THE BOCHNER INTEGRAL

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1. Basic notions

In this lecture, we present an overview of the theory of Bochner integration, a vector-valued generalization of the theory of Lebesgue integration. Specifically, we introduce an appropriate notion of measurability for maps on a measure space that take values in a Banach space and develop a basic theory of integration for these maps.

In order to develop a theory of integration, we must first decide on a class of maps to be integrated. Since a Banach space always has plenty of bounded linear functionals, we could easily turn vector-valued maps into scalar-valued functions by letting linear functionals act on them.

We fix a σ-finite complete measure space $(X, \mathcal{M}, \mu)$ and a Banach space $\mathbb{B}$.

Definition 1.1. *$f : X \to \mathbb{B}$ is weakly measurable if $l \circ f$ is measurable for each $l \in \mathbb{B}^*$.*

Is this a good notion of measurability? We expect, for example, to be able to approximate measurable maps with simple maps, just as in the theory of Lebesgue integration. A map $f : X \to \mathbb{B}$ is said to be simple provided it can be written $f = \sum_{i=1}^{n} \chi_{E_i} v_i$, where $v_1, \cdots, v_n \in \mathbb{B}$, and $E_1, \cdots, E_n$ are disjoint measurable subsets of X with $\mu(E_i) < \infty$ for each $i = 1, \cdots, n$.

Definition 1.2. *$f : X \to \mathbb{B}$ is strongly measurable if there exists a sequence of simple maps $\{f_n\}$ such that $f_n \to f$ a.e.*

Definition 1.3. *$f : X \to \mathbb{B}$ is essentially separably valued if there exists a null set $N \subset X$ such that $f(N^c)$ is a separable subset of $\mathbb{B}$.*

Remark 1.1. *Unfortunately, not all weakly measurable maps are strongly measurable.*

Theorem 1.2. (*Pettis measurability theorem*) *$f : X \to \mathbb{B}$ is strongly measurable if and only if it is weakly measurable and essentially separably valued.*

We shall make use of the following lemma:

Lemma 1.3. *Suppose that $\mathbb{B}$ is separable. If $f : X \to \mathbb{B}$ is a weakly measurable map, then $x \mapsto \|f(x)\|_{\mathbb{B}}$ is measurable.*

Proof. We fix $a \in \mathbb{R}$ and let

$$E = \{x : \|f(x)\|_{\mathbb{B}} \leq a\}, E_l = \{x : |l(f(x))| \leq a\}, l \in \mathbb{B}^*.$$

If we can find a sequence $\{l_n\}$ in $\mathbb{B}^*$ such that

$$E = \bigcap_{n=1}^{\infty} E_{l_n},$$

then the claim will follows from the weak measurability of f.

Step 1: claim: there exists a sequence $\{l_k\}$ in $B_{\mathbb{B}^*} = \{l \in \mathbb{B}^* : \|l\| \leq 1\}$ such that each $l \in B_{\mathbb{B}^*}$ has a corresponding subsequence of $\{l_k\}$ that converges pointwise to l on $\mathbb{B}$.

Let $\{v_n\}$ be a dense subset of $\mathbb{B}$. The mapping

$$l \mapsto \varphi_N(l) = (l(v_1), \cdots, l(v_N))$$

send $B_{\mathbb{B}^*}$ to l_2^N. Since l_2^N is separable, there exists a sequence of linear functionals $\{l_{N,k}\}_k$ is $B_{\mathbb{B}^*}$ such that $\{\varphi_N(l_{N,k})\}$ is dense in $\varphi_N(B_{\mathbb{B}^*})$. Therefore, each $l \in B_{\mathbb{B}^*}$ furnishes a sequence $\{k_N\}_N$ of indices such that

$$\|\varphi_N(l_{N,k_N}) - \varphi_N(l)\|_{l_2^N} < \frac{1}{N}.$$

This, in particular, implies that

$$|l_{N,k_N}(v_n) - l(v_n)| < \frac{1}{N},$$

for all $1 \leq n \leq N$, whence

$$\lim_N l_{N,k_N}(v_n) = l(v_n),$$

for all $n \in \mathbb{N}_+$. Fix $v \in \mathbb{B}$, given $\epsilon > 0$, there exists $v_n \in \mathbb{B}$ such that $\|v_n - v\|_{\mathbb{B}} < \epsilon$, by the density. Hence

$$|l_{N,k_N}(v) - l(v)| \leq |l_{N,k_N}(v) - l_{N,k_N}(v_n)| + |l_{N,k_N}(v_n) - l(v_n)| + |l(v_n) - l(v)| < 3\epsilon,$$

for N enough large.

Step 2: By Step 1, there exists a sequence $\{l_n\}$ in $B_{\mathbb{B}^*}$ such that

$$\bigcap_{\|l\| \leq 1} E_l = \bigcap_{n=1}^{\infty} E_{l_n}.$$

Step 3: By the Hahn-Banach theorem, we have

$$E = \bigcap_{\|l\| \leq 1} E_l.$$

□

Proof of Theorem 1.2. ($\Rightarrow$) Let $\{f_n\}$ be a sequence of simple maps on X such that $f_n \to f$ a.e. We note that simple maps are evidently weakly measurable. For each $l \in \mathbb{B}^*$, the continuity of l shows that

$$\lim_n l(f_n(x)) = l(f(x)), \text{ a.e.}$$

Since each $l \circ f_n$ is measurable, the a.e. limit $l \circ f$ is also measurable, whence f is weakly measurable.

