \geq 1 - \sigma. \tag{2}$$

It is easy to see that $|u|_{L^\infty(\Omega)} \leq 1$ implies

$$G_c^-(u,\Omega) \cap Q_3 \neq \emptyset,$$

for some constant c depending only on n. We now apply Lemma 5.16 to u/c to get (2). By a simple adjustment, we assume that δ_0, M and σ in Step 1 are the same as those in Lemma 5.15.

Step 2. We extend f by zero outside $B_{6\sqrt{n}}$ and set for $k = 0, 1, \cdots$,

$$\begin{aligned} A &= A^-_{M^{k+1}}(u,\Omega) \cap Q_1, \\ B &= \left(A^-_{M^k}(u,\Omega) \cap Q_1\right) \cup \left\{x \in Q_1; m(f^n)(x) \geq (c_1 M^k)^n\right\}, \end{aligned}$$

for some $c_1 > 0$ to be determined. Then we claim

$$|A| \leq \sigma |B|,$$

where $M > 1$ and $0 < \sigma < 1$ are as before. Recall that $m(f^n)$ denotes the maximal function of f^n.

We prove it by Lemma 5.9. It is easy to see $|A| \leq \sigma$ since we have $|G^-_{M^{k+1}}(u,\Omega) \cap Q_1| \geq |G_M^-(u,\Omega) \cap Q_1| \geq 1 - \sigma$ by Step 1. Next, we claim that, if $Q = Q_r(x_0)$ is a cube in Q_1 such that

$$|A^-_{M^{k+1}}(u,\Omega) \cap Q| = |A \cap Q| > \sigma |Q|, \tag{3}$$

then $\tilde{Q} \cap Q_1 \subset B$ for $\tilde{Q} = Q_{3r}(x_0)$. We prove it by contradiction. If not, we take an $\tilde{x}$ such that

$$\tilde{x} \in G^-_{M^k}(u,\Omega) \cap \tilde{Q},$$

and

$$\sup_{r>0} \frac{1}{|Q_r(\tilde{x})|} \int_{Q_r(\tilde{x})} |f|^n \leq (c_1 M^k)^n.$$

Consider the transformation

$$x = x_0 + ry \quad \text{for any } y \in Q_1 \text{ and } x \in Q = Q_r(x_0),$$

and the function

$$\tilde{u}(y) = \frac{1}{r^2 M^k} u(x).$$

It is easy to check that $B_{6\sqrt{n}} \subset \tilde{\Omega}$, the image of Ω under the transformation above, and that $\tilde{u} \in \mathcal{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{6\sqrt{n}}$ with

$$\tilde{f}(y) = \frac{1}{M^k} f(x) \quad \text{for any } y \in B_{6\sqrt{n}}.$$

By the choice of $\tilde{x}$, we have

$$G_1^-(\tilde{u}, \tilde{\Omega}) \cap Q_3 \neq \emptyset.$$

Since $B_{6\sqrt{n}r}(x_0) \subset Q_{15\sqrt{n}r}(\tilde{x})$, there holds

$$\|\tilde{f}\|_{L^n(B_{6\sqrt{n}})} \leq \frac{1}{rM^k} \|f\|_{L^n(Q_{15\sqrt{n}r}(\tilde{x}))} \leq c(n)c_1 \leq \delta_0,$$

if we take c_1 small enough, depending only on n, λ and Λ.

Hence $\tilde{u}$ satisfies the assumption of Lemma 5.16 with Ω replaced by $\tilde{\Omega}$. We apply Lemma 5.16 to $\tilde{u}$ to get

$$|G_M^-(\tilde{u}, \tilde{\Omega}) \cap Q_1| \geq 1 - \sigma,$$

or

$$|G_{M^{k+1}}^-(u, \Omega) \cap Q| > (1 - \sigma)|Q|.$$

This contradicts (3). We are in a position to apply Lemma 5.9.