We now let N be the set on which f_n does not converge pointwise to f. Observe that

$$f(N^c) \subset \overline{\bigcup_{n=1}^{\infty} f_n(N^c)},$$

it follows that $f(N^c)$ is separable.

($\Leftarrow$) Suppose now that f is weakly measurable and essentially separably valued. We assume without loss of generality that im$f$ is separable. define

$$\hat{\mathbb{B}} = \overline{span(imf)},$$

then $\hat{\mathbb{B}}$ is separable. We fix a dense subset $\{v_n\}$ of $\hat{\mathbb{B}}$. For each $b \in \hat{\mathbb{B}}$, we define $k(N,b)$ to be the smallest integer $1 \le k(N,b) \le N$ such that

$$\|b - v_{k(N,b)}\|_{\mathbb{B}} = \min_{1\le j\le N} \|b - v_j\|_{\mathbb{B}}.$$

By the density of $\{v_n\}$,

$$\lim_N \|b - v_{k(N,b)}\|_{\mathbb{B}} = 0.$$

We set $s'_N(b) = v_{k(N,b)}$ so that $s'_N(b) \to b$, for any $b \in \hat{\mathbb{B}}$. We now define

$$s_N = s'_N \circ f,$$

so that $s_N \to f$ pointwise. Note that im$s_N \subset \{v_1, \cdots, v_N\}$, whence

$$s_N = \sum_{n=1}^{N} v_n \chi_{E_N^n},$$

where $E_N^n = \{x \in X : s_N(x) = v_n\}$. Therefore, showing that f can be approximated by simple maps amounts to proving that each E_N^n is measurable. To this end, we observe that

$$\begin{aligned} E_N^n &= \{x : v_n = v_{k(N,f(x))}\} \\ &= \{x : \|f(x) - v_n\| = \min_{1\le j\le N} \|f(x) - v_j\|\} \\ &\quad \cap (\cap_{p=1}^{n-1}\{x : \|f(x) - v_p\| > \min_{1\le j\le N} \|f(x) - v_j\|\}). \end{aligned}$$

We now let

$$\phi_n(x) = \|f(x) - v_n\|, \varphi_N(x) = \min_{1\le j\le N} \phi_j(x),$$

which are measurable function by Lemma 1.3. Therefore,

$$(\phi_n - \varphi_N)^{-1}(\{0\}) = \{x : \|f(x) - v_n\| = \min_{1\le j\le N} \|f(x) - v_j\|\}$$

and

$$(\phi_p - \varphi_N)^{-1}((0,\infty)) = \{x : \|f(x) - v_p\| > \min_{1\le j\le N} \|f(x) - v_j\|\}$$

are measurable, whence so is E_N^n. Since μ is σ-finite, there exists an increasing sequence $\{E_n\}_n$ of finite measure sets whose union is the whole space. set $f_N = s_N\chi_{E_N}$. This completes the proof. $\square$

Corollary 1.4. *If $\{f_n\}$ is a sequence of strongly measurable maps and $f_n \to f$ a.e. Then f is strongly measurable.*

Proof. By the Pettis measurability theorem, each f_n is weakly measurable and essentially separably valued. It is easy to see that f is also essentially separably valued. For $l \in \mathbb{B}^*$, $l \circ f$ is a a.e. limit of $l \circ f_n$, hence it is measurable. □

Corollary 1.5. *A essentially separably valued map $f : X \to \mathbb{B}$ is strongly measurable if and only if f is measurable, i.e. the preimages of open subsets of $\mathbb{B}$ are measurable.*

Proof. ($\Rightarrow$) Consider an arbitrary open bounded subset U of $\mathbb{B}$. If $\{f_n\}$ is a sequence of simple functions that converge pointwise to f, and if

$$U_r = \{v \in U : d(v, \partial U) > r\},$$

then $f_n^{-1}(U_r)$ is measurable regardless of n and r. Fix $n \in \mathbb{N}_+$. If $x \in \cap_{k\geq n} f_k^{-1}(U_r)$, then $f_n(x) \in U_r$ for all $n \geq k$, whence $f(x) \in \overline{U_r} \subset U_{r/2}$. Therefore, $f^{-1}(U_{r/2}) \supset \cap_{k\geq n} f_k^{-1}(U_r)$, whence

$$f^{-1}(U_{r/2}) \supset \bigcup_{n\geq 1} \bigcap_{k\geq n} f_k^{-1}(U_r).$$

This implies that

$$f^{-1}(U) = \bigcup_{m\geq 1} f^{-1}(U_{1/2m}) \supset \bigcup_{m\geq 1} \bigcup_{n\geq 1} \bigcap_{k\geq n} f_k^{-1}(U_{1/m}).$$