Step 3. We finish the proof of Lemma 5.15. Define for $k = 0, 1, \cdots$,

$$\begin{aligned}\alpha_k &= |A_{M^k}^-(u, \Omega) \cap Q_1|,\\ \beta_k &= |\{x \in Q_1; m(f^n)(x) \geq (c_1 M^k)^n\}|.\end{aligned}$$

Then Step 2 implies $\alpha_{k+1} \leq \sigma(\alpha_k + \beta_k)$ for any $k = 0, 1, \cdots$. Hence by an iteration, we have

$$\alpha_k \leq \sigma^k + \sum_{i=0}^{k-1} \sigma^{k-i} \beta_i.$$

Since $\|f^n\|_{L^1} \leq \delta_0^n$ and the maximal operator is of the weak type (1, 1), we conclude

$$\beta_k \leq c(n) \delta_0^n (c_1 M^k)^{-n} = C M^{-nk},$$

where C is a positive constant depending only on n, λ and Λ. This implies

$$\sum_{i=0}^{k-1} \sigma^{k-i} \beta_i \leq C \sum_{i=0}^{k-1} \sigma^{k-i} M^{-ni} \leq C k \gamma_0^k,$$

with $\gamma_0 = \max\{\sigma, M^{-n}\} < 1$. Therefore, we obtain for k large

$$\alpha_k \leq \sigma^k + C k \gamma_0^k \leq (1 + Ck)\gamma_0^k \leq \gamma^k,$$

for some constant $\gamma \in (0, 1)$, depending only on n, λ and Λ. This finishes the proof. □

Remark 5.17. The polynomial decay of the function

$$\mu(t) = |A_t(u, \Omega) \cap Q_1|$$

for $u \in \mathcal{S}(\lambda, \Lambda, f)$ implies that $D^2 u$ is L^p-integrable in Q_1 for small $p > 0$, depending only on n, λ and Λ. In order to show the L^p-integrability for large p, we need to speed up the convergence in the proof of Lemma 5.15. We will discuss $W^{2,p}$-estimates in Section 5.4.

5.3. Schauder Estimates

In this section, we prove the Schauder estimates for viscosity solutions.

Throughout this section, we always assume that $a_{ij} \in C(B_1)$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for any } x \in B_1 \text{ and any } \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ and that f is a continuous function in B_1.

The following approximation result plays an important role in the discussion of the regularity theory.

LEMMA 5.18. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_1,$$

with $|u| \le 1$ in B_1. Assume for some $0 < \varepsilon < 1/16$,

$$\|a_{ij} - a_{ij}(0)\|_{L^n(B_{3/4})} \le \varepsilon.$$

Then there exists a function $h \in C(\bar{B}_{3/4})$, with $a_{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $|h| \le 1$ in $B_{3/4}$, such that

$$|u - h|_{L^\infty(B_{\frac{1}{2}})} \le C\{\varepsilon^\gamma + \|f\|_{L^n(B_1)}\},$$

where C and $\gamma \in (0,1)$ are constants depending only on n, λ and Λ.

PROOF. Solve for $h \in C(\bar{B}_{3/4}) \cap C^\infty(B_{3/4})$ such that

$$\begin{aligned} a_{ij}(0)D_{ij}h &= 0 \quad \text{in } B_{3/4} \\ h &= u \quad \text{on } \partial B_{3/4}. \end{aligned}$$

The maximum principle implies $|h| \le 1$ in $B_{3/4}$. Note that u belongs to $\mathcal{S}(\lambda, \Lambda, f)$ in B_1. Corollary 5.11 implies $u \in C^\alpha(\bar{B}_{3/4})$ for some $\alpha \in (0,1)$ depending only on n, λ and Λ, with the estimate

$$\|u\|_{C^\alpha(\bar{B}_{3/4})} \le C\{1 + \|f\|_{L^n(B_1)}\},$$

where C is a positive constant depending only on n, λ and Λ. By Lemma 1.36 in Chapter 1, we have

$$\|h\|_{C^{\frac{\alpha}{2}}(\bar{B}_{3/4})} \le C\|u\|_{C^\alpha(\bar{B}_{3/4})} \le C\{1 + \|f\|_{L^n(B_1)}\}.$$

Since $u - h = 0$ on $\partial B_{3/4}$, we get for any $0 < \delta < 1/4$

$$|u - h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} \le C\delta^{\frac{\alpha}{2}}\{1 + \|f\|_{L^n(B_1)}\}. \tag{1}$$

We claim for any $0 < \delta < 1$

$$|D^2h|_{L^\infty(B_{\frac{3}{4}-\delta})} \le C\delta^{\frac{\alpha}{2}-2}\{1 + \|f\|_{L^n(B_1)}\}. \tag{2}$$

In fact, for any $x_0 \in B_{3/4-\delta}$, we apply interior C^2-estimate to $h - h(x_1)$ in $B_\delta(x_0) \subset B_{3/4}$ for some $x_1 \in \partial B_\delta(x_0)$ and obtain

$$|D^2h(x_0)| \le C\delta^{-2} \sup_{B_\delta(x_0)} |h - h(x_1)| \le C\delta^{-2}\delta^{\frac{\alpha}{2}}\{1 + \|f\|_{L^n(B_1)}\}.$$

Note that $u - h$ is a viscosity solution of

$$a_{ij}D_{ij}(u - h) = f - (a_{ij} - a_{ij}(0))D_{ij}h \equiv F \quad \text{in } B_{3/4}.$$

By Theorem 5.8, Alexandroff maximum principle, we have with (1) and (2)

$$\begin{aligned} &|u - h|_{L^\infty(B_{\frac{3}{4}-\delta})} \le |u - h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} + C\|F\|_{L^n(B_{\frac{3}{4}-\delta})} \\ \le &|u - h|_{L^\infty(\partial B_{\frac{3}{4}-\delta})} + C|D^2h|_{L^\infty(B_{\frac{3}{4}-\delta})}\|a_{ij} - a_{ij}(0)\|_{L^n(B_{\frac{3}{4}})} + C\|f\|_{L^n(B_1)} \\ \le &C(\delta^{\frac{\alpha}{2}} + \delta^{\frac{\alpha}{2}-2}\varepsilon)\{1 + \|f\|_{L^n(B_1)}\} + C\|f\|_{L^n(B_1)}. \end{aligned}$$

Take $\delta = \varepsilon^{1/2} < 1/4$ and then $\gamma = \alpha/4$. This finishes the proof. □

For the next result, we need to introduce the following concept.

DEFINITION 5.19. A function g is Hölder continuous at 0 with exponent α in L^n-sense if

$$[g]_{C^{\alpha}_{L^n}}(0) \equiv \sup_{0<r<1} \frac{1}{r^{\alpha}} \left(\frac{1}{|B_r|} \int_{B_r} |g - g(0)|^n \right)^{\frac{1}{n}} < \infty.$$

Now we state the Schauder estimates.

THEOREM 5.20. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij} D_{ij} u = f \quad \text{in } B_1.$$

Assume $\{a_{ij}\}$ is Hölder continuous at 0 with exponent α in L^n-sence for some $\alpha \in (0,1)$. If f is Hölder continuous at 0 with exponent α in L^n-sense, then u is $C^{2,\alpha}$ at 0. Moreover, there exists a polynomial P of degree 2 such that

$$\begin{aligned} &|u - P|_{L^\infty(B_r(0))} \le C_* r^{2+\alpha} \quad \text{for any } 0 < r < 1,\\ &|P(0)| + |DP(0)| + |D^2 P(0)| \le C_*, \end{aligned}$$

and

$$C_* \le C\big\{ |u|_{L^\infty(B_1)} + |f(0)| + [f]_{C^{\alpha}_{L^n}}(0) \big\},$$

where C is a positive constant depending only on $n, \lambda, \Lambda, \alpha$ and $[a_{ij}]_{C^{\alpha}_{L^n}}(0)$.