Conversely, if $x \in f^{-1}(U_{1/m})$, then $f(x) \in U_{1/m}$. Since $U_{1/m}$ is open, there exists an $n \in \mathbb{N}_+$ such that $f_k(x) \in U_{1/m}$ for all $k \geq n$. Therefore,

$$f^{-1}(U_{1/m}) \subset \bigcup_{n\geq 1} \bigcap_{k\geq n} f_k^{-1}(U_{1/m}),$$

and it follows that

$$f^{-1}(U) = \bigcup_{m\geq 1} f^{-1}(U_{1/m}) \subset \bigcup_{m\geq 1} \bigcup_{n\geq 1} \bigcap_{k\geq n} f_k^{-1}(U_{1/m}).$$

we conclude that $f^{-1}(U)$ is measurable. This completes the proof. □

2. The Bochner integral

Having studied the basic properties of measurable Banach-valued maps, we are now in a position to study the Bochner integral.

Definition 2.1. *Given a simple map $f : X \to \mathbb{B}$ and a measurable set $E \subset X$, we define*

$$\int_E f d\mu = \sum_{i=1}^{n} \mu(E_i \cap E) v_i \in \mathbb{B},$$

for $f = \sum_{i=1}^n \chi_{E_i} v_i$.

Remark 2.1. *The integral is well defined, i.e. it does not depend on the particular way of writing f, and that it is linear.*

Definition 2.2. *$f : X \to \mathbb{B}$ is Bochner integrable if there exists a sequence $\{f_n\}$ of simple maps such that $f_n \to f$ a.e. (hence $x \mapsto \|f_n - f\|_{\mathbb{B}}$ is measurable) and $\int \|f_n - f\|_{\mathbb{B}} d\mu \to 0$. In this case, we define the Bochner integral of f to be*

$$\int_E f d\mu = \lim_n \int_E f_n d\mu,$$

for any measurable set $E \subset X$.

Proposition 2.2. *A strongly measurable map $f : X \to \mathbb{B}$ is Bochner integrable if and only if $\int \|f\|_{\mathbb{B}} d\mu < \infty$.*

Proof. ($\Leftarrow$) It f is strongly measurable and if $\int \|f\|_{\mathbb{B}} d\mu < \infty$, then we can find a sequence $\{f'_n\}$ of simple maps such that $f'_n \to f$ a.e. We set

$$f_n = (\chi_{\|f'_n\|_{\mathbb{B}} \leq 2\|f\|_{\mathbb{B}}}) f'_n$$

for each n, so that $\{f_n\}$ is a sequence of simple maps such that $f_n \to f$ a.e. Since $\|f_n - f\|_{\mathbb{B}} \leq 3\|f\|_{\mathbb{B}}$ for all n, we invoke the dominated convergence theorem to conclude that $\int \|f_n - f\|_{\mathbb{B}} d\mu \to 0$. □

Definition 2.3. *The Lebesgue-Bochner space of order p, $1 \leq p < \infty$, is the space of all strongly measurable maps $f : X \to \mathbb{B}$ with the condition*

$$\|f\|_{L^p(\mathbb{B})} = \left(\int \|f\|_{\mathbb{B}}^p d\mu \right)^{1/p} < \infty,$$

quotiented out by the almost-everywhere equivalence relation. The Lebesgue-Bochner space of order ∞ is the space of all strongly measurable maps $f : X \to \mathbb{B}$ with the condition

$$\|f\|_{L^\infty(\mathbb{B})} = \inf\{r > 0 : \mu(\{x \in X : \|f(x)\|_{\mathbb{B}} > r\}) = 0\} < \infty,$$

quotiented out by the almost-everywhere equivalence relation.

Because we cannot make sense of or monotone convergence of vector-valued functions, there are no analogues of Fatou's lemma or the monotone convergence theorem. We can, on the other hand, generalize the dominated convergence theorem:

Proposition 2.3. *Fix $1 \leq p < \infty$, and let $\{f_n\}$ be a sequence in $L^p(\mathbb{B})$. If $f_n \to f$ a.e. and if there exists $g \in L^p(X)$ such that $\|f_n\|_{\mathbb{B}} \leq |g|$ a.e. for all n, then $f \in L^p(\mathbb{B})$, and*

$$\|f_n - f\|_{L^p(\mathbb{B})} \to 0.$$

Proposition 2.4. *$L^p(\mathbb{B})$ is complete for all $1 \leq p \leq \infty$.*

3. Furthermore

Now assume that (X, d, μ) is a metric measure space, μ is finite on bounded sets.

Proposition 3.1. *Let $E \subset X$ be measurable. Let $\mathbb{V}$ be another Banach space. Then*
i) For every $f \in L^1(\mathbb{B})$, it holds that

$$\left\| \int_E f d\mu \right\|_{\mathbb{B}} \leq \int_E \|f\|_{\mathbb{B}} d\mu.$$

In particular, the map $L^1(\mathbb{B}) \to \mathbb{B}$ sending f to $\int f d\mu$ is linear and continuous.

ii) The space $Lip_{bs}(X, \mathbb{B})$ is contained and dense in $L^1(\mathbb{B})$.

Proof. To prove its density, it suffices to approximate just the maps of the form $\chi_E v$, where E is bounded. First choose a increasing sequence $\{C_n\}$ of closed subsets of E with $\mu(E \backslash C_n) \to 0$, so that $\chi_{C_n} v \to \chi_E v$ with respect to the $L^1(\mathbb{B})$-norm, then for each n notice that the maps $(1 - kd(\cdot, C_n))^+ v$ belong to $Lip_{bs}(X, \mathbb{B})$ and $L^1(\mathbb{B})$-converge to $\chi_{C_n} v$ as $k \to \infty$. So $Lip_{bs}(X, \mathbb{B})$ is dense in $L^1(\mathbb{B})$. □

iii) If $l : \mathbb{B} \to \mathbb{V}$ is linear continuous and $f \in L^1(\mathbb{B})$, one has that $l \circ f \in L^1(\mathbb{V})$ and

$$l\left(\int_E f d\mu \right) = \int_E l \circ f d\mu.$$

Lemma 3.2. *Let $f \in L^1(\mathbb{B})$ be given. Suppose there exists a closed subspace V of $\mathbb{B}$ such that $f(x) \in V$ holds for a.e.$x \in X$. Then $\int_E f d\mu \in V$ for every $E \subset X$ measurable.*

Proof. We argue by contradiction: suppose $\int_E f d\mu \in V^c$, then we can choose $l \in \mathbb{B}^*$ with $l = 0$ on V and $l(\int_E f d\mu) = 1$ by Hahn-Banach theorem. But the fact that $l(f(x)) = 0$ holds a.e. implies $l(\int_E f d\mu) = \int_E l \circ f d\mu$=0, giving a contradiction. □

Theorem 3.3. (*Hille*) *Let $T : \mathbb{B} \to \mathbb{V}$ be a closed operator. Consider a map $f \in L^1(\mathbb{B})$ that satisfies $f(x) \in D(T)$ for a.e. $x \in X$ and $T \circ f \in L^1(\mathbb{V})$. Then for every $E \subset X$ measurable it holds that $\int_E f d\mu \in D(T)$ and that*

$$T\left(\int_E f d\mu \right) = \int_E T \circ f d\mu.$$

Proof. Define the map $\Phi : X \to \mathbb{B} \times \mathbb{V}$ as $\Phi(x) = (f(x), (T \circ f)(x))$ for a.e. $x \in X$. One can readily check that $\Phi \in L^1(\mathbb{B} \times \mathbb{V})$. Moreover, $\Phi(x) \in Gragh(T)$ for a.e. $x \in X$, whence

$$\left(\int_E f d\mu, \int_E T \circ f d\mu \right) = \int_E \Phi(x) d\mu \in Graph(T).$$

This means that $\int_E f d\mu \in D(T)$ and that $T(\int_E f d\mu) = \int_E T \circ f d\mu$. □

Proposition 3.4. *Let $v : [0,1] \to \mathbb{B}$ be an absolutely continuous curve. Suppose that*

$$v'_t = \lim_{h \to 0} \frac{v_{t+h} - v_t}{h} \in \mathbb{B}$$

exists for a.e. $t \in [0,1]$. *Then the map* $v' : [0,1] \to \mathbb{B}$ *is Bochner integrable and satisfies*

$$v_t - v_s = \int_s^t v'_r dr$$

for any $0 \le s < t \le 1$.

Proof. v is continuous $\Rightarrow im(v)$ is separable $\Rightarrow \overline{span(im(v))}$ is separable

$im(v') \subset \overline{span(im(v))}$ a.e. $\Rightarrow v'$ is essential separable valued

Define w:

$$w(t) = \begin{cases} \lim_{h\to 0} \dfrac{v_{t+h} - v_t}{h}, & \text{if such limit exists,} \\ +\infty, & \text{otherwise.} \end{cases}$$

Claim: w is Borel. To prove it, consider a dense subset $\{r_n\}$ of $\overline{span(im(v))}$. Given any $\epsilon, h > 0$ and $n \in \mathbb{N}_+$, we define the Borel sets $A(\epsilon, n, h)$ and $B(\epsilon, n)$ as follows:

$$A(\epsilon, n, h) := \left\{ t : \left| \frac{v_{t+h} - v_t}{h} - r_n \right| < \epsilon \right\}, B(\epsilon, n) = \bigcup_{0<\delta\in\mathbb{Q}} \bigcap_{h\in(-\delta,\delta)\cap\mathbb{Q}} A(\epsilon, n, h).$$

Hence $\lim_{h\to 0} \dfrac{v_{t+h} - v_t}{h}$ exists if and only if $t \in \bigcap_{j\in\mathbb{N}_+} \bigcup_{n\in\mathbb{N}_+} B(2^{-j}, n)$. Now let us call $C(j,n) = B(2^{-j}, n) \setminus \cup_{i<n} B(2^{-j}, i)$ for any $j, n \in \mathbb{N}_+$. Then the map f_j, defined as

$$w_j(t) := \begin{cases} r_n. & t \in C(j,n) \\ +\infty, & t \notin \cup_n C(j,n) \end{cases}$$

is Borel by construction. Given that $w_j(t) \to w(t)$ for all t, we finally conclude that the map w is Borel. Hence v' is strongly measurable. Since the function $\|v'\|_{\mathbb{B}}$ coincides a.e. with the metric speed $|\dot{v}|$, which belongs to $L^1(0,1)$, we conclude that v' is Bochner integrable.