PROOF. First we assume $f(0) = 0$. For this, we consider $v = u - b_{ij} x_i x_j f(0)/2$ for a constant matrix $\{b_{ij}\}$ such that $a_{ij}(0) b_{ij} = 1$. By scaling, we also assume that $[a_{ij}]_{C^{\alpha}_{L^n}}(0)$ is small. Next, by considering for $\delta > 0$

$$\frac{u}{|u|_{L^\infty(B_1)} + \frac{1}{\delta} [f]_{C^{\alpha}_{L^n}}(0)},$$

we may assume $|u|_{L^\infty(B_1)} \le 1$ and $[f]_{C^{\alpha}_{L^n}}(0) \le \delta$.

In the following, we prove that there is a constant $\delta > 0$, depending only on n, λ, Λ and α, such that, if $u \in C(B_1)$ is a viscosity solution of

$$a_{ij} D_{ij} u = f \quad \text{in } B_1,$$

with

$$\begin{gathered} |u|_{L^\infty(B_1)} \le 1, \qquad [a_{ij}]_{C^{\alpha}_{L^n}}(0) \le \delta, \\ \left(\frac{1}{|B_r|} \int_{B_r} |f|^n \right)^{\frac{1}{n}} \le \delta r^{\alpha} \quad \text{for any } 0 < r < 1, \end{gathered}$$

then there exists a polynomial P of degree 2 such that

$$|u - P|_{L^\infty(B_r(0))} \le C r^{2+\alpha} \quad \text{for any } 0 < r < 1, \tag{1}$$

and

$$|P(0)| + |DP(0)| + |D^2 P(0)| \le C, \tag{2}$$

where C is a positive constant depending only on n, λ, Λ and α.

We claim that there exist a $\mu \in (0,1)$, depending only on n, λ, Λ and α, and a sequence of polynomials of degree 2 of the form

$$P_k(x) = a_k + b_k \cdot x + \frac{1}{2} x^t C_k x,$$

such that for any $k = 0, 1, 2, \cdots$,

$$
\begin{aligned}
&a_{ij}(0)D_{ij}P_k = 0,\\
&|u - P_k|_{L^\infty(B_{\mu^k})} \le \mu^{k(2+\alpha)},
\end{aligned}
\tag{3}
$$

and

$$
|a_k - a_{k-1}| + \mu^{k-1}|b_k - b_{k-1}| + \mu^{2(k-1)}|C_k - C_{k-1}| \le C\mu^{(k-1)(2+\alpha)},
\tag{4}
$$

where $P_0 = P_{-1} \equiv 0$, and C is a positive constant depending only on n, λ, Λ and α.

We first prove that Theorem 5.20 follows from (3) and (4). It is easy to see that a_k, b_k and C_k converge and that the limiting polynomial

$$p(x) = a_\infty + b_\infty \cdot x + \frac{1}{2}x^t C_\infty x$$

satisfies

$$|P_k(x) - p(x)| \le C\big(|x|^2\mu^{\alpha k} + |x|\mu^{(\alpha+1)k} + \mu^{(\alpha+2)k}\big) \le C\mu^{(2+\alpha)k},$$

for any $|x| \le \mu^k$. Hence, we have for $|x| \le \mu^k$

$$|u(x) - p(x)| \le |u(x) - P_k(x)| + |P_k(x) - p(x)| \le C\mu^{(2+\alpha)k},$$

and hence

$$|u(x) - p(x)| \le C|x|^{2+\alpha} \quad \text{for any } x \in B_1.$$

Now we prove (3) and (4). Clearly (3) and (4) hold for $k = 0$. We assume they hold for $k = 0, 1, 2, \cdots, l$ and proceed to prove for $k = l + 1$. Consider the function

$$\tilde{u}(y) = \frac{1}{\mu^{l(2+\alpha)}}(u - P_l)(\mu^l y) \quad \text{for any } y \in B_1.$$

Then $\tilde{u} \in C(B_1)$ is a viscosity solution of

$$\tilde{a}_{ij}D_{ij}\tilde{u} = \tilde{f} \quad \text{in } B_1,$$

with

$$
\begin{aligned}
\tilde{a}_{ij}(y) &= \frac{1}{\mu^{l\alpha}}a_{ij}(\mu^l y),\\
\tilde{f}(y) &= \frac{1}{\mu^{l\alpha}}\big(f(\mu^l y) - a_{ij}(\mu^l y)D_{ij}P_k\big).
\end{aligned}
$$