Claim: $v_t = v_0 + \int_0^t v'_s ds$ for any $t \in [0,1]$.

For every $l \in \mathbb{B}^*$ it holds that $t \mapsto l(v_t) \in \mathbb{R}$ is absolutely continuous, with $\frac{d}{dt} l(v_t) = l(v'_t)$ for a.e. $t \in [0,1]$. Therefore

$$l(v_t) = l(v_0) + \int_0^t \frac{d}{ds} l(v_s) ds = l(v_0 + \int_0^t v'_s ds),$$

which implies that $v_t = v_0 + \int_0^t v'_s ds$. □

Proposition 3.5. (*Lebesgue Points*) *Let* $v : [0,1] \to \mathbb{B}$ *be Bochner integrable. Then*

$$\lim_{h\to 0} \fint_{t-h}^{t+h} \|v_s - v_t\|_{\mathbb{B}} ds = 0$$

for a.e. $t \in [0,1]$.

Proof. Choose a separable set $V \subset \mathbb{B}$ such that $v_t \in V$ for a.e. $t \in [0,1]$ and a sequence $\{w_n\}$ that is dense in V. For any n, the map $t \mapsto \|v_t - w_n\|_{\mathbb{B}}$ belongs to $L^1(0,1)$, hence there exists a measurable set $N_n \subset [0,1]$, with $\mathcal{L}^1(N_n) = 0$, such that

$$\|v_t - w_n\|_{\mathbb{B}} = \lim_{h\to 0} \fint_{t-h}^{t+h} \|v_s - w_n\|_{\mathbb{B}} ds$$

holds for every $t \in [0,1]\backslash N_n$, by Lebesgue differentiation theorem. Call $N := \cup_n N_n$, which is an $\mathcal{L}^1$-negligible measurable subset of $[0,1]$. Therefore for every $t \in [0,1]\backslash N$ one has that

$$\limsup_{h\to 0} \fint_{t-h}^{t+h} \|v_s - v_t\|_{\mathbb{B}} ds \le \limsup_{h\to 0} \fint_{t-h}^{t+h} \|v_s - w_n\|_{\mathbb{B}} + \|v_t - w_n\|_{\mathbb{B}} ds = 2\|v_t - w_n\|_{\mathbb{B}}.$$

By density of $\{w_n\}$ in V, we get the statement. □

Fix two metric measure spaces (X, d_X, μ), (Y, d_Y, ν), with μ finite and ν finite on bounded sets. In the following three results we will distinguish real-valued functions from their equivalence classes up to a.e. equality: namely, we will denote by $f : Y \to \mathbb{R}$ the ν-measurable maps and by $[f]$ the elements of $L^p(\nu)$.

Proposition 3.6. *Let $X \ni x \mapsto [f_x] \in L^p(\nu)$ be any μ-measurable map, $1 \le p < \infty$. Then there exists a choice $(x,y) \mapsto \tilde{f}(x,y)$ of representatives, i.e. $[\tilde{f}(x,\cdot)] = [f_x]$ holds for μ-a.e. $x \in X$, which is $\mu \times \nu$-measurable, Moreover, any two such choices agree $(\mu \times \nu)$-a.e. in $X \times Y$.*

Proof. The statement is clearly verified when $x \mapsto [f_x]$ is a simple map. $x \in E_i, [f_x] = [f^i]$ for some $[f^i] \in L^p(\nu)$, $i = 1, \cdots, n$. Define $\tilde{f}(x,y) = \sum_{i=1}^n \chi_{E_i}(x) f^i(y)$.

For $x \mapsto [f_x]$ generic, define $[f_x^k] := \chi_{A_k}(x)[f_x]$ for $x \in X$, where we set $A_k := \{x \in X : \|[f_x]\|_{L^p(\nu)} \le k\}$. Given that $[f^k]$ belongs to $L^p(L^p(\nu))$, we can choose a sequence of simple maps $[g^n] : X \to L^p(\nu)$ such that $\|[g^n] - [f^k]\|^p_{L^p(L^p(\nu))} \le 2^{-2n}$ for every n. As observed in the first part of the proof, we can choose a representative $\tilde{g}^n : X \times Y \to \mathbb{R}$ for every n. Hence

$$\mu(\{x \in X : \|[g_x^n] - [f_x^k]\|^p_{L^p(\nu)}\} > 2^{-n}) \le 2^{-n}$$

holds for every n. Therefore we have that

$$\mu\bigg(\bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} \{x \in X : \|[g_x^n] - [f_x^k]\|^p_{L^p(\nu)} \le 2^{-n}\}\bigg) = \mu(X).$$

Then the functions $\tilde{g}^n$ converge $(\mu \times \nu)$-a.e. to some limit function $\tilde{f}^k : X \times Y \to \mathbb{R}$, $\tilde{f}^k : X \times Y \to \mathbb{R}$ is $\mu \times \nu$-measurable, which is accordingly a representative of $[f^k]$. To conclude, let us define

$$\tilde{f}(x,y) = \sum_k \chi_{A_k \backslash \cup_{i<k} A_i}(x) \tilde{f}^k(x,y)$$

for every $(x,y) \in X \times Y$. Therefore $\tilde{f}$ is the desired representative of $x \mapsto [f_x]$. □