Now we check that $\tilde{u}$ satisfies the assumptions of Lemma 5.18. For this, we note

$$\|\tilde{a}_{ij} - \tilde{a}_{ij}(0)\|_{L^n(B_1)} \le \frac{1}{\mu^{l\alpha}}\|a_{ij} - a_{ij}(0)\|_{L^n(B_{\mu^l})} \le [a_{ij}]_{C^\alpha_{L^n}}(0) \le \delta,$$

and

$$\|\tilde{f}\|_{L^n(B_1)} \le \frac{1}{\mu^{l\alpha}}\|f\|_{L^n(B_{\mu^l})} + \frac{1}{\mu^{l\alpha}}\sup|D^2P_l|\|a_{ij} - a_{ij}(0)\|_{L^n(B_{\mu^l})} \le \delta + C\delta,$$

where we used

$$|D^2P_l| \le \sum_{k=1}^{l}|D^2P_k - D^2P_{k-1}| \le \sum_{k=1}^{l}\mu^{(k-1)\alpha} \le C.$$

Hence we take $\varepsilon = C(n, \lambda, \Lambda)\delta$ as in Lemma 5.18. Then by Lemma 5.18, there exists a function $h \in C(\bar{B}_{3/4})$ with $\tilde{a}_{ij}(0)D_{ij}h = 0$ in $B_{3/4}$ and $|h| \le 1$ in $B_{3/4}$ such that

$$|\tilde{u} - h|_{L^\infty(B_{\frac{1}{2}})} \le C\big(\varepsilon^\gamma + \varepsilon\big) \le 2C\varepsilon^\gamma.$$

Write $\tilde{P}(y) = h(0) + Dh(0) + y^t D^2 h(0) y/2$. Then by interior estimates for h we have

$$|\tilde{u} - \tilde{P}|_{L^\infty(B_\mu)} \le |\tilde{u} - h|_{L^\infty(B_\mu)} + |h - \tilde{P}|_{L^\infty(B_\mu)} \le 2C\varepsilon^\gamma + C\mu^3 \le \mu^{2+\alpha},$$

by choosing μ small and then ε small accordingly. Rescaling back, we have

$$|u(x) - P_l(x) - \mu^{l(2+\alpha)}\tilde{P}(\mu^{-l}x)| \le \mu^{(l+1)(2+\alpha)} \quad \text{for any } x \in B_{\mu^{l+1}}.$$

This implies (3) for $k = l+1$, if we define

$$P_{k+1}(x) = P_k(x) + \mu^{l(2+\alpha)}\tilde{P}(\mu^{-l}x).$$

The estimate (4) follows easily. □

To finish this section, we state a Cordes-Nirenberg type estimate. The proof is similar to that of Theorem 5.20.

THEOREM 5.21. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_1.$$

Then for any $\alpha \in (0,1)$, there exists an $\theta > 0$, depending only on n, λ, Λ and α, such that if

$$\left(\frac{1}{|B_r|}\int_{B_r} |a_{ij} - a_{ij}(0)|^n\right)^{\frac{1}{n}} \le \theta \quad \text{for any } 0 < r \le 1,$$

then u is $C^{1,\alpha}$ at 0; that is, there exists an affine function L such that

$$\begin{aligned} &|u - L|_{L^\infty(B_r(0))} \le C_* r^{1+\alpha} \quad \text{for any } 0 < r < 1, \\ &|L(0)| + |DL(0)| \le C_*, \end{aligned}$$

and

$$C_* \le C\left\{|u|_{L^\infty(B_1)} + \sup_{0<r<1} r^{1-\alpha}\Big(\frac{1}{|B_r|}\int_{B_r}|f|^n\Big)^{\frac{1}{n}}\right\},$$

where C is a positive constant depending only on n, λ, Λ and α.

5.4. $W^{2,p}$ Estimates

In this section, we prove $W^{2,p}$-estimates for viscosity solutions.