Proposition 3.7. *Consider the operator* $\Phi : L^p(L^p(\nu)) \mapsto L^p(\mu\times\nu)$ *sending* $x \mapsto [f_x]$ *to (the equivalent class of) one of its representatives* $\tilde{f}$ *found in Proposition* (3.6)*. Then the map is an isometric isomorphism.*

Proof.

$$\begin{aligned}\|[f_\cdot]\|^p_{L^p(L^p(\nu))} &= \int \|[f_x]\|^p_{L^p(\nu)} d\mu(x)\\ &= \int\int |f_x|^p d\nu d\mu(x)\\ &= \int\int |\tilde{f}(x,y)|^p d\nu(y) d\mu(x)\\ &= \int |\tilde{f}(x,y)|^p d(\mu\times\nu),\end{aligned}$$

hence Φ is an isometry. Moreover, the map Φ is linear continuous and injective. In order to conclude, it suffices to show that the image of Φ is dense in $L^p(X\times Y)$. Given any $\tilde{f}\in Lip_{bs}(X\times Y)$, we have that $\lim_{x'\to x}\int|\tilde{f}(x',y)-\tilde{f}(x,y)|^p d\nu(y)=0$ for every $x\in X$, so that $x\mapsto \tilde{f}(x,\cdot)\in L^p(\nu)$ is continuous and accordingly in $L^p(L^p(\nu))$. In other words, we proved that any $\tilde{f}\in Lip_{bs}(X\times Y)$ belongs to the image of Φ. Since $Lip_{bs}(X\times Y)$ is dense in $L^p(X\times Y)$, we thus obtained the statement. □

Proposition 3.8. *Let* $(x\mapsto[f_x])\in L^1(L^1(\nu))$ *and call* $[\tilde{f}]$ *its image under* Φ*. Then*

$$\left(\int [f_x] d\mu(x)\right)(y) = \int \tilde{f}(x,y) d\mu(x)$$

holds for ν*-a.e.* $y\in Y$*.*

Proof. First of all, we define the linear and continuous operator $T_1 : L^1(L^1(\nu)) \to L^1(\nu)$ as $T_1(f) := \int [f_x] d\mu(x)$.

On the other hand, we define the linear and continuous operator $T_2 : L^1(\mu\times\nu)\to L^1(\nu)$ as $T_2(\tilde{f}) = [\int \tilde{f}(x,y)d\mu(x)]$. T_1 and $T_2\circ\Phi$ agree on simple maps. □

Lemma 3.9 (Easy Version of Dunford-Pettis)**.** *Assume that* ν *is finite. Let* $\{f_n\}\subset L^1(\nu)$ *be a sequence with the property: there exists* $g\in L^1(\nu)$ *such that* $|f_n|\le g$ *holds* ν*-a.e. for every* n*. Then there exists a subsequence* $\{n_k\}_k$ *and some function* $f\in L^1(\nu)$ *such that* $f_{n_k}\rightharpoonup f$ *weakly in* $L^1(\nu)$ *and* $|f|\le g$ *holds* ν*-a.e. in* Y*.*

Proof. For any $k\in\mathbb{N}$, denote $f_n^k = \min\{\max\{f_n,-k\},k\}$ and $g_k=\min\{\max\{g,-k\},k\}$. The sequence $\{f_n^k\}$ is bounded in $L^2(\nu)$ for any fixed $k\in\mathbb{N}_+$, thus a diagonalisation argument shows the existence of $\{n_i\}$ and $\{h_k\}\subset L^2(\nu)$ such that $f_{n_i}^k\rightharpoonup h_k$ weakly in $L^2(\nu)$ for all k. In particular, $f_{n_i}^k\rightharpoonup h_k$ weakly in $L^1(\nu)$ for all k. Moreover, one can readily check that

$$|f_{n_i}^k - f_{n_i}^{k'}| \le |g_k - g_{k'}| \tag{3.1}$$

holds ν-a.e. for every $i,k,k'\in\mathbb{N}$. By using (3.1), the lower semicontinuity of $\|\cdot\|_{L^1(\nu)}$ with respect to the weak topology and the dominated convergence theorem, we then deduce that

$$\int |h_k - h_{k'}| d\nu \le \liminf_i \int |f_{n_i}^k - f_{n_i}^{k'}| d\nu \le \int |g_k - g_{k'}| d\nu \to 0, \text{ as } k, k' \to 0,$$

which grants that the sequence $\{h_k\} \subset L^1(\nu)$ is Cauchy. Call $f \in L^1(\nu)$ its limit. To prove that $f_{n_i} \rightharpoonup f$ weakly in $L^1(\nu)$ as $i \to +\infty$, observe that for any $l \in L^\infty(\nu)$ it holds that

$$\begin{aligned}\limsup_i \left| \int (f_{n_i} - f) l d\nu \right| &\le \limsup_i \Big[\int |f_{n_i} - f_{n_i}^k||l| d\nu + \left| \int (h_k - f_{n_i}^k) l d\nu \right| \\ &\qquad + \int |h_k - f||l| d\nu \Big] \\ &\le 2\|g - g_k\|_{L^1(\nu)} \|l\|_{L^\infty(\nu)}.\end{aligned}$$