Throughout this section, we always assume that $a_{ij} \in C(B_1)$ satisfies

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } x \in B_1 \text{ and any } \xi \in \mathbb{R}^n,$$

for some positive constants λ and Λ and that f is a continuous function in B_1.

The main result in this section is the following theorem.

THEOREM 5.22. *Suppose $u \in C(B_1)$ is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_1.$$

Then for any $p \in (n, \infty)$, there exists an $\varepsilon > 0$, depending only on n, λ, Λ and p, such that if

$$\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |a_{ij} - a_{ij}(x_0)|^n\right)^{\frac{1}{n}} \le \varepsilon \quad \text{for any } B_r(x_0) \subset B_1,$$

then $u \in W^{2,p}_{loc}(B_1)$. *Moreover,*

$$\|u\|_{W^{2,p}(B_{\frac{1}{2}})} \le C\big\{|u|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\big\},$$

where C *is a positive constant depending only on* n, λ, Λ *and* p.

As before, we prove the following result instead.

LEMMA 5.23. *Suppose* $u \in C(B_{8\sqrt{n}})$ *is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_{8\sqrt{n}}.$$

Then for any $p \in (n, \infty)$, *there exist positive constants* ε *and* C, *depending only on* n, λ, Λ *and* p, *such that if*

$$\|u\|_{L^\infty(B_{8\sqrt{n}})} \le 1, \quad \|f\|_{L^p(B_{8\sqrt{n}})} \le \varepsilon,$$

and

$$\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |a_{ij} - a_{ij}(x_0)|^n\right)^{\frac{1}{n}} \le \varepsilon \quad \text{for any } B_r(x_0) \subset B_{8\sqrt{n}},$$

then $u \in W^{2,p}(B_1)$ *and* $\|u\|_{W^{2,p}(B_1)} \le C$.

We first describe the strategy of the proof. Let Ω be a bounded domain and u be a continuous function in Ω. As in Section 5.2, we define for $M > 0$

$$\begin{aligned} G_M(u,\Omega) = \{x_0 \in \Omega;\ & \text{there exists an affine function } L \text{ such that} \\ & L(x) - \frac{M}{2}|x-x_0|^2 \le u(x) \le L(x) + \frac{M}{2}|x-x_0|^2 \\ & \text{for } x \in \Omega \text{ with equality at } x_0\}, \\ A_M(u,\Omega) = \Omega &\setminus G_M(u,\Omega). \end{aligned}$$

We consider the function

$$\theta(x) = \theta(u,\Omega)(x) = \inf\{M; x \in G_M(u,\Omega)\} \in [0,\infty] \quad \text{for } x \in \Omega.$$

It is straightforward to verify that for $p \in (1,\infty]$ the condition $\theta \in L^p(\Omega)$ implies $D^2u \in L^p(\Omega)$ and

$$\|D^2u\|_{L^p(\Omega)} \le 2\|\theta\|_{L^p(\Omega)}.$$

In order to study the integrability of the function θ, we discuss its distribution function

$$\mu_\theta(t) = |\{x \in \Omega; \theta(x) > t\}| \quad \text{for any } t > 0.$$

It is clear that

$$\mu_\theta(t) \le |A_t(u,\Omega)| \quad \text{for any } t > 0.$$

Hence we need to study the decay of $|A_t(u,\Omega)|$.

LEMMA 5.24. *Suppose that* Ω *is a bounded domain with* $B_{8\sqrt{n}} \subset \Omega$ *and that* $u \in C(\Omega)$ *is a viscosity solution of*

$$a_{ij}D_{ij}u = f \quad \text{in } B_{8\sqrt{n}}.$$

Then for any $\varepsilon_0 \in (0,1)$, *there exist an* $M > 1$, *depending only on* n, λ *and* Λ, *and an* $\varepsilon \in (0,1)$, *depending only on* n, λ, Λ *and* ε_0, *such that if*

$$\|f\|_{L^n(B_{8\sqrt{n}})} \le \varepsilon, \quad \|a_{ij} - a_{ij}(0)\|_{L^n(B_{7\sqrt{n}})} \le \varepsilon, \tag{1}$$

and

$$G_1(u,\Omega)\cap Q_3 \neq \emptyset, \tag{2}$$

then

$$|G_M(u,\Omega)\cap Q_1| \geq 1-\varepsilon_0.$$

PROOF. Let $x_1 \in G_1(u,\Omega)\cap Q_3$. Then, there exists an affine function L such that

$$-\frac{1}{2}|x-x_1|^2 \leq u(x)-L(x) \leq \frac{1}{2}|x-x_1|^2 \quad \text{in } \Omega.$$

By considering $(u-L)/c$ instead of u, for a constant $c>1$ large enough, depending only on n, we may assume

$$|u| \leq 1 \quad \text{in } B_{8\sqrt{n}}, \tag{3}$$

and hence

$$-|x|^2 \leq u(x) \leq |x|^2 \quad \text{for any } x \in \Omega \setminus B_{6\sqrt{n}}. \tag{4}$$

Solve for $h \in C(\bar{B}_{7\sqrt{n}}) \cap C^\infty(B_{7\sqrt{n}})$ such that

$$\begin{aligned} a_{ij}(0)D_{ij}h &= 0 \quad \text{in } B_{7\sqrt{n}} \\ h &= u \quad \text{on } \partial B_{7\sqrt{n}}. \end{aligned}$$

Then Lemma 5.18 implies

$$|u-h|_{L^\infty(B_{6\sqrt{n}})} \leq C\big\{\varepsilon^\gamma + \|f\|_{L^n(B_{8\sqrt{n}})}\big\}, \tag{5}$$

and

$$\|h\|_{C^2(B_{6\sqrt{n}})} \leq C, \tag{6}$$

where $C>0$ and $\gamma \in (0,1)$, as in Lemma 5.18, depending only on n, λ and Λ. Consider $h|_{\bar{B}_{6\sqrt{n}}}$. Extend h outside $\bar{B}_{6\sqrt{n}}$ continuously such that $h=u$ in $\Omega \setminus B_{7\sqrt{n}}$ and $|u-h|_{L^\infty(\Omega)} = |u-h|_{L^\infty(B_{6\sqrt{n}})}$. Note $|h|\leq 1$ in Ω. It follows that $|u-h|_{L^\infty(\Omega)} \leq 2$ and hence with (4)

$$-2-|x|^2 \leq h(x) \leq 2+|x|^2 \quad \text{for any } x \in \Omega \setminus \bar{B}_{6\sqrt{n}}.$$

Then there exists an $N>1$, depending only on n, λ and Λ, such that

$$Q_1 \subset G_N(h,\Omega). \tag{7}$$

Consider

$$w = \frac{\min\{1,\delta_0\}}{2C\varepsilon^\gamma}(u-h),$$

where δ_0 is the constant in Lemma 5.15, and C and γ are constants in (5) and (6). It is easy to check that w satisfies the hypothesis of Lemma 5.15 in Ω. We now apply Lemma 5.15 to get

$$|A_t(w,\Omega)\cap Q_1| \leq Ct^{-\mu} \quad \text{for any } t>0.$$

Therefore, we have

$$|A_s(u-h,\Omega)\cap Q_1| \leq C\varepsilon^{\gamma\mu}s^{-\mu} \quad \text{for any } s>0.$$

It follows that

$$|G_N(u-h,\Omega)\cap Q_1| \geq 1-C_1\varepsilon^{\gamma\mu} \geq 1-\varepsilon_0,$$

if we choose $\varepsilon \in (0,1)$ small, depending only on n, λ, Λ and ε_0. With (7) we get

$$|G_{2N}(u,\Omega) \cap Q_1| \geq 1 - \varepsilon_0.$$

This finishes the proof. $\square$

REMARK 5.25. In fact, we proved Lemma 5.24 with the assumption (2) replaced by (3).