Finally, in order to prove the ν-a.e. inequality $|f| \le g$ it is clearly sufficient to show that

$$\left| \int f l d\nu \right| \le \int g l d\nu,$$

for every $l \in L^\infty(\nu)$ with $l \ge 0$. Since

$$\left| \int f l d\nu \right| = \lim_i \left| \int f_{n_i} l d\nu \right| \le \liminf_i \int |f_{n_i}| l d\nu \le \int g l d\nu.$$

□

Remark 3.10 (Dunford-Pettis theorem)**.** *For a finite measure space $(X, \mathcal{M}, \mu)$ and bounded sequence $\{f_n\}$ in $L^1(X, \mu)$, the following two properties are equivalent:*

(i) $\{f_n\}$ is uniformly integrable over X.

(ii) Every subsequence of $\{f_n\}$ has a further subsequence that converges weakly in $L^1(\mu)$.

Here, a sequence $\{f_n\}$ in $L^1(\mu)$ is uniformly integrable over X provided for each $\epsilon > 0$, there is $\delta > 0$ such that for any measurable set $E \subset X$, if $\mu(E) < \delta$, then

$$\int_E |f_n| d\mu < \epsilon$$

for all n.

Proposition 3.11. *Assume that ν is finite. Let $f : [0,1] \to L^1(\nu)$ and $g \in L^1(\mathcal{L}_1; L^1(\nu))$ be given. Suppose that*

$$|f_t(y) - f_s(y)| \le \int_s^t g_r(y) dr$$

holds for ν-a.e. $y \in Y$, for all $0 \le s < t \le 1$. Then f is absolutely continuous and $\mathcal{L}_1$-a.e. differentiable. Moreover, its derivative satisfies

$$|f_t'|(y) \le g_t(y),$$

for $\mathcal{L}_1 \times \nu$-a.e. $(t, y) \in [0,1] \times Y$.

Proof. $g \in L^1(L^1(\nu)) \rightsquigarrow \tilde{g} : [0,1] \times Y \to \mathbb{R}$ is $\mathcal{L}_1 \times \nu$-measurable

By assumption,

$$|f_t(y) - f_s(y)| \leq \int \chi_{(s,t)} \tilde{g}(r,y) dr \leq \int \chi_{(s,t)} |\tilde{g}|(r,y) dr$$

holds for ν-a.e. $y \in Y$, for all $0 \leq s < t \leq 1$.

By integrating, we get that

$$\|f_t - f_s\|_{L^1(\nu)} \leq \int_s^t \int |\tilde{g}|(r,y) d\nu(y) dr \leq \int_s^t \|g_r\|_{L^1(\nu)} dr$$

for all $0 \leq s < t \leq 1$.

This proves that $t \to f_t \in L^1(\nu)$ is AC.

We proceed in the following way: let us define $g_t^\epsilon := \frac{1}{\epsilon} \int_t^{t+\epsilon} [g_r] dr$ for every $\epsilon > 0$ and $t \in (0,1)$. Observe that

$$g_t^\epsilon(y) = \frac{1}{\epsilon} \int_t^{t+\epsilon} [g_r] dr(y) = \int \frac{1}{\epsilon} \chi_{(t,t+\epsilon)}(r) \tilde{g}(r,y) dr$$

for ν-a.e. Therefore, we have

$$\begin{aligned} \|g^\epsilon_\cdot\|_{L^1(\mathcal{L}_1 \times \nu)} &= \int_0^1 \|[g_t^\epsilon]\|_{L^1(\nu)} dt \\ &\leq \int \int |\tilde{g}(r,y)| dr d\nu(y) \\ &= \|g_\cdot\|_{L^1(\mathcal{L}_1 \times \nu)}. \end{aligned}$$

Given any map $h \in C_b([0,1] \times Y)$, it clearly holds that $h^\epsilon \to h$ in $L^1(\mathcal{L}_1 \times \nu)$.

$$\int \left| \int \frac{1}{\epsilon} \chi_{(t,t+\epsilon)}(r) \tilde{h}(r,y) - \tilde{h}(t,y) dr \right| d(\mathcal{L}_1 \times \nu) \to 0, \epsilon \to 0.$$

Therefore for any such h one has that

$$\begin{aligned} \|g^\epsilon - g\|_{L^1(\mathcal{L}_1 \times \nu)} &\leq \|g^\epsilon - h^\epsilon\|_{L^1(\mathcal{L}_1 \times \nu)} + \|h^\epsilon - h\|_{L^1(\mathcal{L}_1 \times \nu)} + \|g - h\|_{L^1(\mathcal{L}_1 \times \nu)} \\ &\leq 2\|g - h\|_{L^1(\mathcal{L}_1 \times \nu)} + \|h^\epsilon - h\|_{L^1(\mathcal{L}_1 \times \nu)} \end{aligned}$$