PROOF OF LEMMA 5.23. *Step 1.* For any $\varepsilon_0 \in (0,1)$, there exist an $M > 1$, depending only on n, λ and Λ, and an $\varepsilon \in (0,1)$, depending only on n, λ, Λ and ε_0, such that, under the assumptions of Lemma 5.23, there holds

$$|G_M(u, B_{8\sqrt{n}}) \cap Q_1| \geq 1 - \varepsilon_0. \tag{1}$$

We remark that M does not depend on ε_0. In fact, we have $|u| \leq 1 \leq |x|^2$ in $B_{8\sqrt{n}} \setminus B_{6\sqrt{n}}$. We may apply Lemma 5.24 to get (1) with $\Omega = B_{8\sqrt{n}}$ (see Remark 5.25).

Step 2. We set, for $k = 0, 1, \cdots$,

$$A = A_{M^{k+1}}(u, B_{8\sqrt{n}}) \cap Q_1,$$
$$B = \big(A_{M^k}(u, B_{8\sqrt{n}}) \cap Q_1\big) \cup \big\{x \in Q_1; m(f^n)(x) \geq (c_1 M^k)^n\big\},$$

for some $c_1 > 0$ to be determined, depending only on n, λ, Λ and ε_0. Then there holds

$$|A| \leq \varepsilon_0 |B|.$$

The proof is identical to that of Lemma 5.15.

Step 3. We finish the proof of Lemma 5.24. We take ε_0 such that

$$\varepsilon_0 M^p = \frac{1}{2},$$

where M, depending only on n, λ and Λ, is as in Step 1. Hence the constants ε and c_1 depend only on n, λ, Λ and p. Define for $k = 0, 1, \cdots$,

$$\alpha_k = |A_{M^k}(u, B_{8\sqrt{n}}) \cap Q_1|,$$
$$\beta_k = |\big\{x \in Q_1; m(f^n)(x) \geq (c_1 M^k)^n\big\}|.$$

Then Step 2 implies $\alpha_{k+1} \leq \varepsilon_0(\alpha_k + \beta_k)$ for any $k = 0, 1, \cdots$. Hence, by an iteration we have

$$\alpha_k \leq \varepsilon_0^k + \sum_{i=1}^{k-1} \varepsilon_0^{k-i} \beta_i.$$

Since $f^n \in L^{p/n}$ and the maximal operator is of strong type (p,p), we conclude that $m(f^n) \in L^{p/n}$ and

$$\|m(f^n)\|_{L^{\frac{p}{n}}} \leq C \|f\|_{L^p}^n \leq C.$$

Then the definition of β_k implies

$$\sum_{k \geq 0} M^{pk} \beta_k \leq C.$$

As before, we set

$$\theta(x) = \theta(u, B_{\frac{1}{2}})(x) = \inf\{M; x \in G_M(u, B_{\frac{1}{2}})\} \in [0, \infty] \quad \text{for } x \in B_{\frac{1}{2}},$$

and
$$\mu_\theta(t) = |\{x \in B_{\frac{1}{2}}; \theta(x) > t\}| \quad \text{for any } t > 0.$$
The proof will be finished if we show
$$\|\theta\|_{L^p(B_{\frac{1}{2}})} \leq C.$$
It is clear that
$$\mu_\theta(t) \leq |A_t(u, B_{\frac{1}{2}})| \leq |A_t(u, B_{8\sqrt{n}}) \cap Q_1| \quad \text{for any } t > 0.$$
It suffices to prove, with the definition of α_k, that
$$\sum_{k \geq 1} M^{pk} \alpha_k \leq C.$$
In fact, we have
$$\begin{aligned}
\sum_{k \geq 1} M^{pk} \alpha_k &\leq \sum_{k \geq 1} (\varepsilon_0 M^p)^k + \sum_{k \geq 1} \sum_{i=0}^{k-1} \varepsilon_0^{k-i} M^{p(k-i)} M^{pi} \beta_i \\
&\leq \sum_{k \geq 1} 2^{-k} + \big(\sum_{i \geq 0} M^{pi} \beta_i\big)\big(\sum_{j \geq 1} 2^{-j}\big) \leq C.
\end{aligned}$$
This finishes the proof. □

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