Since $C_b([0,1] \times Y)$ is dense in $L^1(\mathcal{L}_1 \times \nu)$, we conclude that $\|g^\epsilon - g\|_{L^1(\mathcal{L}_1 \times \nu)} \to 0$. Hence, g^ϵ is uniformly integrable. Since

$$\left| \frac{f_{t+\epsilon_n} - f_t}{\epsilon_n} \right| \leq \frac{1}{\epsilon_n} \int_t^{t+\epsilon_n} g_r(y) dr = g_t^{\epsilon_n}(y),$$

we know that $(f_{\cdot+\epsilon_n} - f_\cdot)/\epsilon_n$ is uniformly integrable. By the Dunford-Pettis theorem: up to a not relabeled subsequence, we have that $(f_{\cdot+\epsilon_n} - f_\cdot)/\epsilon_n$ weakly converges in $L^1(\mathcal{L}_1 \times \nu)$ to some function $f' \in L^1(\mathcal{L}_1 \times \nu)$. Moreover, simple computations yield

$$\int_s^t \frac{f_{r+\epsilon_n} - f_r}{\epsilon_n} dr = \fint_t^{t+\epsilon_n} f_r dr - \fint_s^{s+\epsilon_n} f_r dr$$

for every $0 < s < t < 1$. The continuity of $r \mapsto f_r \in L^1(\nu)$ grant that the right hand side in above equality converges to $f_t - f_s$ in $L^1(\nu)$. On the other hand, for every $l \in L^\infty(\nu)$ it holds that

$$\int l(y) \int_s^t \frac{f_{r+\epsilon_n} - f_r}{\epsilon_n} dr(y) d\nu(y) = \int l(y) \chi_{[s,t]}(r) \frac{f_{r+\epsilon_n}(y) - f_r(y)}{\epsilon_n} d(\mathcal{L}_1 \times \nu)(r, y),$$

which in turn converges to $\int l(y)(\int_s^t f'_r dr)(y) d\nu(y)$ as $n \to \infty$. In other words, we showed that $\int_s^t \frac{f_{r+\epsilon_n} - f_r}{\epsilon_n} dr \rightharpoonup \int_s^t f'_r dr$ weakly in $L^1(\nu)$. We get

$$\int_s^t f'_r dr = f_t - f_s,$$

for every $0 < s < t < 1$. By the Lebesgue differential theorem, we know that f'_t is the strong derivative in $L^1(\nu)$ of the map $t \to f_t$ for a.e. $t \in [0, 1]$. $\square$

Lemma 3.12. *Let $h \in L^1(0, 1)$ be given. Then $h \in W^{1,1}(0, 1)$ if and only if there exists a function $g \in L^1(0, 1)$ such that*

$$h_t - h_s = \int_s^t g_r dr$$

holds for $\mathcal{L}^2$-a.e. $(t, s) \in \Delta$. Moreover, in such case it holds that $h' = g$.

Proof. ($\Rightarrow$) Fix any family of convolution kernels $\rho_\epsilon \in C_c^\infty(\mathbb{R})$, i.e. $\int \rho_\epsilon(x) dx = 1$, the support of ρ_ϵ is contained in $[-\epsilon, \epsilon]$. Let us define $h^\epsilon = h * \rho_\epsilon$ for all $\epsilon > 0$. Choose a sequence $\epsilon_n \to 0$ and a negligible Borel set $N \subset [0, 1]$ such that $h_t^{\epsilon_n} \to h_t$ as $n \to +\infty$ for every $t \in [0, 1] \backslash N$. Given $0 < s < t < 1$, for n large, we have

$$h_t^{\epsilon_n} - h_s^{\epsilon_n} = \int_s^t (h^{\epsilon_n})'_r dr = \int_s^t (h')_r^{\epsilon_n} dr,$$

letting $n \to +\infty$, we have

$$h_t - h_s = \int_s^t h'_r dr,$$

for $\mathcal{L}^2$-a.e. $(t, s) \in \Delta$.

($\Leftarrow$) By Fubini theorem, we see that for a.e. $\epsilon > 0$ it holds that $h_{t+\epsilon} - h_t = \int_t^{t+\epsilon} g_r dr$ for a.e. $t \in [0, 1]$. In particular, there is a sequence $\epsilon_n \to 0$ such that

$$h_{t+\epsilon_n} - h_t = \int_t^{t+\epsilon_n} g_r dr$$

for a.e. $t \in [0, 1]$. Now fix $\varphi \in C_c^\infty(0, 1)$. Then

$$\int \frac{\varphi_{t-\epsilon_n} - \varphi_t}{\epsilon_n} h_t dt = \int \frac{h_{t+\epsilon_n - h_t}}{\epsilon_n} \varphi_t dt = \int ⨍_t^{t+\epsilon_n} g_r dr \varphi_t dt. \tag{3.2}$$

By applying the dominated convergence theorem and Dunford-Pettis theorem, we finally deduce by letting $n \to \infty$ in the equation (3.2) that $-\int \varphi'_t h_t dt = \int g_t \varphi_t dt$. Hence $h \in W^{1,1}(0, 1)$ and $h' = g$. $\square$

References

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